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RECENT DEVELOPMENTS OF SCHWARZ'S TYPE TRACE INEQUALITIES FOR OPERATORS IN HILBERT SPACES

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Abstract. In this paper, we survey some recent trace inequalities for operators in Hilbert spaces that are connected to Schwarz's, Buzano's and Kato's inequalities and the reverses of Schwarz inequality known in the literature as Cassels' inequality and Shisha–Mond's inequality. Applications for some functionals that are naturally associated to some of these inequalities and for functions of operators defined by power series are given. Examples for fundamental functions such as the power, logarithmic, resolvent and exponential functions are provided as well.

1. Introduction

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^* f_j\|^2 \tag{1.2}$$

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showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator if and only if A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of Hilbert-Schmidt operators in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2 \right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H. This definition does not depend on the choice of the orthonormal basis.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt if and only if $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1.1. *We have*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^*Ae_i, e_i\rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$;

(ii) We have the inequalities

$$\|A\| \le \|A\|_2 \tag{1.5}$$

for any $A \in \mathcal{B}_2(H)$ *and*

$$\|AT\|_2, \|TA\|_2 \le \|T\| \|A\|_2 \tag{1.6}$$

for any $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$;

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_2(H)\mathcal{B}(H) \subseteq \mathcal{B}_2(H);$$

(iv) $\mathcal{B}_{fin}(H)$, *the space of operators of finite rank, is a dense subspace of* $\mathcal{B}_2(H)$;

(v) $\mathcal{B}_2(H) \subseteq \mathcal{K}(H)$, *where* $\mathcal{K}(H)$ *denotes the algebra of compact operators on* H.

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1.2. *If $A \in \mathcal{B}(H)$, then the following are equivalent:*
(i) $A \in \mathcal{B}_1(H)$;
(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$;
(ii) A (or $|A|$) is the product of two elements of $\mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 1.3. *With the above notations:*
(i) We have

$$\|A\|_1 = \|A^*\|_1 \text{ and } \|A\|_2 \leq \|A\|_1 \tag{1.8}$$

for any $A \in \mathcal{B}_1(H)$;
(ii) $\mathcal{B}_1(H)$ is an operator ideal in $\mathcal{B}(H)$, i.e.

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\{|\langle A, B\rangle_2| \mid B \in \mathcal{B}_2(H), \ \|B\|_2 \leq 1\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ is a Banach space.
(iv) We have the following isometric isomorphisms

$$\mathcal{B}_1(H) \cong K(H)^* \text{ and } \mathcal{B}_1(H)^* \cong \mathcal{B}(H),$$

where $K(H)^*$ is the dual space of $K(H)$ and $\mathcal{B}_1(H)^*$ is the dual space of $\mathcal{B}_1(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i \in I} \langle Ae_i, e_i\rangle \tag{1.9}$$

where $\{e_i\}_{i \in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 1.4. *We have*
(i) If $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A \in \mathcal{B}_1(H)$ and $T \in \mathcal{B}(H)$, then AT, $TA \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \leq \|A\|_1 \|T\|; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|\operatorname{tr}\| = 1$;
(iv) If A, $B \in \mathcal{B}_2(H)$ then AB, $BA \in \mathcal{B}_1(H)$ and $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;
(v) $\mathcal{B}_{fin}(H)$ is a dense subspace of $\mathcal{B}_1(H)$.

Utilizing the trace notation we obviously have that

$$\langle A, B\rangle_2 = \operatorname{tr}(B^*A) = \operatorname{tr}(AB^*) \text{ and } \|A\|_2^2 = \operatorname{tr}(A^*A) = \operatorname{tr}(|A|^2)$$

for any A, $B \in \mathcal{B}_2(H)$.

Now, for the finite dimensional case, it is well known that the trace functional is *submultiplicative*, that is, for *positive semidefinite matrices* A and B in $M_n(\mathbb{C})$,

$$0 \leq \operatorname{tr}(AB) \leq \operatorname{tr}(A) \operatorname{tr}(B).$$

Therefore

$$0 \leq \operatorname{tr}(A^k) \leq [\operatorname{tr}(A)]^k,$$

where k is any positive integer.

In 2000, Yang [83] proved a matrix trace inequality

$$\operatorname{tr}\left[(AB)^k\right] \leq (\operatorname{tr}A)^k(\operatorname{tr}B)^k, \tag{1.12}$$

where A and B are positive semidefinite matrices over $\mathbb{C}$ of the same order n and k is any positive integer. For related works the reader can refer to [18], [19], [70] and [85], which are continuations of the work of Bellman [6].

If $(H, \langle \cdot, \cdot \rangle)$ is a separable infinite-dimensional Hilbert space then the inequality (1.12) is also valid for any positive operators $A, B \in \mathcal{B}_1(H)$. This result was obtained by L. Liu in 2007, see [59].

In 2001, Yang et al. [84] improved (1.12) as follows:

$$\operatorname{tr}[(AB)^m] \leq \left[\operatorname{tr}\left(A^{2m}\right) \operatorname{tr}\left(B^{2m}\right)\right]^{1/2}, \tag{1.13}$$

where A and B are positive semidefinite matrices over $\mathbb{C}$ of the same order and m is any positive integer.

In [75] the authors have proved many trace inequalities for sums and products of matrices. For instance, if A and B are positive semidefinite matrices in $M_n(\mathbb{C})$ then

$$\operatorname{tr}\left[(AB)^k\right] \leq \min\left\{\|A\|^k \operatorname{tr}\left(B^k\right), \|B\|^k \operatorname{tr}\left(A^k\right)\right\} \tag{1.14}$$

for any positive integer k. Also, if $A, B \in M_n(\mathbb{C})$ then for $r \geq 1$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have the following *Young type inequality*

$$\operatorname{tr}(|AB^*|^r) \leq \operatorname{tr}\left[\left(\frac{|A|^p}{p} + \frac{|B|^q}{q}\right)^r\right]. \tag{1.15}$$

Ando [4] proved a very strong form of Young's inequality - it was shown that if A and B are in $M_n(\mathbb{C})$, then there is a *unitary matrix* U such that

$$|AB^*| \leq U\left(\frac{1}{p}|A|^p + \frac{1}{q}|B|^q\right)U^*,$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, which immediately gives the trace inequality

$$\operatorname{tr}(|AB^*|) \leq \frac{1}{p}\operatorname{tr}(|A|^p) + \frac{1}{q}\operatorname{tr}(|B|^q). \tag{1.16}$$

This inequality can also be obtained from (1.15) by taking $r = 1$.

Another Hölder type inequality has been proved by Manjegani in [68] and can be stated as follows:

$$\operatorname{tr}(AB) \leq [\operatorname{tr}(A^p)]^{1/p} [\operatorname{tr}(B^q)]^{1/q}, \tag{1.17}$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and A and B are positive semidefinite matrices.

For the theory of trace functionals and their applications the reader is referred to [77].

For other trace inequalities see [7], [18], [41], [33], [51], [58], [74] and [80].

In this paper we survey some recent trace inequalities obtained by the author for operators in Hilbert spaces that are connected to Schwarz's, Buzano's and Kato's inequalities and the reverses of Schwarz inequality known in the literature as Cassels' inequality and Shisha–Mond's inequality. Applications for some functionals that are naturally associated to some of these inequalities and for functions of operators defined by power series are given. Examples for fundamental functions such as the power, logarithmic, resolvent and exponential functions are provided as well.

Although some of these inequalities have been established for the general concept of positive linear map instead of trace, we would like to state them in this survey for trace to unify our approach to trace inequalities.

For Grüss' type inequalities for positive maps, see [5], [65] and [71]. For Cassels, Diaz–Metcalf and Shisha–Mond type inequalities, see [69]. For other inequalities for positive maps see [8], [9], [17], [78] and [86].

For trace inequalities for Hilbert space operators that appeared in information theory and quantum information theory we refer to [20], [42], [67] and [82].

2. Schwarz Type Trace Inequalities

2.1. **Some Trace Inequalities Via Hermitian Forms.** Let P a selfadjoint operator with $P \geq 0$. For $A \in \mathcal{B}_2(H)$ and $\{e_i\}_{i\in I}$ an orthonormal basis of H we have

$$\|A\|_{2,P}^2 := \operatorname{tr}(A^*PA) = \sum_{i\in I} \langle PAe_i, Ae_i\rangle \leq \|P\| \sum_{i\in I} \|Ae_i\|^2 = \|P\| \, \|A\|_2^2 ,$$

which shows that $\langle\cdot,\cdot\rangle_{2,P}$ defined by

$$\langle A, B\rangle_{2,P} := \operatorname{tr}(B^*PA) = \sum_{i\in I} \langle PAe_i, Be_i\rangle = \sum_{i\in I} \langle B^*PAe_i, e_i\rangle$$

is a *nonnegative Hermitian form* on $\mathcal{B}_2(H)$, i.e. $\langle\cdot,\cdot\rangle_{2,P}$ satisfies the properties:

(h) $\langle A, A\rangle_{2,P} \geq 0$ for any $A \in \mathcal{B}_2(H)$;

(hh) $\langle\cdot,\cdot\rangle_{2,P}$ is linear in the first variable;

(hhh) $\langle B, A\rangle_{2,P} = \overline{\langle A, B\rangle_{2,P}}$ for any $A,\ B \in \mathcal{B}_2(H)$.

Using the properties of the trace we also have the following representations

$$\|A\|_{2,P}^2 := \operatorname{tr}\left(P\,|A^*|^2\right) = \operatorname{tr}\left(|A^*|^2\,P\right)$$

and

$$\langle A, B\rangle_{2,P} := \operatorname{tr}(PAB^*) = \operatorname{tr}(AB^*P) = \operatorname{tr}(B^*PA)$$

for any $A,\ B \in \mathcal{B}_2(H)$.

We start with the following result:

Theorem 2.1 (Dragomir, 2014, [35])**.** *Let P a selfadjoint operator with $P \geq 0$, i.e. $\langle Px, x\rangle \geq 0$ for any $x \in H$.*

(i) For any $A,\ B\in\mathcal{B}_2(H)$

$$|\operatorname{tr}(PAB^*)|^2\le\operatorname{tr}\left(P|A^*|^2\right)\operatorname{tr}\left(P|B^*|^2\right)\tag{2.1}$$

and

$$\begin{aligned}&\left[\operatorname{tr}\left(P|A^*|^2\right)+2\operatorname{Re}\operatorname{tr}(PAB^*)+\operatorname{tr}\left(P|B^*|^2\right)\right]^{1/2}\\&\le\left[\operatorname{tr}\left(P|A^*|^2\right)\right]^{1/2}+\left[\operatorname{tr}\left(P|B^*|^2\right)\right]^{1/2};\end{aligned}\tag{2.2}$$

(ii) For any $A,\ B,\ C\in\mathcal{B}_2(H)$

$$\begin{aligned}&\left|\operatorname{tr}(PAB^*)\operatorname{tr}\left(P|C^*|^2\right)-\operatorname{tr}(PAC^*)\operatorname{tr}(PCB^*)\right|^2\\&\le\left[\operatorname{tr}\left(P|A^*|^2\right)\operatorname{tr}\left(P|C^*|^2\right)-|\operatorname{tr}(PAC^*)|^2\right]\\&\quad\times\left[\operatorname{tr}\left(P|B^*|^2\right)\operatorname{tr}\left(P|C^*|^2\right)-|\operatorname{tr}(PBC^*)|^2\right],\end{aligned}\tag{2.3}$$

$$\begin{aligned}&|\operatorname{tr}(PAB^*)|\operatorname{tr}\left(P|C^*|^2\right)\\&\le\left|\operatorname{tr}(PAB^*)\operatorname{tr}\left(P|C^*|^2\right)-\operatorname{tr}(PAC^*)\operatorname{tr}(PCB^*)\right|+|\operatorname{tr}(PAC^*)\operatorname{tr}(PCB^*)|\\&\le\left[\operatorname{tr}\left(P|A^*|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(P|B^*|^2\right)\right]^{1/2}\operatorname{tr}\left(P|C^*|^2\right)\end{aligned}\tag{2.4}$$

and

$$\begin{aligned}&|\operatorname{tr}(PAC^*)\operatorname{tr}(PCB^*)|\\&\le\frac{1}{2}\left[\left[\operatorname{tr}\left(P|A^*|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(P|B^*|^2\right)\right]^{1/2}+|\operatorname{tr}(PAB^*)|\right]\operatorname{tr}\left(P|C^*|^2\right).\end{aligned}\tag{2.5}$$

Proof. (i) Making use of the Schwarz inequality for the nonnegative hermitian form $\langle\cdot,\cdot\rangle_{2,P}$ we have

$$\left|\langle A,B\rangle_{2,P}\right|^2\le\langle A,A\rangle_{2,P}\langle B,B\rangle_{2,P}$$

for any $A,\ B\in\mathcal{B}_2(H)$ and the inequality (2.1) is proved.

We observe that $\|\cdot\|_{2,P}$ is a seminorm on $\mathcal{B}_2(H)$ and by the triangle inequality we have

$$\|A+B\|_{2,P}\le\|A\|_{2,P}+\|B\|_{2,P}$$

for any $A,\ B\in\mathcal{B}_2(H)$ and the inequality (2.2) is proved.

(ii) Let $C\in\mathcal{B}_2(H)$, $C\neq0$. Define the mapping $[\cdot,\cdot]_{2,P,C}:\mathcal{B}_2(H)\times\mathcal{B}_2(H)\to\mathbb{C}$ by

$$[A,B]_{2,P,C}:=\langle A,B\rangle_{2,P}\|C\|_{2,P}^2-\langle A,C\rangle_{2,P}\langle C,B\rangle_{2,P}.$$

Observe that $[\cdot,\cdot]_{2,P,C}$ is a nonnegative Hermitian form on $\mathcal{B}_2(H)$ and by Schwarz inequality we have

$$\begin{aligned}&\left|\langle A,B\rangle_{2,P}\|C\|_{2,P}^2-\langle A,C\rangle_{2,P}\langle C,B\rangle_{2,P}\right|^2\\&\le\left[\|A\|_{2,P}^2\|C\|_{2,P}^2-\left|\langle A,C\rangle_{2,P}\right|^2\right]\left[\|B\|_{2,P}^2\|C\|_{2,P}^2-\left|\langle B,C\rangle_{2,P}\right|^2\right]\end{aligned}\tag{2.6}$$

for any $A,\ B\in\mathcal{B}_2(H)$, which proves (2.3).

The case $C=0$ is obvious.

Utilizing the elementary inequality for real numbers m, n, p, q

$$\left(m^2-n^2\right)\left(p^2-q^2\right)\le (mp-nq)^2,$$

we can easily see that

$$\left[\|A\|_{2,P}^2\|C\|_{2,P}^2-\left|\langle A,C\rangle_{2,P}\right|^2\right]\left[\|B\|_{2,P}^2\|C\|_{2,P}^2-\left|\langle B,C\rangle_{2,P}\right|^2\right] \tag{2.7a}$$
$$\le\left(\|A\|_{2,P}\|B\|_{2,P}\|C\|_{2,P}^2-\left|\langle A,C\rangle_{2,P}\right|\left|\langle B,C\rangle_{2,P}\right|\right)^2$$

for any $A,\ B,\ C\in\mathcal{B}_2(H)$.

Since, by Schwarz's inequality we have

$$\|A\|_{2,P}\|C\|_{2,P}\ge\left|\langle A,C\rangle_{2,P}\right|$$

and

$$\|B\|_{2,P}\|C\|_{2,P}\ge\left|\langle B,C\rangle_{2,P}\right|,$$

then by multiplying these inequalities we have

$$\|A\|_{2,P}\|B\|_{2,P}\|C\|_{2,P}^2\ge\left|\langle A,C\rangle_{2,P}\right|\left|\langle B,C\rangle_{2,P}\right|$$

for any $A,\ B,\ C\in\mathcal{B}_2(H)$.

Utilizing the inequalities (2.6) and (2.7a) and taking the square root we get

$$\left|\langle A,B\rangle_{2,P}\|C\|_{2,P}^2-\langle A,C\rangle_{2,P}\langle C,B\rangle_{2,P}\right| \tag{2.8}$$
$$\le\|A\|_{2,P}\|B\|_{2,P}\|C\|_{2,P}^2-\left|\langle A,C\rangle_{2,P}\right|\left|\langle B,C\rangle_{2,P}\right|$$

for any $A,\ B,\ C\in\mathcal{B}_2(H)$, which proves the second inequality in (2.4).

The first inequality is obvious by the modulus properties.

By the triangle inequality for modulus we also have

$$\left|\langle A,C\rangle_{2,P}\langle C,B\rangle_{2,P}\right|-\left|\langle A,B\rangle_{2,P}\right|\|C\|_{2,P}^2 \tag{2.9}$$
$$\le\left|\langle A,B\rangle_{2,P}\|C\|_{2,P}^2-\langle A,C\rangle_{2,P}\langle C,B\rangle_{2,P}\right|$$

for any $A,\ B,\ C\in\mathcal{B}_2(H)$.

On making use of (2.8) and (2.9) we have

$$\left|\langle A,C\rangle_{2,P}\langle C,B\rangle_{2,P}\right|-\left|\langle A,B\rangle_{2,P}\right|\|C\|_{2,P}^2$$
$$\le\|A\|_{2,P}\|B\|_{2,P}\|C\|_{2,P}^2-\left|\langle A,C\rangle_{2,P}\right|\left|\langle B,C\rangle_{2,P}\right|,$$

which is equivalent to the desired inequality (2.5). □

Remark 2.2. By the triangle inequality for the hermitian form $[\cdot,\cdot]_{2,P,C}:\mathcal{B}_2(H)\times\mathcal{B}_2(H)\to\mathbb{C}$,

$$[A,B]_{2,P,C}:=\langle A,B\rangle_{2,P}\|C\|_{2,P}^2-\langle A,C\rangle_{2,P}\langle C,B\rangle_{2,P}$$

we get

$$\left(\|A+B\|_{2,P}^2\|C\|_{2,P}^2-\left|\langle A+B,C\rangle_{2,P}\right|^2\right)^{1/2}$$
$$\leq\left(\|A\|_{2,P}^2\|C\|_{2,P}^2-\left|\langle A,C\rangle_{2,P}\right|^2\right)^{1/2}+\left(\|B\|_{2,P}^2\|C\|_{2,P}^2-\left|\langle B,C\rangle_{2,P}\right|^2\right)^{1/2},$$

which can be written as

$$\begin{aligned}&\left(\operatorname{tr}\left[P\left|(A+B)^*\right|^2\right]\operatorname{tr}\left(P\left|C^*\right|^2\right)-\left|\operatorname{tr}\left[P\left(A+B\right)C^*\right]\right|^2\right)^{1/2}\\&\quad\leq\left(\operatorname{tr}\left(P\left|A^*\right|^2\right)\operatorname{tr}\left(P\left|C^*\right|^2\right)-\left|\operatorname{tr}\left(PAC^*\right)\right|^2\right)^{1/2}\\&\qquad+\left(\operatorname{tr}\left(P\left|B^*\right|^2\right)\operatorname{tr}\left(P\left|C^*\right|^2\right)-\left|\operatorname{tr}\left(PBC^*\right)\right|^2\right)^{1/2}\end{aligned}\tag{2.10}$$

for any $A,\ B,\ C\in\mathcal{B}_2(H)$.

Remark 2.3. If we take $B=\lambda C$ in (2.10), then we get

$$\begin{aligned}0&\leq\operatorname{tr}\left[P\left|(A+\lambda C)^*\right|^2\right]\operatorname{tr}\left(P\left|C^*\right|^2\right)-\left|\operatorname{tr}\left[P\left(A+\lambda C\right)C^*\right]\right|^2\\&\leq\operatorname{tr}\left(P\left|A^*\right|^2\right)\operatorname{tr}\left(P\left|C^*\right|^2\right)-\left|\operatorname{tr}\left(C^*PA\right)\right|^2\end{aligned}\tag{2.11}$$

for any $\lambda\in\mathbb{C}$ and $A,\ C\in\mathcal{B}_2(H)$.

Therefore, we have the bound

$$\begin{aligned}&\sup_{\lambda\in\mathbb{C}}\left\{\operatorname{tr}\left[P\left|(A+\lambda C)^*\right|^2\right]\operatorname{tr}\left(P\left|C^*\right|^2\right)-\left|\operatorname{tr}\left[P\left(A+\lambda C\right)C^*\right]\right|^2\right\}\\&\quad=\operatorname{tr}\left(P\left|A^*\right|^2\right)\operatorname{tr}\left(P\left|C^*\right|^2\right)-\left|\operatorname{tr}\left(PAC^*\right)\right|^2.\end{aligned}\tag{2.12}$$

We also have the inequalities

$$\begin{aligned}0&\leq\operatorname{tr}\left[P\left|(A\pm C)^*\right|^2\right]\operatorname{tr}\left(P\left|C^*\right|^2\right)-\left|\operatorname{tr}\left[P\left(A\pm C\right)C^*\right]\right|^2\\&\leq\operatorname{tr}\left(P\left|A^*\right|^2\right)\operatorname{tr}\left(P\left|C^*\right|^2\right)-\left|\operatorname{tr}\left(PAC^*\right)\right|^2\end{aligned}\tag{2.13}$$

for any $A,\ C\in\mathcal{B}_2(H)$.

Remark 2.4. We observe that, by replacing A^* by A and B^* by B etc above, we can get the dual inequalities, like, for instance

$$\begin{aligned}&\left|\operatorname{tr}\left(PA^*C\right)\operatorname{tr}\left(PC^*B\right)\right|\\&\quad\leq\frac{1}{2}\left[\left[\operatorname{tr}\left(P\left|A\right|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(P\left|B\right|^2\right)\right]^{1/2}+\left|\operatorname{tr}\left(PA^*B\right)\right|\right]\operatorname{tr}\left(P\left|C\right|^2\right),\end{aligned}\tag{2.14}$$

that holds for any $A,\ B,\ C\in\mathcal{B}_2(H)$.

This is an operator version of Buzano's inequality in inner product spaces, namely

$$\left|\langle x,e\rangle\langle e,y\rangle\right|\leq\frac{1}{2}\left[\|x\|\|y\|+\left|\langle x,y\rangle\right|\right]\tag{2.15}$$

for $x,\ y,\ e\in H$ with $\|e\|=1$.

Since

$$\left|\operatorname{tr}\left(PA^*C\right)\right|=\left|\overline{\operatorname{tr}\left(PA^*C\right)}\right|=\left|\operatorname{tr}\left[\left(PA^*C\right)^*\right]\right|=\left|\operatorname{tr}\left(C^*AP\right)\right|=\left|\operatorname{tr}\left(PC^*A\right)\right|,$$

$$|\text{tr}(PC^*B)| = |\text{tr}(PB^*C)|$$

and

$$|\text{tr}(PA^*B)| = |\text{tr}(PB^*A)|$$

then the inequality (2.14) can be also written as

$$\begin{aligned} &|\text{tr}(PC^*A)\,\text{tr}(PB^*C)| \\ &\leq \frac{1}{2}\left[\left[\text{tr}\left(P|A|^2\right)\right]^{1/2}\left[\text{tr}\left(P|B|^2\right)\right]^{1/2} + |\text{tr}(PB^*A)|\right]\text{tr}\left(P|C|^2\right), \end{aligned} \tag{2.16}$$

that holds for any $A, B, C \in \mathcal{B}_2(H)$.

If we take in (2.16) $B = A^*$ then we get the following inequality

$$\begin{aligned} &|\text{tr}(PC^*A)\,\text{tr}(PAC)| \\ &\leq \frac{1}{2}\left[\left[\text{tr}\left(P|A|^2\right)\right]^{1/2}\left[\text{tr}\left(P|A^*|^2\right)\right]^{1/2} + \left|\text{tr}\left(PA^2\right)\right|\right]\text{tr}\left(P|C|^2\right), \end{aligned} \tag{2.17}$$

for any $A, B, C \in \mathcal{B}_2(H)$.

If A is a normal operator, i.e. $|A|^2 = |A^*|^2$ then we have from (2.17) that

$$|\text{tr}(PC^*A)\,\text{tr}(PAC)| \leq \frac{1}{2}\left[\text{tr}\left(P|A|^2\right) + \left|\text{tr}\left(PA^2\right)\right|\right]\text{tr}\left(P|C|^2\right), \tag{2.18}$$

In particular, if C is selfadjoint and $C \in \mathcal{B}_2(H)$, then

$$|\text{tr}(PAC)|^2 \leq \frac{1}{2}\left[\text{tr}\left(P|A|^2\right) + \left|\text{tr}\left(PA^2\right)\right|\right]\text{tr}\left(PC^2\right), \tag{2.19}$$

for any $A \in \mathcal{B}_2(H)$ a normal operator.

We notice that (2.19) is a trace operator version of *de Bruijn inequality* obtained in 1960 in [10], which gives the following refinement of the Cauchy–Bunyakovsky–Schwarz inequality:

$$\left|\sum_{i=1}^{n} a_i z_i\right|^2 \leq \frac{1}{2}\sum_{i=1}^{n} a_i^2\left[\sum_{i=1}^{n}|z_i|^2 + \left|\sum_{i=1}^{n} z_i^2\right|\right], \tag{2.20}$$

provided that a_i are real numbers while z_i are complex for each $i \in \{1, \cdots, n\}$.

We notice that, if $P \in \mathcal{B}_1(H)$, $P \geq 0$ and $A, B \in \mathcal{B}(H)$, then

$$\langle A, B\rangle_{2,P} := \text{tr}(PAB^*) = \text{tr}(AB^*P) = \text{tr}(B^*PA)$$

is a *nonnegative Hermitian form* on $\mathcal{B}(H)$ and all the inequalities above will hold for $A, B, C \in \mathcal{B}(H)$. The details are left to the reader.

2.2. **Some Functional Properties.** We consider now the convex cone $\mathcal{B}_+(H)$ of nonnegative operators on the complex Hilbert space H and, for $A, B \in \mathcal{B}_2(H)$ define the functional $\sigma_{A,B} : \mathcal{B}_+(H) \to [0, \infty)$ by

$$\sigma_{A,B}(P) := \left[\text{tr}\left(P|A|^2\right)\right]^{1/2}\left[\text{tr}\left(P|B|^2\right)\right]^{1/2} - |\text{tr}(PA^*B)|\,(\geq 0). \tag{2.21}$$

The following theorem collects some fundamental properties of this functional.

Theorem 2.5 (Dragomir, 2014, [35])**.** *Let $A,\ B\in\mathcal{B}_2(H)$.*

(i) For any $P,\ Q\in\mathcal{B}_+(H)$

$$\sigma_{A,B}(P+Q)\geq\sigma_{A,B}(P)+\sigma_{A,B}(Q)\,(\geq 0)\tag{2.22}$$

,namely, $\sigma_{A,B}$ is a superadditive functional on $\mathcal{B}_+(H)$;

(ii) For any $P,\ Q\in\mathcal{B}_+(H)$ with $P\geq Q$

$$\sigma_{A,B}(P)\geq\sigma_{A,B}(Q)\,(\geq 0),\tag{2.23}$$

namely, $\sigma_{A,B}$ is a monotonic nondecreasing functional on $\mathcal{B}_+(H)$;

(iii) If $P,\ Q\in\mathcal{B}_+(H)$ and there exist the constants $M>m>0$ such that $MQ\geq P\geq mQ$ then

$$M\sigma_{A,B}(Q)\geq\sigma_{A,B}(P)\geq m\sigma_{A,B}(Q)\,(\geq 0).\tag{2.24}$$

Proof. (i) Let $P,\ Q\in\mathcal{B}_+(H)$. On utilizing the elementary inequality

$$\left(a^2+b^2\right)^{1/2}\left(c^2+d^2\right)^{1/2}\geq ac+bd,\ \ a,b,c,d\geq 0$$

and the triangle inequality for the modulus, we have

$$\begin{aligned}
&\sigma_{A,B}(P+Q)\\
&=\left[\operatorname{tr}\left((P+Q)|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left((P+Q)|B|^2\right)\right]^{1/2}-\left|\operatorname{tr}\left((P+Q)A^*B\right)\right|\\
&=\left[\operatorname{tr}\left(P|A|^2+Q|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(P|B|^2+Q|B|^2\right)\right]^{1/2}\\
&\quad-\left|\operatorname{tr}\left(PA^*B+QA^*B\right)\right|\\
&=\left[\operatorname{tr}\left(P|A|^2\right)+\operatorname{tr}\left(Q|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(P|B|^2\right)+\operatorname{tr}\left(Q|B|^2\right)\right]^{1/2}\\
&\quad-\left|\operatorname{tr}\left(PA^*B\right)+\operatorname{tr}\left(QA^*B\right)\right|\\
&\geq\left[\operatorname{tr}\left(P|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}+\left[\operatorname{tr}\left(Q|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(Q|B|^2\right)\right]^{1/2}\\
&\quad-\left|\operatorname{tr}\left(PA^*B\right)\right|-\left|\operatorname{tr}\left(QA^*B\right)\right|\\
&=\sigma_{A,B}(P)+\sigma_{A,B}(Q)
\end{aligned}$$

and the inequality (2.22) is proved.

(ii) Let $P,\ Q\in\mathcal{B}_+(H)$ with $P\geq Q$. Utilizing the superadditivity property we have

$$\sigma_{A,B}(P)=\sigma_{A,B}((P-Q)+Q)\geq\sigma_{A,B}(P-Q)+\sigma_{A,B}(Q)\geq\sigma_{A,B}(Q)$$

and the inequality (2.23) is obtained.

(iii) From the monotonicity property we have

$$\sigma_{A,B}(P)\geq\sigma_{A,B}(mQ)=m\sigma_{A,B}(Q)$$

and a similar inequality for M, which prove the desired result (2.24). □

Corollary 2.6. *Let A, $B \in \mathcal{B}_2(H)$ and $P \in \mathcal{B}(H)$ such that there exist the constants $M > m > 0$ with $M1_H \geq P \geq m1_H$. Then*

$$M\left(\left[\operatorname{tr}\left(|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|B|^2\right)\right]^{1/2} - |\operatorname{tr}(A^*B)|\right) \tag{2.25}$$
$$\geq \left[\operatorname{tr}\left(P|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2} - |\operatorname{tr}(PA^*B)|$$
$$\geq m\left(\left[\operatorname{tr}\left(|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|B|^2\right)\right]^{1/2} - |\operatorname{tr}(A^*B)|\right).$$

Let $P = |V|^2$ with $V \in \mathcal{B}(H)$. If A, $B \in \mathcal{B}_2(H)$ then

$$\begin{aligned}\sigma_{A,B}\left(|V|^2\right) &= \left[\operatorname{tr}\left(|V|^2|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|V|^2|B|^2\right)\right]^{1/2} - \left|\operatorname{tr}\left(|V|^2A^*B\right)\right| \\ &= [\operatorname{tr}(V^*VA^*A)]^{1/2}[\operatorname{tr}(V^*VB^*B)]^{1/2} - |\operatorname{tr}(V^*VA^*B)| \\ &= [\operatorname{tr}(VA^*AV^*)]^{1/2}[\operatorname{tr}(VB^*BV^*)]^{1/2} - |\operatorname{tr}(VA^*BV^*)| \\ &= [\operatorname{tr}((AV^*)^*AV^*)]^{1/2}[\operatorname{tr}((BV^*)^*BV^*)]^{1/2} - |\operatorname{tr}((AV^*)^*BV^*)| \\ &= \left[\operatorname{tr}\left(|AV^*|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|BV^*|^2\right)\right]^{1/2} - |\operatorname{tr}((AV^*)^*BV^*)|.\end{aligned}$$

On utilizing the property (2.22) for $P = |V|^2$, $Q = |U|^2$ with V, $U \in \mathcal{B}(H)$, then we have for any A, $B \in \mathcal{B}_2(H)$ the following trace inequality

$$\left[\operatorname{tr}\left(|AV^*|^2 + |AU^*|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|BV^*|^2 + |BU^*|^2\right)\right]^{1/2} \tag{2.26}$$
$$- |\operatorname{tr}((AV^*)^*BV^* + (AU^*)^*BU^*)|$$
$$\geq \left[\operatorname{tr}\left(|AV^*|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|BV^*|^2\right)\right]^{1/2} - |\operatorname{tr}((AV^*)^*BV^*)|$$
$$+ \left[\operatorname{tr}\left(|AU^*|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|BU^*|^2\right)\right]^{1/2} - |\operatorname{tr}((AU^*)^*BU^*)|\ (\geq 0).$$

Also, if $|V|^2 \geq |U|^2$ with V, $U \in \mathcal{B}(H)$, then we have for any A, $B \in \mathcal{B}_2(H)$ that

$$\left[\operatorname{tr}\left(|AV^*|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|BV^*|^2\right)\right]^{1/2} - |\operatorname{tr}((AV^*)^*BV^*)| \tag{2.27}$$
$$\geq \left[\operatorname{tr}\left(|AU^*|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|BU^*|^2\right)\right]^{1/2} - |\operatorname{tr}((AU^*)^*BU^*)|\ (\geq 0).$$

If $U \in \mathcal{B}(H)$ is invertible, then

$$\frac{1}{\|U^{-1}\|}\|x\| \leq \|Ux\| \leq \|U\|\|x\| \text{ for any } x \in H,$$

which implies that

$$\frac{1}{\|U^{-1}\|^2}1_H \leq |U|^2 \leq \|U\|^2 1_H.$$

By making use of (2.25) we have the following trace inequality

$$\|U\|^2\left(\left[\operatorname{tr}\left(|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|B|^2\right)\right]^{1/2} - |\operatorname{tr}(A^*B)|\right) \tag{2.28}$$
$$\geq \left[\operatorname{tr}\left(|AU^*|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|BU^*|^2\right)\right]^{1/2} - |\operatorname{tr}((AU^*)^*BU^*)|$$
$$\geq \frac{1}{\|U^{-1}\|^2}\left(\left[\operatorname{tr}\left(|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|B|^2\right)\right]^{1/2} - |\operatorname{tr}(A^*B)|\right)$$

for any $A, B \in \mathcal{B}_2(H)$.

Similar results may be stated for $P \in \mathcal{B}_1(H)$, $P \geq 0$ and $A, B \in \mathcal{B}(H)$. The details are omitted.

2.3. Inequalities for Sequences of Operators.

For $n \geq 2$, define the Cartesian products $\mathcal{B}^{(n)}(H) := \mathcal{B}(H) \times \cdots \times \mathcal{B}(H)$, $\mathcal{B}_2^{(n)}(H) := \mathcal{B}_2(H) \times \cdots \times \mathcal{B}_2(H)$ and $\mathcal{B}_+^{(n)}(H) := \mathcal{B}_+(H) \times \cdots \times \mathcal{B}_+(H)$ where $\mathcal{B}_+(H)$ denotes the convex cone of nonnegative selfadjoint operators on H, i.e. $P \in \mathcal{B}_+(H)$ if $\langle Px, x\rangle \geq 0$ for any $x \in H$.

Proposition 2.7 (Dragomir, 2014, [35]). *Let* $\mathbf{P} = (P_1, \cdots, P_n) \in \mathcal{B}_+^{(n)}(H)$ *and* $\mathbf{A} = (A_1, \cdots, A_n)$, $\mathbf{B} = (B_1, \cdots, B_n) \in \mathcal{B}_2^{(n)}(H)$ *and* $\mathbf{z} = (z_1, \cdots, z_n) \in \mathbb{C}^n$ *with* $n \geq 2$. *Then*

$$\left|\operatorname{tr}\left(\sum_{k=1}^{n} z_k P_k A_k^* B_k\right)\right|^2 \leq \operatorname{tr}\left(\sum_{k=1}^{n} |z_k| P_k |A_k|^2\right) \operatorname{tr}\left(\sum_{k=1}^{n} |z_k| P_k |B_k|^2\right). \quad (2.29)$$

Proof. Using the properties of modulus and the inequality (2.1) we have

$$\begin{aligned}
\left|\operatorname{tr}\left(\sum_{k=1}^{n} z_k P_k A_k^* B_k\right)\right| &= \left|\sum_{k=1}^{n} z_k \operatorname{tr}(P_k A_k^* B_k)\right| \\
&\leq \sum_{k=1}^{n} |z_k| \left|\operatorname{tr}(P_k A_k^* B_k)\right| \\
&\leq \sum_{k=1}^{n} |z_k| \left[\operatorname{tr}\left(P_k |A_k|^2\right)\right]^{1/2} \left[\operatorname{tr}\left(P_k |B_k|^2\right)\right]^{1/2}.
\end{aligned}$$

Utilizing the weighted discrete Cauchy–Bunyakovsky–Schwarz inequality we also have

$$\begin{aligned}
&\sum_{k=1}^{n} |z_k| \left[\operatorname{tr}\left(P_k |A_k|^2\right)\right]^{1/2} \left[\operatorname{tr}\left(P_k |B_k|^2\right)\right]^{1/2} \\
&\quad \leq \left(\sum_{k=1}^{n} |z_k| \left(\left[\operatorname{tr}\left(P_k |A_k|^2\right)\right]^{1/2}\right)^2\right)^{1/2} \left(\sum_{k=1}^{n} |z_k| \left(\left[\operatorname{tr}\left(P_k |B_k|^2\right)\right]^{1/2}\right)^2\right)^{1/2} \\
&\quad = \left(\sum_{k=1}^{n} |z_k| \operatorname{tr}\left(P_k |A_k|^2\right)\right)^{1/2} \left(\sum_{k=1}^{n} |z_k| \operatorname{tr}\left(P_k |B_k|^2\right)\right)^{1/2} \\
&\quad = \left(\operatorname{tr}\left(\sum_{k=1}^{n} |z_k| P_k |A_k|^2\right)\right)^{1/2} \left(\operatorname{tr}\left(\sum_{k=1}^{n} |z_k| P_k |B_k|^2\right)\right)^{1/2},
\end{aligned}$$

which is equivalent to the desired result (2.29). □

We consider the functional for n-tuples of nonnegative operators as follows:

$$\sigma_{\mathbf{A},\mathbf{B}}(\mathbf{P}) := \left[\operatorname{tr}\left(\sum_{k=1}^{n} P_k |A_k|^2\right)\right]^{1/2} \left[\operatorname{tr}\left(\sum_{k=1}^{n} P_k |B_k|^2\right)\right]^{1/2} - \left|\operatorname{tr}\left(\sum_{k=1}^{n} P_k A_k^* B_k\right)\right|. \tag{2.30}$$

Utilizing a similar argument to the one in Theorem 2.5 we can state:

Proposition 2.8. *Let* $\mathbf{A} = (A_1, \cdots, A_n)$, $\mathbf{B} = (B_1, \cdots, B_n) \in \mathcal{B}_2^{(n)}(H)$.

(i) For any $\mathbf{P}$, $\mathbf{Q} \in \mathcal{B}_+^{(n)}(H)$

$$\sigma_{\mathbf{A},\mathbf{B}}(\mathbf{P}+\mathbf{Q}) \geq \sigma_{\mathbf{A},\mathbf{B}}(\mathbf{P}) + \sigma_{\mathbf{A},\mathbf{B}}(\mathbf{Q})\,(\geq 0), \tag{2.31}$$

namely, $\sigma_{\mathbf{A},\mathbf{B}}$ *is a superadditive functional on* $\mathcal{B}_+^{(n)}(H)$;

(ii) For any $\mathbf{P}$, $\mathbf{Q} \in \mathcal{B}_+^{(n)}(H)$ *with* $\mathbf{P} \geq \mathbf{Q}$, *namely* $P_k \geq Q_k$ *for all* $k \in \{1, \cdots, n\}$

$$\sigma_{\mathbf{A},\mathbf{B}}(\mathbf{P}) \geq \sigma_{\mathbf{A},\mathbf{B}}(\mathbf{Q})\,(\geq 0), \tag{2.32}$$

namely, $\sigma_{\mathbf{A},\mathbf{B}}$ *is a monotonic nondecreasing functional on* $\mathcal{B}_+^{(n)}(H)$;

(iii) If $\mathbf{P}$, $\mathbf{Q} \in \mathcal{B}_+^{(n)}(H)$ *and there exist the constants* $M > m > 0$ *such that* $M\mathbf{Q} \geq \mathbf{P} \geq m\mathbf{Q}$ *then*

$$M\sigma_{\mathbf{A},\mathbf{B}}(\mathbf{Q}) \geq \sigma_{\mathbf{A},\mathbf{B}}(\mathbf{P}) \geq m\sigma_{\mathbf{A},\mathbf{B}}(\mathbf{Q})\,(\geq 0). \tag{2.33}$$

If $\mathbf{P} = (p_1 1_H, \cdots, p_n 1_H)$ with $p_k \geq 0$, $k \in \{1, \cdots, n\}$ then the functional of nonnegative weights $\mathbf{p} = (p_1, \cdots, p_n)$ defined by

$$\sigma_{\mathbf{A},\mathbf{B}}(\mathbf{p}) := \left[\operatorname{tr}\left(\sum_{k=1}^{n} p_k |A_k|^2\right)\right]^{1/2} \left[\operatorname{tr}\left(\sum_{k=1}^{n} p_k |B_k|^2\right)\right]^{1/2} - \left|\operatorname{tr}\left(\sum_{k=1}^{n} p_k A_k^* B_k\right)\right|. \tag{2.34}$$

has the same properties as in (2.31)-(2.33).

Moreover, we have the simple bounds:

$$\begin{aligned} &\max_{k\in\{1,\cdots,n\}}\{p_k\}\left\{\left[\operatorname{tr}\left(\sum_{k=1}^{n}|A_k|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(\sum_{k=1}^{n}|B_k|^2\right)\right]^{1/2}\right. \\ &\qquad\left.-\left|\operatorname{tr}\left(\sum_{k=1}^{n}A_k^*B_k\right)\right|\right\} \\ &\quad\geq\left[\operatorname{tr}\left(\sum_{k=1}^{n}p_k|A_k|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(\sum_{k=1}^{n}p_k|B_k|^2\right)\right]^{1/2}-\left|\operatorname{tr}\left(\sum_{k=1}^{n}p_kA_k^*B_k\right)\right| \\ &\quad\geq\min_{k\in\{1,\cdots,n\}}\{p_k\}\left\{\left[\operatorname{tr}\left(\sum_{k=1}^{n}|A_k|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(\sum_{k=1}^{n}|B_k|^2\right)\right]^{1/2}\right. \\ &\qquad\left.-\left|\operatorname{tr}\left(\sum_{k=1}^{n}A_k^*B_k\right)\right|\right\}. \end{aligned} \tag{2.35}$$

2.4. **Inequalities for Power Series of Operators.**

Denote by:

$$D(0,R)=\begin{cases}\{z\in\mathbb{C}:|z|<R\}, & \text{if } R<\infty\\ \mathbb{C}, & \text{if } R=\infty,\end{cases}$$

and consider the functions:

$$\lambda\mapsto f(\lambda):D(0,R)\to\mathbb{C},\ f(\lambda):=\sum_{n=0}^{\infty}\alpha_n\lambda^n$$

and

$$\lambda\mapsto f_a(\lambda):D(0,R)\to\mathbb{C},\ f_a(\lambda):=\sum_{n=0}^{\infty}|\alpha_n|\lambda^n.$$

As some natural examples that are useful for applications, we can point out that, if

$$\begin{aligned} f(\lambda)&=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\lambda^n=\ln\frac{1}{1+\lambda},\ \lambda\in D(0,1); \\ g(\lambda)&=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\lambda^{2n}=\cos\lambda,\ \lambda\in\mathbb{C}; \\ h(\lambda)&=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\lambda^{2n+1}=\sin\lambda,\ \lambda\in\mathbb{C}; \\ l(\lambda)&=\sum_{n=0}^{\infty}(-1)^n\lambda^n=\frac{1}{1+\lambda},\ \lambda\in D(0,1); \end{aligned} \tag{2.36}$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_a(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n}\lambda^n = \ln\frac{1}{1-\lambda}, \;\; \lambda \in D(0,1); \tag{2.37}$$
$$g_a(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n)!}\lambda^{2n} = \cosh\lambda, \;\; \lambda \in \mathbb{C};$$
$$h_a(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\lambda^{2n+1} = \sinh\lambda, \;\; \lambda \in \mathbb{C};$$
$$l_a(\lambda) = \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}, \;\; \lambda \in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\lambda^n \qquad \lambda \in \mathbb{C}, \tag{2.38}$$
$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda \in D(0,1);$$
$$\sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}\lambda^{2n+1}, \qquad \lambda \in D(0,1);$$
$$\tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda \in D(0,1)$$
$${}_2F_1(\alpha,\beta,\gamma,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^n, \alpha,\beta,\gamma > 0,$$
$$\lambda \in D(0,1);$$

where Γ is *Gamma function.*

Proposition 2.9 (Dragomir, 2014, [35])**.** *Let* $f(\lambda) := \sum_{n=0}^{\infty} \alpha_n \lambda^n$ *be a power series with complex coefficients and convergent on the open disk* $D(0,R)$, $R > 0$. *If* $(H, \langle\cdot,\cdot\rangle)$ *is a separable infinite-dimensional Hilbert space and* A, $B \in \mathcal{B}_1(H)$ *are positive operators with* $\operatorname{tr}(A)$, $\operatorname{tr}(B) < R^{1/2}$, *then*

$$|\operatorname{tr}(f(AB))|^2 \leq f_a^2(\operatorname{tr}A\operatorname{tr}B) \leq f_a\left((\operatorname{tr}A)^2\right) f_a\left((\operatorname{tr}B)^2\right). \tag{2.39}$$

Proof. By the inequality (1.12) for the positive operators A, $B \in \mathcal{B}_1(H)$ we have

$$\left|\operatorname{tr}\left[\sum_{k=0}^{n} \alpha_k (AB)^k\right]\right| = \left|\sum_{k=0}^{n} \alpha_k \operatorname{tr}\left[(AB)^k\right]\right| \tag{2.40}$$
$$\leq \sum_{k=0}^{n} |\alpha_k| \left|\operatorname{tr}\left[(AB)^k\right]\right| = \sum_{k=0}^{n} |\alpha_k| \operatorname{tr}\left[(AB)^k\right]$$
$$\leq \sum_{k=0}^{n} |\alpha_k| (\operatorname{tr}A)^k (\operatorname{tr}B)^k = \sum_{k=0}^{n} |\alpha_k| (\operatorname{tr}A \operatorname{tr}B)^k.$$

Utilizing the weighted Cauchy–Bunyakovsky–Schwarz inequality for sums we have

$$\sum_{k=0}^{n} |\alpha_k| (\operatorname{tr}A)^k (\operatorname{tr}B)^k \leq \left(\sum_{k=0}^{n} |\alpha_k| (\operatorname{tr}A)^{2k}\right)^{1/2} \left(\sum_{k=0}^{n} |\alpha_k| (\operatorname{tr}B)^{2k}\right)^{1/2}. \tag{2.41}$$

Then by (2.40) and (2.41) we have

$$\left|\operatorname{tr}\left[\sum_{k=0}^{n} \alpha_k (AB)^k\right]\right|^2 \leq \left[\sum_{k=0}^{n} |\alpha_k| (\operatorname{tr}A \operatorname{tr}B)^k\right]^2 \tag{2.42}$$
$$\leq \sum_{k=0}^{n} |\alpha_k| \left[(\operatorname{tr}A)^2\right]^k \sum_{k=0}^{n} |\alpha_k| \left[(\operatorname{tr}B)^2\right]^k$$

for $n \geq 1$.

Since $0 \leq \operatorname{tr}(A), \operatorname{tr}(B) < R^{1/2}$, the numerical series

$$\sum_{k=0}^{\infty} |\alpha_k| (\operatorname{tr}A \operatorname{tr}B)^k, \quad \sum_{k=0}^{\infty} |\alpha_k| \left[(\operatorname{tr}A)^2\right]^k \text{ and } \sum_{k=0}^{\infty} |\alpha_k| \left[(\operatorname{tr}B)^2\right]^k$$

are convergent.

Also, since $0 \leq \operatorname{tr}(AB) \leq \operatorname{tr}(A)\operatorname{tr}(B) < R$, the operator series $\sum_{k=0}^{\infty} \alpha_k (AB)^k$ is convergent in $\mathcal{B}_1(H)$.

Letting $n \to \infty$ in (2.42) and utilizing the continuity property of $\operatorname{tr}(\cdot)$ on $\mathcal{B}_1(H)$ we get the desired result (2.39). □

Example 2.10. a) If we take in (2.39) $f(\lambda) = (1 \pm \lambda)^{-1}$, $|\lambda| < 1$ then we get the inequality

$$\left|\operatorname{tr}\left((1_H \pm AB)^{-1}\right)\right|^2 \leq \left(1 - (\operatorname{tr}A)^2\right)^{-1} \left(1 - (\operatorname{tr}B)^2\right)^{-1} \tag{2.43}$$

for any A, $B \in \mathcal{B}_1(H)$ positive operators with $\operatorname{tr}(A), \operatorname{tr}(B) < 1$.

b) If we take in (2.39) $f(\lambda) = \ln(1 \pm \lambda)^{-1}$, $|\lambda| < 1$, then we get the inequality

$$\left|\operatorname{tr}\left(\ln(1_H \pm AB)^{-1}\right)\right|^2 \leq \ln\left(1 - (\operatorname{tr}A)^2\right)^{-1} \ln\left(1 - (\operatorname{tr}B)^2\right)^{-1} \tag{2.44}$$

for any A, $B \in \mathcal{B}_1(H)$ positive operators with $\operatorname{tr}(A), \operatorname{tr}(B) < 1$.

We have the following result as well:

Theorem 2.11 (Dragomir, 2014, [35])**.** *Let $f(\lambda) := \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$, $R > 0$. If A, $B \in \mathcal{B}_2(H)$ are normal operators with $A^*B = BA^*$ and $\operatorname{tr}(|A|^2)$, $\operatorname{tr}(|B|^2) < R$ then the inequality*

$$|\operatorname{tr}(f(A^*B))|^2 \leq \operatorname{tr}(f_a(|A|^2)) \operatorname{tr}(f_a(|B|^2)). \tag{2.45}$$

Proof. From the inequality (2.29) we have

$$\left|\operatorname{tr}\left(\sum_{k=0}^{n} \alpha_k (A^*)^k B^k\right)\right|^2 \leq \operatorname{tr}\left(\sum_{k=0}^{n} |\alpha_k| \left|A^k\right|^2\right) \operatorname{tr}\left(\sum_{k=0}^{n} |\alpha_k| \left|B^k\right|^2\right). \tag{2.46}$$

Since A, B are normal operators, then we have $\left|A^k\right|^2 = |A|^{2k}$ and $\left|B^k\right|^2 = |B|^{2k}$ for any $k \geq 0$. Also, since $A^*B = BA^*$ then we also have $(A^*)^k B^k = (A^*B)^k$ for any $k \geq 0$.

Due to the fact that A, $B \in \mathcal{B}_2(H)$ and $\operatorname{tr}(|A|^2)$, $\operatorname{tr}(|B|^2) < R$, it follows that $\operatorname{tr}(A^*B) \leq R$ and the operator series

$$\sum_{k=0}^{\infty} \alpha_k (A^*B)^k, \quad \sum_{k=0}^{\infty} |\alpha_k| |A|^{2k} \text{ and } \sum_{k=0}^{\infty} |\alpha_k| |B|^{2k}$$

are convergent in the Banach space $\mathcal{B}_1(H)$.

Taking the limit over $n \to \infty$ in (2.46) and using the continuity of the $\operatorname{tr}(\cdot)$ on $\mathcal{B}_1(H)$ we deduce the desired result (2.45). □

Example 2.12. a) If we take in (2.45) $f(\lambda) = (1 \pm \lambda)^{-1}$, $|\lambda| < 1$ then we get the inequality

$$\left|\operatorname{tr}\left((1_H \pm A^*B)^{-1}\right)\right|^2 \leq \operatorname{tr}\left(\left(1 - |A|^2\right)^{-1}\right) \operatorname{tr}\left(\left(1 - |B|^2\right)^{-1}\right) \tag{2.47}$$

for any A, $B \in \mathcal{B}_2(H)$ normal operators with $A^*B = BA^*$ and $\operatorname{tr}(|A|^2)$, $\operatorname{tr}(|B|^2) < 1$.

b) If we take in (2.45) $f(\lambda) = \exp(\lambda)$, $\lambda \in \mathbb{C}$ then we get the inequality

$$|\operatorname{tr}(\exp(A^*B))|^2 \leq \operatorname{tr}(\exp(|A|^2)) \operatorname{tr}(\exp(|B|^2)) \tag{2.48}$$

for any A, $B \in \mathcal{B}_2(H)$ normal operators with $A^*B = BA^*$.

Theorem 2.13 (Dragomir, 2014, [35])**.** *Let $f(z) := \sum_{j=0}^{\infty} p_j z^j$ and $g(z) := \sum_{j=0}^{\infty} q_j z^j$ be two power series with nonnegative coefficients and convergent on the open disk $D(0,R)$, $R > 0$. If T and V are two normal and commuting operators from $\mathcal{B}_2(H)$ with $\operatorname{tr}(|T|^2)$, $\operatorname{tr}(|V|^2) < R$, then*

$$\begin{aligned}
&\left[\operatorname{tr}\left(f\left(|T|^2\right) + g\left(|T|^2\right)\right)\right]^{1/2} \left[\operatorname{tr}\left(f\left(|V|^2\right) + g\left(|V|^2\right)\right)\right]^{1/2} \\
&\qquad - |\operatorname{tr}(f(T^*V) + g(T^*V))| \\
&\quad \geq \left[\operatorname{tr}\left(f\left(|T|^2\right)\right)\right]^{1/2} \left[\operatorname{tr}\left(f\left(|V|^2\right)\right)\right]^{1/2} - |\operatorname{tr}(f(T^*V))| \\
&\qquad + \left[\operatorname{tr}\left(g\left(|T|^2\right)\right)\right]^{1/2} \left[\operatorname{tr}\left(g\left(|V|^2\right)\right)\right]^{1/2} - |\operatorname{tr}(g(T^*V))| \, (\geq 0).
\end{aligned} \tag{2.49}$$

Moreover, if $p_j \geq q_j$ for any $j \in \mathbb{N}$, then, with the above assumptions on T and V,

$$\left[\operatorname{tr}\left(f\left(|T|^2\right)\right)\right]^{1/2}\left[\operatorname{tr}\left(f\left(|V|^2\right)\right)\right]^{1/2}-\left|\operatorname{tr}\left(f\left(T^*V\right)\right)\right| \geq \left[\operatorname{tr}\left(g\left(|T|^2\right)\right)\right]^{1/2}\left[\operatorname{tr}\left(g\left(|V|^2\right)\right)\right]^{1/2}-\left|\operatorname{tr}\left(g\left(T^*V\right)\right)\right| (\geq 0). \tag{2.50}$$

Proof. Utilizing the superadditivity property of the functional $\sigma_{\mathbf{A},\mathbf{B}}(\cdot)$ above as a function of weights $\mathbf{p}$ and the fact that T and V are two normal and commuting operators we can state that

$$\begin{aligned}
&\left[\operatorname{tr}\left(\sum_{k=0}^{n}(p_k+q_k)|T|^{2k}\right)\right]^{1/2}\left[\operatorname{tr}\left(\sum_{k=0}^{n}(p_k+q_k)|V|^{2k}\right)\right]^{1/2} \\
&\quad -\left|\operatorname{tr}\left(\sum_{k=0}^{n}(p_k+q_k)(T^*V)^k\right)\right| \\
&\geq \left[\operatorname{tr}\left(\sum_{k=0}^{n}p_k|T|^{2k}\right)\right]^{1/2}\left[\operatorname{tr}\left(\sum_{k=0}^{n}p_k|V|^{2k}\right)\right]^{1/2}-\left|\operatorname{tr}\left(\sum_{k=0}^{n}p_k(T^*V)^k\right)\right| \\
&\quad +\left[\operatorname{tr}\left(\sum_{k=0}^{n}q_k|T|^{2k}\right)\right]^{1/2}\left[\operatorname{tr}\left(\sum_{k=0}^{n}q_k|V|^{2k}\right)\right]^{1/2}-\left|\operatorname{tr}\left(\sum_{k=0}^{n}q_k(T^*V)^k\right)\right|
\end{aligned} \tag{2.51}$$

for any $n \geq 1$.

Since all the series whose partial sums are involved in (2.51) are convergent in $\mathcal{B}_1(H)$, by letting $n \to \infty$ in (2.51) we get (2.49).

The inequality (2.50) follows by the monotonicity property of $\sigma_{\mathbf{A},\mathbf{B}}(\cdot)$ and the details are omitted. □

Example 2.14. Now, observe that if we take

$$f(\lambda)=\sinh\lambda=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\lambda^{2n+1}$$

and

$$g(\lambda)=\cosh\lambda=\sum_{n=0}^{\infty}\frac{1}{(2n)!}\lambda^{2n},$$

then

$$f(\lambda)+g(\lambda)=\exp\lambda=\sum_{n=0}^{\infty}\frac{1}{n!}\lambda^{n}$$

for any $\lambda \in \mathbb{C}$.

If T and V are two normal and commuting operators from $\mathcal{B}_2(H)$, then by (2.11)

$$\begin{aligned}
&\left[\operatorname{tr}\left(\exp\left(|T|^2\right)\right)\right]^{1/2}\left[\operatorname{tr}\left(\exp\left(|V|^2\right)\right)\right]^{1/2}-\left|\operatorname{tr}\left(\exp\left(T^*V\right)\right)\right| \\
&\geq \left[\operatorname{tr}\left(\sinh\left(|T|^2\right)\right)\right]^{1/2}\left[\operatorname{tr}\left(\sinh\left(|V|^2\right)\right)\right]^{1/2}-\left|\operatorname{tr}\left(\sinh\left(T^*V\right)\right)\right| \\
&\quad +\left[\operatorname{tr}\left(\cosh\left(|T|^2\right)\right)\right]^{1/2}\left[\operatorname{tr}\left(\cosh\left(|V|^2\right)\right)\right]^{1/2}-\left|\operatorname{tr}\left(\cosh\left(T^*V\right)\right)\right| (\geq 0).
\end{aligned} \tag{2.52}$$

Now, consider the series $\frac{1}{1-\lambda}=\sum_{n=0}^{\infty}\lambda^{n}$, $\lambda\in D(0,1)$ and $\ln\frac{1}{1-\lambda}=\sum_{n=1}^{\infty}\frac{1}{n}\lambda^{n}$, $\lambda\in D(0,1)$ and define $p_n=1$, $n\geq 0$, $q_0=0$, $q_n=\frac{1}{n}$, $n\geq 1$, then we observe that for any $n\geq 0$, $p_n\geq q_n$.

If T and V are two normal and commuting operators from $\mathcal{B}_2(H)$ with $\operatorname{tr}\left(|T|^2\right)$, $\operatorname{tr}\left(|V|^2\right)<1$, then by (2.12)

$$\begin{aligned}
&\left[\operatorname{tr}\left(\left(1_H-|T|^2\right)^{-1}\right)\right]^{1/2}\left[\operatorname{tr}\left(\left(1_H-|V|^2\right)^{-1}\right)\right]^{1/2} \\
&\qquad-\left|\operatorname{tr}\left((1_H-T^*V)^{-1}\right)\right| \\
&\quad\geq\left[\operatorname{tr}\left(\ln\left(1_H-|T|^2\right)^{-1}\right)\right]^{1/2}\left[\operatorname{tr}\left(\ln\left(1_H-|V|^2\right)^{-1}\right)\right]^{1/2} \\
&\qquad-\left|\operatorname{tr}\left(\ln\left(1_H-T^*V\right)^{-1}\right)\right|(\geq 0).
\end{aligned}\tag{2.53}$$

2.5. **Inequalities for Matrices.** We have the following result for matrices.

Proposition 2.15 (Dragomir, 2014, [35])**.** *Let* $f(\lambda):=\sum_{n=0}^{\infty}\alpha_n\lambda^n$ *be a power series with complex coefficients and convergent on the open disk* $D(0,R)$, $R>0$. *If* A *and* B *are positive semidefinite matrices in* $M_n(\mathbb{C})$ *with* $\operatorname{tr}(A^2)$, $\operatorname{tr}(B^2)<R$, *then the inequality*

$$|\operatorname{tr}[f(AB)]|^2\leq\operatorname{tr}\left[f_a\left(A^2\right)\right]\operatorname{tr}\left[f_a\left(B^2\right)\right].\tag{2.54}$$

If $\operatorname{tr}(A)$, $\operatorname{tr}(B)<\sqrt{R}$, *then also*

$$|\operatorname{tr}[f(AB)]|\leq\min\left\{\operatorname{tr}\left(f_a\left(\|A\|B\right)\right),\operatorname{tr}\left(f_a\left(\|B\|A\right)\right)\right\}.\tag{2.55}$$

Proof. We observe that (1.13) holds for $m=0$ with equality.

By utilizing the generalized triangle inequality for the modulus and the inequality (1.13) we have

$$\begin{aligned}
&\left|\operatorname{tr}\left[\sum_{n=0}^{m}\alpha_n(AB)^n\right]\right| \\
&\quad=\left|\sum_{n=0}^{m}\alpha_n\operatorname{tr}\left[(AB)^n\right]\right|\leq\sum_{n=0}^{m}|\alpha_n|\left|\operatorname{tr}\left[(AB)^n\right]\right| \\
&\quad=\sum_{n=0}^{m}|\alpha_n|\operatorname{tr}\left[(AB)^n\right]\leq\sum_{n=0}^{m}|\alpha_n|\left[\operatorname{tr}\left(A^{2n}\right)\right]^{1/2}\left[\operatorname{tr}\left(B^{2n}\right)\right]^{1/2},
\end{aligned}\tag{2.56}$$

for any $m\geq 1$.

Applying the weighted Cauchy–Bunyakowsky–Schwarz discrete inequality we also have

$$\sum_{n=0}^{m}|\alpha_n|\left[\operatorname{tr}\left(A^{2n}\right)\right]^{1/2}\left[\operatorname{tr}\left(B^{2n}\right)\right]^{1/2} \tag{2.57}$$
$$\leq\left(\sum_{n=0}^{m}|\alpha_n|\left(\left[\operatorname{tr}\left(A^{2n}\right)\right]^{1/2}\right)^2\right)^{1/2}\left(\sum_{n=0}^{m}|\alpha_n|\left(\left[\operatorname{tr}\left(B^{2n}\right)\right]^{1/2}\right)^2\right)^{1/2}$$
$$=\left(\sum_{n=0}^{m}|\alpha_n|\left[\operatorname{tr}\left(A^{2n}\right)\right]\right)^{1/2}\left(\sum_{n=0}^{m}|\alpha_n|\left[\operatorname{tr}\left(B^{2n}\right)\right]\right)^{1/2}$$
$$=\left[\operatorname{tr}\left(\sum_{n=0}^{m}|\alpha_n|A^{2n}\right)\right]^{1/2}\left[\operatorname{tr}\left(\sum_{n=0}^{m}|\alpha_n|B^{2n}\right)\right]^{1/2}$$

for any $m\geq 1$.

Therefore, by (2.56) and (2.57) we get

$$\left|\operatorname{tr}\left[\sum_{n=0}^{m}\alpha_n(AB)^n\right]\right|^2\leq\operatorname{tr}\left(\sum_{n=0}^{m}|\alpha_n|A^{2n}\right)\operatorname{tr}\left(\sum_{n=0}^{m}|\alpha_n|B^{2n}\right) \tag{2.58}$$

for any $m\geq 1$.

Since $\operatorname{tr}(A^2),\operatorname{tr}(B^2)<R$, then $\operatorname{tr}(AB)\leq\sqrt{\operatorname{tr}(A^2)\operatorname{tr}(B^2)}<R$ and the series

$$\sum_{n=0}^{\infty}\alpha_n(AB)^n,\ \ \sum_{n=0}^{\infty}|\alpha_n|A^{2n}\text{ and }\sum_{n=0}^{\infty}|\alpha_n|B^{2n}$$

are convergent in $M_n(\mathbb{C})$.

Taking the limit over $m\to\infty$ in (2.58) and utilizing the continuity property of $\operatorname{tr}(\cdot)$ on $M_n(\mathbb{C})$ we get (2.54).

The inequality (2.55) follows from (1.14) in a similar way and the details are omitted. □

Example 2.16. a) If we take $f(\lambda)=(1\pm\lambda)^{-1}$, $|\lambda|<1$ then we get the inequality

$$\left|\operatorname{tr}\left[(I_n\pm AB)^{-1}\right]\right|^2\leq\operatorname{tr}\left[\left(I_n-A^2\right)^{-1}\right]\operatorname{tr}\left[\left(I_n-B^2\right)^{-1}\right] \tag{2.59}$$

for any A and B positive semidefinite matrices in $M_n(\mathbb{C})$ with $\operatorname{tr}(A^2),\operatorname{tr}(B^2)<1$. Here I_n is the identity matrix in $M_n(\mathbb{C})$.

We also have the inequality

$$\left|\operatorname{tr}\left[(I_n\pm AB)^{-1}\right]\right|\leq\min\left\{\operatorname{tr}\left((I_n-\|A\|B)^{-1}\right),\operatorname{tr}\left((I_n-\|B\|A)^{-1}\right)\right\} \tag{2.60}$$

for any A and B positive semidefinite matrices in $M_n(\mathbb{C})$ with $\operatorname{tr}(A),\operatorname{tr}(B)<1$.

b) If we take $f(\lambda)=\exp\lambda$, then

$$(\operatorname{tr}[\exp(AB)])^2\leq\operatorname{tr}\left[\exp\left(A^2\right)\right]\operatorname{tr}\left[\exp\left(B^2\right)\right] \tag{2.61}$$

and

$$\operatorname{tr}[\exp(AB)]\leq\min\left\{\operatorname{tr}(\exp(\|A\|B)),\operatorname{tr}(\exp(\|B\|A))\right\} \tag{2.62}$$

for any A and B positive semidefinite matrices in $M_n(\mathbb{C})$.

Proposition 2.17 (Dragomir, 2014, [35])**.** *Let $f(\lambda) := \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$, $R > 0$. If A and B are matrices in $M_n(\mathbb{C})$ with $\operatorname{tr}(|A|^p)$, $\operatorname{tr}(|B|^q) < R$, where p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\begin{aligned} |\operatorname{tr}(f(|AB^*|))| &\leq \operatorname{tr}\left[f_a\left(\frac{|A|^p}{p} + \frac{|B|^q}{q}\right)\right] \\ &\leq \operatorname{tr}\left[\frac{1}{p}f_a(|A|^p) + \frac{1}{q}f_a(|B|^q)\right]. \end{aligned} \tag{2.63}$$

Proof. The inequality (1.15) holds with equality for $r = 0$.

By utilizing the generalized triangle inequality for the modulus and the inequality (1.15) we have

$$\begin{aligned} \left|\operatorname{tr}\left(\sum_{n=0}^{m} \alpha_n |AB^*|^n\right)\right| &= \left|\sum_{n=0}^{m} \alpha_n \operatorname{tr}(|AB^*|^n)\right| \\ &\leq \sum_{n=0}^{m} |\alpha_n| \left|\operatorname{tr}(|AB^*|^n)\right| = \sum_{n=0}^{m} |\alpha_n| \operatorname{tr}(|AB^*|^n) \\ &\leq \sum_{n=0}^{m} |\alpha_n| \operatorname{tr}\left[\left(\frac{|A|^p}{p} + \frac{|B|^q}{q}\right)^n\right] \\ &= \operatorname{tr}\left[\sum_{n=0}^{m} |\alpha_n| \left(\frac{|A|^p}{p} + \frac{|B|^q}{q}\right)^n\right] \end{aligned} \tag{2.64}$$

for any $m \geq 1$ and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

It is know that if $f : [0,\infty) \to \mathbb{R}$ is a convex function, then $\operatorname{tr} f(\cdot)$ is convex on the cone $M_n^+(\mathbb{C})$ of positive semidefinite matrices in $M_n(\mathbb{C})$. Therefore, for $n \geq 1$ we have

$$\operatorname{tr}\left[\left(\frac{|A|^p}{p} + \frac{|B|^q}{q}\right)^n\right] \leq \frac{1}{p}\operatorname{tr}(|A|^{pn}) + \frac{1}{q}\operatorname{tr}(|B|^{qn}) \tag{2.65}$$

where p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

The inequality reduces to equality if $n = 0$.

Then we have

$$\begin{aligned} \sum_{n=0}^{m} |\alpha_n| \operatorname{tr}\left[\left(\frac{|A|^p}{p} + \frac{|B|^q}{q}\right)^n\right] &\leq \sum_{n=0}^{m} |\alpha_n| \left[\frac{1}{p}\operatorname{tr}(|A|^{pn}) + \frac{1}{q}\operatorname{tr}(|B|^{qn})\right] \\ &= \operatorname{tr}\left[\frac{1}{p}\sum_{n=0}^{m} |\alpha_n| |A|^{pn} + \frac{1}{q}\sum_{n=0}^{m} |\alpha_n| |B|^{qn}\right] \end{aligned} \tag{2.66}$$

for any $m \geq 1$ and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

From (2.64) and (2.66) we get

$$\left|\operatorname{tr}\left(\sum_{n=0}^{m}\alpha_n\left|AB^*\right|^r\right)\right| \le \operatorname{tr}\left[\sum_{n=0}^{m}\left|\alpha_n\right|\left(\frac{|A|^p}{p}+\frac{|B|^q}{q}\right)^n\right] \le \operatorname{tr}\left[\frac{1}{p}\sum_{n=0}^{m}\left|\alpha_n\right|\left|A\right|^{pn}+\frac{1}{q}\sum_{n=0}^{m}\left|\alpha_n\right|\left|B\right|^{qn}\right] \tag{2.67}$$

for any $m \ge 1$ and $p,\, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Since $\operatorname{tr}\left(|A|^p\right), \operatorname{tr}\left(|B|^q\right) < R$, then all the series whose partial sums are involved in (2.67) are convergent, then by letting $m \to \infty$ in (2.67) we deduce the desired inequality (2.63). □

Example 2.18. a) If we take $f(\lambda) = (1 \pm \lambda)^{-1}$, $|\lambda| < 1$ then we get the inequalities

$$\left|\operatorname{tr}\left(\left(I_n \pm \left|AB^*\right|\right)^{-1}\right)\right| \le \operatorname{tr}\left(\left[I_n - \left(\frac{|A|^p}{p}+\frac{|B|^q}{q}\right)\right]^{-1}\right) \le \operatorname{tr}\left[\frac{1}{p}\left(I_n-|A|^p\right)^{-1}+\frac{1}{q}\left(I_n-|B|^q\right)^{-1}\right], \tag{2.68}$$

where A and B are matrices in $M_n(\mathbb{C})$ with $\operatorname{tr}\left(|A|^p\right), \operatorname{tr}\left(|B|^q\right) < 1$ and $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$.

b) If we take $f(\lambda) = \exp\lambda$, then

$$\operatorname{tr}\left(\exp\left(\left|AB^*\right|\right)\right) \le \operatorname{tr}\left[\exp\left(\frac{|A|^p}{p}+\frac{|B|^q}{q}\right)\right] \le \operatorname{tr}\left[\frac{1}{p}\exp\left(|A|^p\right)+\frac{1}{q}\exp\left(|B|^q\right)\right], \tag{2.69}$$

where A and B are matrices in $M_n(\mathbb{C})$ and $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Finally, we can state the following result:

Proposition 2.19 (Dragomir, 2014, [35]). *Let $f(\lambda) := \sum_{n=0}^{\infty}\alpha_n\lambda^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$, $R>0$. If A and B are commuting positive semidefinite matrices in $M_n(\mathbb{C})$ with $\operatorname{tr}(A^p)$, $\operatorname{tr}(B^q) < R$, where $p,\, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\left|\operatorname{tr}\left(f\left(AB\right)\right)\right| \le \left[\operatorname{tr}(f_a\left(A^p\right))\right]^{1/p}\left[\operatorname{tr}(f_a\left(B^q\right))\right]^{1/q}. \tag{2.70}$$

Proof. Since A and B are commuting positive semidefinite matrices in $M_n(\mathbb{C})$, then by (1.17) we have for any natural number n including $n = 0$ that

$$\operatorname{tr}((AB)^n) = \operatorname{tr}(A^nB^n) \le \left[\operatorname{tr}(A^{np})\right]^{1/p}\left[\operatorname{tr}(B^{nq})\right]^{1/q}, \tag{2.71}$$

where $p,\, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$.

By (2.71) and the weighted Hölder discrete inequality we have

$$\begin{aligned}\left|\operatorname{tr}\left(\sum_{n=0}^{m}\alpha_n(AB)^n\right)\right| &= \left|\sum_{n=0}^{m}\alpha_n\operatorname{tr}(A^nB^n)\right| \le \sum_{n=0}^{m}|\alpha_n|\,|\operatorname{tr}(A^nB^n)| \\ &\le \sum_{n=0}^{m}|\alpha_n|\,[\operatorname{tr}(A^{np})]^{1/p}\,[\operatorname{tr}(B^{nq})]^{1/q} \\ &\le \left(\sum_{n=0}^{m}|\alpha_n|\left([\operatorname{tr}(A^{np})]^{1/p}\right)^p\right)^{1/p} \\ &\quad\times\left(\sum_{n=0}^{m}|\alpha_n|\left([\operatorname{tr}(B^{nq})]^{1/q}\right)^q\right)^{1/q} \\ &= \left(\sum_{n=0}^{m}|\alpha_n|\operatorname{tr}(A^{np})\right)^{1/p}\left(\sum_{n=0}^{m}|\alpha_n|\operatorname{tr}(B^{nq})\right)^{1/q} \\ &= \left(\operatorname{tr}(\sum_{n=0}^{m}|\alpha_n|\,A^{np})\right)^{1/p}\left(\operatorname{tr}(\sum_{n=0}^{m}|\alpha_n|\,B^{nq})\right)^{1/q}\end{aligned}$$

where $p,\, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

The proof follows now in a similar way with the ones from above and the details are omitted. □

Example 2.20. a) If we take $f(\lambda) = (1 \pm \lambda)^{-1}$, $|\lambda| < 1$ then we get the inequality

$$\left|\operatorname{tr}\left((I_n \pm AB)^{-1}\right)\right| \le \left[\operatorname{tr}((I_n - A^p)^{-1})\right]^{1/p}\left[\operatorname{tr}((I_n - B^q)^{-1})\right]^{1/q}, \tag{2.72}$$

for any A and B commuting positive semidefinite matrices in $M_n(\mathbb{C})$ with $\operatorname{tr}(A^p)$, $\operatorname{tr}(B^q) < 1$, where $p,\, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

b) If we take $f(\lambda) = \exp\lambda$, then

$$\operatorname{tr}(\exp(AB)) \le [\operatorname{tr}(\exp(A^p))]^{1/p}\,[\operatorname{tr}(\exp(B^q))]^{1/q}, \tag{2.73}$$

for any A and B commuting positive semidefinite matrices in $M_n(\mathbb{C})$ and $p,\, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

3. Kato's Type Trace Inequalities

3.1. **Kato's Inequality.** We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H; \langle\cdot,\cdot\rangle)$.

If P is a positive selfadjoint operator on H, i.e. $\langle Px, x\rangle \ge 0$ for any $x \in H$, then the following inequality is a generalization of the Schwarz inequality in H

$$|\langle Px, y\rangle|^2 \le \langle Px, x\rangle\,\langle Py, y\rangle, \tag{3.1}$$

for any $x, y \in H$.

The following inequality is of interest as well, see [56, p. 221].

Let P be a positive selfadjoint operator on H. Then

$$\|Px\|^2 \le \|P\|\,\langle Px, x\rangle \tag{3.2}$$

for any $x \in H$.

The *"square root"* of a positive bounded selfadjoint operator on H can be defined as follows, see for instance [56, p. 240]: *If the operator $A \in B(H)$ is selfadjoint and positive, then there exists a unique positive selfadjoint operator $B := \sqrt{A} \in B(H)$ such that $B^2 = A$. If A is invertible, then so is B.*

If $A \in \mathcal{B}(H)$, then the operator A^*A is selfadjoint and positive. Define the *"absolute value"* operator by $|A| := \sqrt{A^*A}$.

In 1952, Kato [57] proved the following celebrated *generalization of Schwarz inequality* for any bounded linear operator T on H:

$$|\langle Tx, y\rangle|^2 \leq \langle (T^*T)^\alpha x, x\rangle \left\langle (TT^*)^{1-\alpha} y, y\right\rangle, \tag{3.3}$$

for any $x, y \in H$, $\alpha \in [0,1]$. Utilizing the modulus notation introduced before, we can write (3.3) as follows

$$|\langle Tx, y\rangle|^2 \leq \left\langle |T|^{2\alpha} x, x\right\rangle \left\langle |T^*|^{2(1-\alpha)} y, y\right\rangle \tag{3.4}$$

for any $x, y \in H$, $\alpha \in [0,1]$.

It is useful to observe that, if $T = N$, a normal operator, i.e., we recall that $NN^* = N^*N$, then the inequality (3.4) can be written as

$$|\langle Nx, y\rangle|^2 \leq \left\langle |N|^{2\alpha} x, x\right\rangle \left\langle |N|^{2(1-\alpha)} y, y\right\rangle, \tag{3.5}$$

and in particular, for selfadjoint operators A we can state it as

$$|\langle Ax, y\rangle| \leq \||A|^\alpha x\| \left\||A|^{1-\alpha} y\right\| \tag{3.6}$$

for any $x, y \in H$, $\alpha \in [0,1]$.

If $T = U$, a unitary operator, i.e., we recall that $UU^* = U^*U = 1_H$, then the inequality (3.4) becomes

$$|\langle Ux, y\rangle| \leq \|x\| \|y\|$$

for any $x, y \in H$, which provides a natural generalization for the Schwarz inequality in H.

The symmetric powers in the inequalities above are natural to be considered, so if we choose in (3.4), (3.5) and in (3.6) $\alpha = 1/2$ then we get for any $x, y \in H$

$$|\langle Tx, y\rangle|^2 \leq \langle |T| x, x\rangle \langle |T^*| y, y\rangle, \tag{3.7}$$

$$|\langle Nx, y\rangle|^2 \leq \langle |N| x, x\rangle \langle |N| y, y\rangle, \tag{3.8}$$

and

$$|\langle Ax, y\rangle| \leq \left\||A|^{1/2} x\right\| \left\||A|^{1/2} y\right\| \tag{3.9}$$

respectively.

It is also worthwhile to observe that, if we take the supremum over $y \in H$, $\|y\| = 1$ in (3.4) then we get

$$\|Tx\|^2 \leq \|T\|^{2(1-\alpha)} \left\langle |T|^{2\alpha} x, x\right\rangle \tag{3.10}$$

for any $x \in H$, or in an equivalent form

$$\|Tx\| \leq \||T|^\alpha x\| \|T\|^{1-\alpha} \tag{3.11}$$

for any $x \in H$.

If we take $\alpha = 1/2$ in (3.10), then we get

$$\|Tx\|^2 \leq \|T\| \langle |T| x, x \rangle \tag{3.12}$$

for any $x \in H$, which in the particular case of $T = P$, a positive operator, provides the result from (3.2).

For various interesting generalizations, extension and Kato related results, see the papers [44]-[54], [59]-[68] and [79].

3.2. **Trace Inequalities Via Kato's Result.** We start with the following result:

Theorem 3.1 (Dragomir, 2014, [34])**.** *Let $T \in \mathcal{B}(H)$.*

(i) If for some $\alpha \in (0,1)$, $|T|^{2\alpha}$, $|T^*|^{2(1-\alpha)} \in \mathcal{B}_1(H)$, then $T \in \mathcal{B}_1(H)$ and

$$|\mathrm{tr}(T)|^2 \leq \mathrm{tr}\left(|T|^{2\alpha}\right) \mathrm{tr}\left(|T^*|^{2(1-\alpha)}\right); \tag{3.13}$$

(ii) If for some $\alpha \in [0,1]$ and an orthonormal basis $\{e_i\}_{i\in I}$ the sum

$$\sum_{i\in I} \|Te_i\|^{\alpha} \|T^* e_i\|^{1-\alpha}$$

is finite, then $T \in \mathcal{B}_1(H)$ and

$$|\mathrm{tr}(T)| \leq \sum_{i\in I} \|Te_i\|^{\alpha} \|T^* e_i\|^{1-\alpha}. \tag{3.14}$$

Moreover, if the sums $\sum_{i\in I} \|Te_i\|$ and $\sum_{i\in I} \|T^* e_i\|$ are finite for an orthonormal basis $\{e_i\}_{i\in I}$, then $T \in \mathcal{B}_1(H)$ and

$$|\mathrm{tr}(T)| \leq \inf_{\alpha\in[0,1]} \left\{ \sum_{i\in I} \|Te_i\|^{\alpha} \|T^* e_i\|^{1-\alpha} \right\} \leq \min\left\{ \sum_{i\in F} \|Te_i\|, \sum_{i\in F} \|T^* e_i\| \right\}. \tag{3.15}$$

Proof. (i) Assume that $\alpha \in (0,1)$. Let $\{e_i\}_{i\in I}$ be an orthonormal basis in H and F a finite part of I. Then by Kato's inequality (3.4) we have

$$\left| \sum_{i\in F} \langle Te_i, e_i \rangle \right| \leq \sum_{i\in F} |\langle Te_i, e_i \rangle| \leq \sum_{i\in F} \left\langle |T|^{2\alpha} e_i, e_i \right\rangle^{1/2} \left\langle |T^*|^{2(1-\alpha)} e_i, e_i \right\rangle^{1/2}. \tag{3.16}$$

By Cauchy–Buniakovski–Schwarz inequality for finite sums we have

$$\begin{aligned} &\sum_{i\in F} \left\langle |T|^{2\alpha} e_i, e_i \right\rangle^{1/2} \left\langle |T^*|^{2(1-\alpha)} e_i, e_i \right\rangle^{1/2} \qquad (3.17)\\ &\leq \left(\sum_{i\in F} \left[\left\langle |T|^{2\alpha} e_i, e_i \right\rangle^{1/2} \right]^2 \right)^{1/2} \left(\sum_{i\in F} \left[\left\langle |T^*|^{2(1-\alpha)} e_i, e_i \right\rangle^{1/2} \right]^2 \right)^{1/2} \\ &= \left(\sum_{i\in F} \left\langle |T|^{2\alpha} e_i, e_i \right\rangle \right)^{1/2} \left(\sum_{i\in F} \left\langle |T^*|^{2(1-\alpha)} e_i, e_i \right\rangle \right)^{1/2}. \end{aligned}$$

Therefore, by (3.16) and (3.17) we have

$$\left|\sum_{i\in F}\langle Te_i,e_i\rangle\right|\le\left(\sum_{i\in F}\left\langle |T|^{2\alpha}e_i,e_i\right\rangle\right)^{1/2}\left(\sum_{i\in F}\left\langle |T^*|^{2(1-\alpha)}e_i,e_i\right\rangle\right)^{1/2} \tag{3.18}$$

for any finite part F of I.

If for some $\alpha\in(0,1)$ we have $|T|^{2\alpha},|T^*|^{2(1-\alpha)}\in\mathcal{B}_1(H)$, then the sums $\sum_{i\in I}\left\langle |T|^{2\alpha}e_i,e_i\right\rangle$ and $\sum_{i\in I}\left\langle |T^*|^{2(1-\alpha)}e_i,e_i\right\rangle$ are finite and by (3.18) we have that $\sum_{i\in I}\langle Te_i,e_i\rangle$ is also finite and we have the inequality (3.13).

(ii) Assume that $\alpha\in[0,1]$. Let $\{e_i\}_{i\in I}$ be an orthonormal basis in H and F a finite part of I. Utilizing McCarthy's inequality for the positive operator P, namely

$$\left\langle P^\beta x,x\right\rangle\le\langle Px,x\rangle^\beta,$$

that holds for $\beta\in[0,1]$ and $x\in H$, $\|x\|=1$, we have

$$\left\langle |T|^{2\alpha}e_i,e_i\right\rangle\le\left\langle |T|^2e_i,e_i\right\rangle^\alpha$$

and

$$\left\langle |T^*|^{2(1-\alpha)}e_i,e_i\right\rangle\le\left\langle |T^*|^2e_i,e_i\right\rangle^{1-\alpha}$$

for any $i\in I$.

Making use of (3.16) we have

$$\begin{aligned}\left|\sum_{i\in F}\langle Te_i,e_i\rangle\right|&\le\sum_{i\in F}|\langle Te_i,e_i\rangle|\le\sum_{i\in F}\left\langle |T|^{2\alpha}e_i,e_i\right\rangle^{1/2}\left\langle |T^*|^{2(1-\alpha)}e_i,e_i\right\rangle^{1/2} \qquad (3.19)\\&\le\sum_{i\in F}\left\langle |T|^2e_i,e_i\right\rangle^{\alpha/2}\left\langle |T^*|^2e_i,e_i\right\rangle^{(1-\alpha)/2}\\&=\sum_{i\in F}\langle T^*Te_i,e_i\rangle^{\alpha/2}\langle TT^*e_i,e_i\rangle^{(1-\alpha)/2}\\&=\sum_{i\in F}\|Te_i\|^\alpha\|T^*e_i\|^{1-\alpha}.\end{aligned}$$

Utilizing Hölder's inequality for finite sums and $p=\frac{1}{\alpha}$, $q=\frac{1}{1-\alpha}$ we also have

$$\begin{aligned}&\sum_{i\in F}\|Te_i\|^\alpha\|T^*e_i\|^{1-\alpha} \qquad (3.20)\\&\le\left[\sum_{i\in F}\left(\|Te_i\|^\alpha\right)^{1/\alpha}\right]^\alpha\left[\sum_{i\in F}\left(\|T^*e_i\|^{1-\alpha}\right)^{1/(1-\alpha)}\right]^{1-\alpha}\\&=\left[\sum_{i\in F}\|Te_i\|\right]^\alpha\left[\sum_{i\in F}\|T^*e_i\|\right]^{1-\alpha}.\end{aligned}$$

Since all the series involved in (3.19) and (3.20) are convergent, then we get

$$\left|\sum_{i\in I}\langle Te_i,e_i\rangle\right|\le\sum_{i\in I}\|Te_i\|^\alpha\|T^*e_i\|^{1-\alpha}\left[\sum_{i\in I}\|Te_i\|\right]^\alpha\left[\sum_{i\in I}\|T^*e_i\|\right]^{1-\alpha} \tag{3.21}$$

for any $\alpha \in [0,1]$.

Taking the infimum over $\alpha \in [0,1]$ in (3.21) produces

$$\begin{aligned}\left|\sum_{i\in I}\langle Te_i, e_i\rangle\right| &\le \inf_{\alpha\in[0,1]}\left\{\sum_{i\in F}\|Te_i\|^{\alpha}\|T^*e_i\|^{1-\alpha}\right\} \\ &\le \inf_{\alpha\in[0,1]}\left[\sum_{i\in F}\|Te_i\|\right]^{\alpha}\left[\sum_{i\in F}\|T^*e_i\|\right]^{1-\alpha} \\ &= \min\left\{\sum_{i\in F}\|Te_i\|, \sum_{i\in F}\|T^*e_i\|\right\}.\end{aligned} \tag{3.22}$$

□

Corollary 3.2. *Let $T \in \mathcal{B}(H)$.*

(i) If $|T|, |T^*| \in \mathcal{B}_1(H)$, then $T \in \mathcal{B}_1(H)$ and

$$|\mathrm{tr}(T)|^2 \le \mathrm{tr}(|T|)\,\mathrm{tr}(|T^*|); \tag{3.23}$$

(ii) If for an orthonormal basis $\{e_i\}_{i\in I}$ the sum $\sum_{i\in I}\sqrt{\|Te_i\|\|T^*e_i\|}$ is finite, then $T \in \mathcal{B}_1(H)$ and

$$|\mathrm{tr}(T)| \le \sum_{i\in I}\sqrt{\|Te_i\|\|T^*e_i\|}. \tag{3.24}$$

Corollary 3.3. *Let $N \in \mathcal{B}(H)$ be a normal operator. If for some $\alpha \in (0,1)$, $|N|^{2\alpha}, |N|^{2(1-\alpha)} \in \mathcal{B}_1(H)$, then $N \in \mathcal{B}_1(H)$ and*

$$|\mathrm{tr}(N)|^2 \le \mathrm{tr}\left(|N|^{2\alpha}\right)\mathrm{tr}\left(|N|^{2(1-\alpha)}\right). \tag{3.25}$$

In particular, if $|N| \in \mathcal{B}_1(H)$, then $N \in \mathcal{B}_1(H)$ and

$$|\mathrm{tr}(N)| \le \mathrm{tr}(|N|). \tag{3.26}$$

The following result also holds.

Theorem 3.4 (Dragomir, 2014, [34])**.** *Let $T \in \mathcal{B}(H)$ and $A, B \in \mathcal{B}_2(H)$.*

(i) For any $\alpha \in [0,1]$, $|A^*|^2|T|^{2\alpha}$, $|B^*|^2|T^*|^{2(1-\alpha)}$ and $B^*TA \in \mathcal{B}_1(H)$ and

$$|\mathrm{tr}(AB^*T)|^2 \le \mathrm{tr}\left(|A^*|^2|T|^{2\alpha}\right)\mathrm{tr}\left(|B^*|^2|T^*|^{2(1-\alpha)}\right); \tag{3.27}$$

(ii) We also have

$$\begin{aligned}&|\mathrm{tr}(AB^*T)|^2 \\ &\le \min\left\{\mathrm{tr}\left(|B|^2\right)\mathrm{tr}\left(|A^*|^2|T|^2\right), \mathrm{tr}\left(|A|^2\right)\mathrm{tr}\left(|B^*|^2|T^*|^2\right)\right\}.\end{aligned} \tag{3.28}$$

Proof. (i) Let $\{e_i\}_{i\in I}$ be an orthonormal basis in H and F a finite part of I. Then by Kato's inequality (3.4) we have

$$|\langle TAe_i, Be_i\rangle|^2 \le \left\langle |T|^{2\alpha}Ae_i, Ae_i\right\rangle\left\langle |T^*|^{2(1-\alpha)}Be_i, Be_i\right\rangle \tag{3.29}$$

for any $i \in I$. This is equivalent to

$$|\langle B^*TAe_i, e_i\rangle| \le \left\langle A^*|T|^{2\alpha}Ae_i, e_i\right\rangle^{1/2}\left\langle B^*|T^*|^{2(1-\alpha)}Be_i, e_i\right\rangle^{1/2} \tag{3.30}$$

for any $i \in I$.

Using the generalized triangle inequality for the modulus and the Cauchy–Bunyakowsky–Schwarz inequality for finite sums we have from (3.30) that

$$\left|\sum_{i\in F}\langle B^*TAe_i,e_i\rangle\right| \tag{3.31}$$
$$\le \sum_{i\in F}|\langle B^*TAe_i,e_i\rangle|$$
$$\le \sum_{i\in F}\left\langle A^*|T|^{2\alpha}Ae_i,e_i\right\rangle^{1/2}\left\langle B^*|T^*|^{2(1-\alpha)}Be_i,e_i\right\rangle^{1/2}$$
$$\le \left[\sum_{i\in F}\left(\left\langle A^*|T|^{2\alpha}Ae_i,e_i\right\rangle^{1/2}\right)^2\right]^{1/2}\times\left[\sum_{i\in F}\left(\left\langle B^*|T^*|^{2(1-\alpha)}Be_i,e_i\right\rangle^{1/2}\right)^2\right]^{1/2}$$
$$= \left[\sum_{i\in F}\left\langle A^*|T|^{2\alpha}Ae_i,e_i\right\rangle\right]^{1/2}\left[\sum_{i\in F}\left\langle B^*|T^*|^{2(1-\alpha)}Be_i,e_i\right\rangle\right]^{1/2}$$

for any F a finite part of I.

Let $\alpha\in[0,1]$. Since $A,\ B\in\mathcal{B}_2(H)$, then $A^*|T|^{2\alpha}A,\ B^*|T^*|^{2(1-\alpha)}B$ and $B^*TA\in\mathcal{B}_1(H)$ and by (3.31) we have

$$|\operatorname{tr}(B^*TA)|\le\left[\operatorname{tr}\left(A^*|T|^{2\alpha}A\right)\right]^{1/2}\left[\operatorname{tr}\left(B^*|T^*|^{2(1-\alpha)}B\right)\right]^{1/2}. \tag{3.32}$$

Since, by the properties of trace we have

$$\operatorname{tr}(B^*TA)=\operatorname{tr}(AB^*T),$$
$$\operatorname{tr}\left(A^*|T|^{2\alpha}A\right)=\operatorname{tr}\left(AA^*|T|^{2\alpha}\right)=\operatorname{tr}\left(|A^*|^2|T|^{2\alpha}\right)$$

and

$$\operatorname{tr}\left(B^*|T^*|^{2(1-\alpha)}B\right)=\operatorname{tr}\left(|B^*|^2|T^*|^{2(1-\alpha)}\right),$$

then by (3.32) we get (3.27).

(ii) Utilizing McCarthy's inequality [68] for the positive operator P

$$\left\langle P^\beta x,x\right\rangle\le\langle Px,x\rangle^\beta$$

that holds for $\beta\in(0,1)$ and $x\in H$, $\|x\|=1$, we have

$$\left\langle P^\beta y,y\right\rangle\le\|y\|^{2(1-\beta)}\langle Py,y\rangle^\beta \tag{3.33}$$

for any $y\in H$.

Let $\{e_i\}_{i\in I}$ be an orthonormal basis in H and F a finite part of I. From (3.33) we have

$$\left\langle |T|^{2\alpha}Ae_i,Ae_i\right\rangle\le\|Ae_i\|^{2(1-\alpha)}\left\langle |T|^2Ae_i,Ae_i\right\rangle^\alpha$$

and

$$\left\langle |T^*|^{2(1-\alpha)}Be_i,Be_i\right\rangle\le\|Be_i\|^{2\alpha}\left\langle |T^*|^2Be_i,Be_i\right\rangle^{1-\alpha}$$

for any $i\in I$.

Making use of the inequality (3.29) we get

$$\begin{aligned}|\langle TAe_i, Be_i\rangle|^2 &\le \|Ae_i\|^{2(1-\alpha)} \left\langle |T|^2 Ae_i, Ae_i\right\rangle^{\alpha} \|Be_i\|^{2\alpha} \left\langle |T^*|^2 Be_i, Be_i\right\rangle^{1-\alpha} \\ &= \|Be_i\|^{2\alpha} \left\langle |T|^2 Ae_i, Ae_i\right\rangle^{\alpha} \|Ae_i\|^{2(1-\alpha)} \left\langle |T^*|^2 Be_i, Be_i\right\rangle^{1-\alpha}\end{aligned}$$

and taking the square root we get

$$|\langle TAe_i, Be_i\rangle| \le \|Be_i\|^{\alpha} \left\langle |T|^2 Ae_i, Ae_i\right\rangle^{\frac{\alpha}{2}} \|Ae_i\|^{1-\alpha} \left\langle |T^*|^2 Be_i, Be_i\right\rangle^{\frac{1-\alpha}{2}} \tag{3.34}$$

for any $i \in I$.

Using the generalized triangle inequality for the modulus and the Hölder's inequality for finite sums and $p = \frac{1}{\alpha}$, $q = \frac{1}{1-\alpha}$ we get from (3.34) that

$$\begin{aligned}&\left|\sum_{i\in F} \langle B^*TAe_i, e_i\rangle\right| \qquad (3.35)\\ &\le \sum_{i\in F} |\langle B^*TAe_i, e_i\rangle| \\ &\le \sum_{i\in F} \|Be_i\|^{\alpha} \left\langle |T|^2 Ae_i, Ae_i\right\rangle^{\frac{\alpha}{2}} \|Ae_i\|^{1-\alpha} \left\langle |T^*|^2 Be_i, Be_i\right\rangle^{\frac{1-\alpha}{2}} \\ &\le \left(\sum_{i\in F} \left[\|Be_i\|^{\alpha} \left\langle |T|^2 Ae_i, Ae_i\right\rangle^{\frac{\alpha}{2}}\right]^{1/\alpha}\right)^{\alpha} \\ &\quad \times \left(\sum_{i\in F} \left[\|Ae_i\|^{1-\alpha} \left\langle |T^*|^2 Be_i, Be_i\right\rangle^{\frac{1-\alpha}{2}}\right]^{1/(1-\alpha)}\right)^{1-\alpha} \\ &= \left(\sum_{i\in F} \|Be_i\| \left\langle |T|^2 Ae_i, Ae_i\right\rangle^{\frac{1}{2}}\right)^{\alpha} \left(\sum_{i\in F} \|Ae_i\| \left\langle |T^*|^2 Be_i, Be_i\right\rangle^{\frac{1}{2}}\right)^{1-\alpha}.\end{aligned}$$

By Cauchy–Bunyakowsky–Schwarz inequality for finite sums we also have

$$\begin{aligned}\sum_{i\in F} \|Be_i\| \left\langle |T|^2 Ae_i, Ae_i\right\rangle^{\frac{1}{2}} &\le \left(\sum_{i\in F} \|Be_i\|^2\right)^{1/2} \left(\sum_{i\in F} \left\langle |T|^2 Ae_i, Ae_i\right\rangle\right)^{1/2} \\ &= \left(\sum_{i\in F} \left\langle |B|^2 e_i, e_i\right\rangle\right)^{1/2} \left(\sum_{i\in F} \left\langle A^* |T|^2 Ae_i, e_i\right\rangle\right)^{1/2}\end{aligned}$$

and

$$\begin{aligned}\sum_{i\in F} \|Ae_i\| \left\langle |T^*|^2 Be_i, Be_i\right\rangle^{\frac{1}{2}} &\le \left(\sum_{i\in F} \|Ae_i\|^2\right)^{1/2} \left(\sum_{i\in F} \left\langle |T^*|^2 Be_i, Be_i\right\rangle\right)^{1/2} \\ &= \left(\sum_{i\in F} \left\langle |A|^2 e_i, e_i\right\rangle\right)^{1/2} \left(\sum_{i\in F} \left\langle B^* |T^*|^2 Be_i, e_i\right\rangle\right)^{1/2}\end{aligned}$$

and by (3.35) we obtain

$$\left|\sum_{i\in F}\langle B^{*}TAe_{i},e_{i}\rangle\right| \le \left(\sum_{i\in F}\left\langle |B|^{2}e_{i},e_{i}\right\rangle\right)^{\alpha/2}\left(\sum_{i\in F}\left\langle A^{*}|T|^{2}Ae_{i},e_{i}\right\rangle\right)^{\alpha/2} \times\left(\sum_{i\in F}\left\langle |A|^{2}e_{i},e_{i}\right\rangle\right)^{(1-\alpha)/2}\left(\sum_{i\in F}\left\langle B^{*}|T^{*}|^{2}Be_{i},e_{i}\right\rangle\right)^{(1-\alpha)/2} \tag{3.36}$$

for any F a finite part of I.

Let $\alpha\in[0,1]$. Since $A,\ B\in\mathcal{B}_{2}(H)$, then $A^{*}|T|^{2}A$ and $B^{*}|T^{*}|^{2}B\in\mathcal{B}_{1}(H)$ and by (3.36) we get

$$\begin{aligned}|\operatorname{tr}(AB^{*}T)|^{2} &\le\left[\operatorname{tr}\left(|B|^{2}\right)\operatorname{tr}\left(A^{*}|T|^{2}A\right)\right]^{\alpha}\left[\operatorname{tr}\left(|A|^{2}\right)\operatorname{tr}\left(B^{*}|T^{*}|^{2}B\right)\right]^{1-\alpha}\\ &=\left[\operatorname{tr}\left(|B|^{2}\right)\operatorname{tr}\left(|A^{*}|^{2}|T|^{2}\right)\right]^{\alpha}\left[\operatorname{tr}\left(|A|^{2}\right)\operatorname{tr}\left(|B^{*}|^{2}|T^{*}|^{2}\right)\right]^{1-\alpha}.\end{aligned} \tag{3.37}$$

Taking the infimum over $\alpha\in[0,1]$ we get (3.28). □

Corollary 3.5. *Let $T\in\mathcal{B}(H)$ and $A,\ B\in\mathcal{B}_{2}(H)$. We have $|A^{*}|^{2}|T|$, $|B^{*}|^{2}|T^{*}|$ and $B^{*}TA\in\mathcal{B}_{1}(H)$ and*

$$|\operatorname{tr}(AB^{*}T)|^{2}\le\operatorname{tr}\left(|A^{*}|^{2}|T|\right)\operatorname{tr}\left(|B^{*}|^{2}|T^{*}|\right). \tag{3.38}$$

Corollary 3.6. *Let $N\in\mathcal{B}(H)$ be a normal operator and $A,\ B\in\mathcal{B}_{2}(H)$.*

(i) For any $\alpha\in[0,1]$, $|A^{*}|^{2}|N|^{2\alpha}$, $|B^{*}|^{2}|N|^{2(1-\alpha)}$ and $B^{*}NA\in\mathcal{B}_{1}(H)$ and

$$|\operatorname{tr}(AB^{*}N)|^{2}\le\operatorname{tr}\left(|A^{*}|^{2}|N|^{2\alpha}\right)\operatorname{tr}\left(|B^{*}|^{2}|N|^{2(1-\alpha)}\right). \tag{3.39}$$

In particular, $|A^{*}|^{2}|N|$, $|B^{*}|^{2}|N|$ and $B^{*}NA\in\mathcal{B}_{1}(H)$ and

$$|\operatorname{tr}(AB^{*}N)|^{2}\le\operatorname{tr}\left(|A^{*}|^{2}|N|\right)\operatorname{tr}\left(|B^{*}|^{2}|N|\right). \tag{3.40}$$

(ii) We also have

$$|\operatorname{tr}(AB^{*}N)|^{2} \le\min\left\{\operatorname{tr}\left(|B|^{2}\right)\operatorname{tr}\left(|A^{*}|^{2}|N|^{2}\right),\operatorname{tr}\left(|A|^{2}\right)\operatorname{tr}\left(|B^{*}|^{2}|N|^{2}\right)\right\}. \tag{3.41}$$

Remark 3.7. Let $\alpha\in[0,1]$. By replacing A by A^{*} and B by B^{*} in (3.27) we get

$$|\operatorname{tr}(A^{*}BT)|^{2}\le\operatorname{tr}\left(|A|^{2}|T|^{2\alpha}\right)\operatorname{tr}\left(|B|^{2}|T^{*}|^{2(1-\alpha)}\right) \tag{3.42}$$

for any $T\in\mathcal{B}(H)$ and $A,\ B\in\mathcal{B}_{2}(H)$.

If in this inequality we take $A=B$, then we get

$$\left|\operatorname{tr}\left(|B|^{2}T\right)\right|^{2}\le\operatorname{tr}\left(|B|^{2}|T|^{2\alpha}\right)\operatorname{tr}\left(|B|^{2}|T^{*}|^{2(1-\alpha)}\right) \tag{3.43}$$

for any $T\in\mathcal{B}(H)$ and $B\in\mathcal{B}_{2}(H)$.

If in (3.42) we take $A = B^*$, then we get

$$\left|\operatorname{tr}\left(B^2T\right)\right|^2 \leq \operatorname{tr}\left(\left|B^*\right|^2 |T|^{2\alpha}\right) \operatorname{tr}\left(|B|^2 |T^*|^{2(1-\alpha)}\right) \tag{3.44}$$

for any $T \in \mathcal{B}(H)$ and $B \in \mathcal{B}_2(H)$.

Also, if $T = N$, a normal operator, then (3.43) and (3.44) become

$$\left|\operatorname{tr}\left(|B|^2 N\right)\right|^2 \leq \operatorname{tr}\left(|B|^2 |N|^{2\alpha}\right) \operatorname{tr}\left(|B|^2 |N|^{2(1-\alpha)}\right) \tag{3.45}$$

and

$$\left|\operatorname{tr}\left(B^2N\right)\right|^2 \leq \operatorname{tr}\left(\left|B^*\right|^2 |N|^{2\alpha}\right) \operatorname{tr}\left(|B|^2 |N|^{2(1-\alpha)}\right), \tag{3.46}$$

for any $B \in \mathcal{B}_2(H)$.

3.3. **Some Functional Properties.** Let $A \in \mathcal{B}_2(H)$ and $P \in \mathcal{B}(H)$ with $P \geq 0$. Then $Q := A^*PA \in \mathcal{B}_1(H)$ with $Q \geq 0$ and writing the inequality (3.43) for $B = (A^*PA)^{1/2} \in \mathcal{B}_2(H)$ we get

$$|\operatorname{tr}(A^*PAT)|^2 \leq \operatorname{tr}\left(A^*PA|T|^{2\alpha}\right) \operatorname{tr}\left(A^*PA|T^*|^{2(1-\alpha)}\right),$$

which, by the properties of trace, is equivalent to

$$|\operatorname{tr}(PATA^*)|^2 \leq \operatorname{tr}\left(PA|T|^{2\alpha}A^*\right) \operatorname{tr}\left(PA|T^*|^{2(1-\alpha)}A^*\right), \tag{3.47}$$

where $T \in \mathcal{B}(H)$ and $\alpha \in [0,1]$.

For a given $A \in \mathcal{B}_2(H)$, $T \in \mathcal{B}(H)$ and $\alpha \in [0,1]$, we consider the functional $\sigma_{A,T,\alpha}$ defined on the cone $\mathcal{B}_+(H)$ of nonnegative operators on $\mathcal{B}(H)$ by

$$\sigma_{A,T,\alpha}(P) := \left[\operatorname{tr}\left(PA|T|^{2\alpha}A^*\right)\right]^{1/2} \left[\operatorname{tr}\left(PA|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2} - |\operatorname{tr}(PATA^*)|.$$

The following theorem collects some fundamental properties of this functional.

Theorem 3.8 (Dragomir, 2014, [34]). *Let* $A \in \mathcal{B}_2(H)$, $T \in \mathcal{B}(H)$ *and* $\alpha \in [0,1]$.

(i) For any $P, Q \in \mathcal{B}_+(H)$

$$\sigma_{A,T,\alpha}(P+Q) \geq \sigma_{A,T,\alpha}(P) + \sigma_{A,T,\alpha}(Q) (\geq 0), \tag{3.48}$$

namely, $\sigma_{A,T,\alpha}$ *is a superadditive functional on* $\mathcal{B}_+(H)$;

(ii) For any $P, Q \in \mathcal{B}_+(H)$ *with* $P \geq Q$

$$\sigma_{A,T,\alpha}(P) \geq \sigma_{A,T,\alpha}(Q) (\geq 0), \tag{3.49}$$

namely, $\sigma_{A,T,\alpha}$ *is a monotonic nondecreasing functional on* $\mathcal{B}_+(H)$;

(iii) If $P, Q \in \mathcal{B}_+(H)$ *and there exist the constants* $M > m > 0$ *such that* $MQ \geq P \geq mQ$ *then*

$$M\sigma_{A,T,\alpha}(Q) \geq \sigma_{A,T,\alpha}(P) \geq m\sigma_{A,T,\alpha}(Q) (\geq 0). \tag{3.50}$$

Proof. (i) Let $P, Q \in \mathcal{B}_+(H)$. On utilizing the elementary inequality

$$\left(a^2+b^2\right)^{1/2}\left(c^2+d^2\right)^{1/2} \geq ac+bd, \; a,b,c,d \geq 0$$

and the triangle inequality for the modulus, we have

$$\begin{aligned}
&\sigma_{A,T,\alpha}(P+Q)\\
&=\left[\operatorname{tr}\left((P+Q)A|T|^{2\alpha}A^*\right)\right]^{1/2}\left[\operatorname{tr}\left((P+Q)A|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2}\\
&\quad-|\operatorname{tr}((P+Q)ATA^*)|\\
&=\left[\operatorname{tr}\left(PA|T|^{2\alpha}A^*+QA|T|^{2\alpha}A^*\right)\right]^{1/2}\\
&\quad\times\left[\operatorname{tr}\left(PA|T^*|^{2(1-\alpha)}A^*+QA|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2}\\
&\quad-|\operatorname{tr}(PATA^*+QATA^*)|\\
&=\left[\operatorname{tr}\left(PA|T|^{2\alpha}A^*\right)+\operatorname{tr}\left(QA|T|^{2\alpha}A^*\right)\right]^{1/2}\\
&\quad\times\left[\operatorname{tr}\left(PA|T^*|^{2(1-\alpha)}A^*\right)+\operatorname{tr}\left(QA|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2}\\
&\quad-|\operatorname{tr}(PATA^*)+\operatorname{tr}(QATA^*)|\\
&\geq\left[\operatorname{tr}\left(PA|T|^{2\alpha}A^*\right)\right]^{1/2}\left[\operatorname{tr}\left(PA|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2}\\
&\quad+\left[\operatorname{tr}\left(QA|T|^{2\alpha}A^*\right)\right]^{1/2}\left[\operatorname{tr}\left(QA|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2}\\
&\quad-|\operatorname{tr}(PATA^*)|-|\operatorname{tr}(QATA^*)|\\
&=\sigma_{A,T,\alpha}(P)+\sigma_{A,T,\alpha}(Q)
\end{aligned}$$

and the inequality (3.48) is proved.

(ii) Let $P,\ Q\in\mathcal{B}_+(H)$ with $P\geq Q$. Utilizing the superadditivity property we have

$$\begin{aligned}
\sigma_{A,T,\alpha}(P) &= \sigma_{A,T,\alpha}((P-Q)+Q)\geq\sigma_{A,T,\alpha}(P-Q)+\sigma_{A,T,\alpha}(Q)\\
&\geq \sigma_{A,T,\alpha}(Q)
\end{aligned}$$

and the inequality (3.49) is obtained.

(iii) From the monotonicity property we have

$$\sigma_{A,T,\alpha}(P)\geq\sigma_{A,T,\alpha}(mQ)=m\sigma_{A,T,\alpha}(Q)$$

and a similar inequality for M, which prove the desired result (3.50). □

Corollary 3.9. *Let $A\in\mathcal{B}_2(H)$, $T\in\mathcal{B}(H)$ and $\alpha\in[0,1]$. If $P\in\mathcal{B}(H)$ is such that there exist the constants $M>m>0$ with $M1_H\geq P\geq m1_H$, then*

$$\begin{aligned}
&M\left(\left[\operatorname{tr}\left(A|T|^{2\alpha}A^*\right)\right]^{1/2}\left[\operatorname{tr}\left(A|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2}-|\operatorname{tr}(ATA^*)|\right) \qquad (3.51)\\
&\geq\left[\operatorname{tr}\left(PA|T|^{2\alpha}A^*\right)\right]^{1/2}\left[\operatorname{tr}\left(PA|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2}-|\operatorname{tr}(PATA^*)|\\
&\geq m\left(\left[\operatorname{tr}\left(A|T|^{2\alpha}A^*\right)\right]^{1/2}\left[\operatorname{tr}\left(A|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2}-|\operatorname{tr}(ATA^*)|\right).
\end{aligned}$$

For a given $A\in\mathcal{B}_2(H)$, $T\in\mathcal{B}(H)$ and $\alpha\in[0,1]$, if we take $P=|V|^2$ with $V\in\mathcal{B}(H)$, we have

$$\begin{aligned}
\sigma_{A,T,\alpha}\left(|V|^2\right) &= \left[\operatorname{tr}\left(|V|^2 A|T|^{2\alpha}A^*\right)\right]^{1/2}\left[\operatorname{tr}\left(|V|^2 A|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2} \\
&\quad -\left|\operatorname{tr}\left(|V|^2 ATA^*\right)\right| \\
&= \left[\operatorname{tr}\left(V^*VA|T|^{2\alpha}A^*\right)\right]^{1/2}\left[\operatorname{tr}\left(V^*VA|T^*|^{2(1-\alpha)}A^*\right)\right]^{1/2} \\
&\quad -\left|\operatorname{tr}\left(V^*VATA^*\right)\right| \\
&= \left[\operatorname{tr}\left(A^*V^*VA|T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(A^*V^*VA|T^*|^{2(1-\alpha)}\right)\right]^{1/2} \\
&\quad -\left|\operatorname{tr}\left(A^*V^*VAT\right)\right| \\
&= \left[\operatorname{tr}\left((VA)^*VA|T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left((VA)^*VA|T^*|^{2(1-\alpha)}\right)\right]^{1/2} \\
&\quad -\left|\operatorname{tr}\left((VA)^*VAT\right)\right| \\
&= \left[\operatorname{tr}\left(|VA|^2|T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(|VA|^2|T^*|^{2(1-\alpha)}\right)\right]^{1/2}-\left|\operatorname{tr}\left(|VA|^2 T\right)\right|.
\end{aligned}$$

Assume that $A\in\mathcal{B}_2(H)$, $T\in\mathcal{B}(H)$ and $\alpha\in[0,1]$.

If we use the superadditivity property of the functional $\sigma_{A,T,\alpha}$ we have for any V, $U\in\mathcal{B}(H)$ that

$$\begin{aligned}
&\left[\operatorname{tr}\left(\left(|VA|^2+|UA|^2\right)|T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(\left(|VA|^2+|UA|^2\right)|T^*|^{2(1-\alpha)}\right)\right]^{1/2} \qquad (3.52)\\
&\quad -\left|\operatorname{tr}\left(\left(|VA|^2+|UA|^2\right)T\right)\right| \\
&\geq \left[\operatorname{tr}\left(|VA|^2|T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(|VA|^2|T^*|^{2(1-\alpha)}\right)\right]^{1/2}-\left|\operatorname{tr}\left(|VA|^2T\right)\right| \\
&\quad +\left[\operatorname{tr}\left(|UA|^2|T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(|UA|^2|T^*|^{2(1-\alpha)}\right)\right]^{1/2}-\left|\operatorname{tr}\left(|UA|^2T\right)\right| \\
&(\geq 0).
\end{aligned}$$

Also, if $|V|^2\geq|U|^2$ with V, $U\in\mathcal{B}(H)$, then

$$\begin{aligned}
&\left[\operatorname{tr}\left(|VA|^2|T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(|VA|^2|T^*|^{2(1-\alpha)}\right)\right]^{1/2}-\left|\operatorname{tr}\left(|VA|^2T\right)\right| \qquad (3.53)\\
&\geq \left[\operatorname{tr}\left(|UA|^2|T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(|UA|^2|T^*|^{2(1-\alpha)}\right)\right]^{1/2}-\left|\operatorname{tr}\left(|UA|^2T\right)\right| \\
&(\geq 0).
\end{aligned}$$

If $U\in\mathcal{B}(H)$ is invertible, then

$$\frac{1}{\|U^{-1}\|}\|x\|\leq\|Ux\|\leq\|U\|\|x\| \text{ for any } x\in H,$$

which implies that

$$\frac{1}{\|U^{-1}\|^2}1_H\leq|U|^2\leq\|U\|^2 1_H.$$

Utilizing (3.51) we get

$$\begin{aligned}
&\|U\|^2\left(\left[\operatorname{tr}\left(|A|^2|T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(|A|^2|T^*|^{2(1-\alpha)}\right)\right]^{1/2}-\left|\operatorname{tr}\left(|A|^2T\right)\right|\right) \qquad (3.54)\\
&\geq\left[\operatorname{tr}\left(|UA|^2|T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(|UA|^2|T^*|^{2(1-\alpha)}\right)\right]^{1/2}-\left|\operatorname{tr}\left(|UA|^2T\right)\right|\\
&\geq\frac{1}{\|U^{-1}\|^2}\\
&\quad\times\left(\left[\operatorname{tr}\left(|A|^2|T|^{2\alpha}\right)\right]^{1/2}\left[\operatorname{tr}\left(|A|^2|T^*|^{2(1-\alpha)}\right)\right]^{1/2}-\left|\operatorname{tr}\left(|A|^2T\right)\right|\right).
\end{aligned}$$

3.4. **Inequalities for n-Tuples of Operators.** We have:

Proposition 3.10 (Dragomir, 2014, [34])**.** *Let* $\mathbf{P}=(P_1,\cdots,P_n)\in\mathcal{B}_+^{(n)}(H)$, $\mathbf{T}=(T_1,\cdots,T_n)\in\mathcal{B}^{(n)}(H)$, $\mathbf{A}=(A_1,\cdots,A_n)\in\mathcal{B}_2^{(n)}(H)$ *and* $\mathbf{z}=(z_1,\cdots,z_n)\in\mathbb{C}^n$ *with* $n\geq 2$. *Then*

$$\begin{aligned}
&\left|\operatorname{tr}\left(\sum_{k=1}^n z_kP_kA_kT_kA_k^*\right)\right|^2 \qquad (3.55)\\
&\quad\leq\operatorname{tr}\left(\sum_{k=1}^n|z_k|P_kA_k|T_k|^{2\alpha}A_k^*\right)\operatorname{tr}\left(\sum_{k=1}^n|z_k|P_kA_k|T_k^*|^{2(1-\alpha)}A_k^*\right)
\end{aligned}$$

for any $\alpha\in[0,1]$.

Proof. Using the properties of modulus and the inequality (3.47) we have

$$\begin{aligned}
&\left|\operatorname{tr}\left(\sum_{k=1}^n z_kP_kA_kT_kA_k^*\right)\right|\\
&\quad=\left|\sum_{k=1}^n z_k\operatorname{tr}\left(P_kA_kT_kA_k^*\right)\right|\leq\sum_{k=1}^n|z_k|\left|\operatorname{tr}\left(P_kA_kT_kA_k^*\right)\right|\\
&\quad\leq\sum_{k=1}^n|z_k|\left[\operatorname{tr}\left(P_kA_k|T_k|^{2\alpha}A_k^*\right)\right]^{1/2}\left[\operatorname{tr}\left(P_kA_k|T_k^*|^{2(1-\alpha)}A_k^*\right)\right]^{1/2}.
\end{aligned}$$

Utilizing the weighted discrete Cauchy–Bunyakowsky–Schwarz inequality we also have

$$\sum_{k=1}^{n}|z_k|\left[\operatorname{tr}\left(P_kA_k|T_k|^{2\alpha}A_k^*\right)\right]^{1/2}\left[\operatorname{tr}\left(P_kA_k|T_k^*|^{2(1-\alpha)}A_k^*\right)\right]^{1/2}$$
$$\leq\left(\sum_{k=1}^{n}|z_k|\left(\left[\operatorname{tr}\left(P_kA_k|T_k|^{2\alpha}A_k^*\right)\right]^{1/2}\right)^2\right)^{1/2}$$
$$\times\left(\sum_{k=1}^{n}|z_k|\left(\left[\operatorname{tr}\left(P_kA_k|T_k^*|^{2(1-\alpha)}A_k^*\right)\right]^{1/2}\right)^2\right)^{1/2}$$
$$=\left(\sum_{k=1}^{n}|z_k|\operatorname{tr}\left(P_kA_k|T_k|^{2\alpha}A_k^*\right)\right)^{1/2}\left(\sum_{k=1}^{n}|z_k|\operatorname{tr}\left(P_kA_k|T_k^*|^{2(1-\alpha)}A_k^*\right)\right)^{1/2},$$

which imply the desired result (3.55). □

Remark 3.11. If we take $P_k=1_H$ for any $k\in\{1,\cdots,n\}$ in (3.55), then

$$\left|\operatorname{tr}\left(\sum_{k=1}^{n}z_k|A_k|^2T_k\right)\right|^2\leq\operatorname{tr}\left(\sum_{k=1}^{n}|z_k||A_k|^2|T_k|^{2\alpha}\right)\operatorname{tr}\left(\sum_{k=1}^{n}|z_k||A_k|^2|T_k^*|^{2(1-\alpha)}\right)\tag{3.56}$$

provided that $\mathbf{T}=(T_1,\cdots,T_n)\in\mathcal{B}^{(n)}(H)$, $\mathbf{A}=(A_1,\cdots,A_n)\in\mathcal{B}_2^{(n)}(H)$, $\alpha\in[0,1]$ and $\mathbf{z}=(z_1,\cdots,z_n)\in\mathbb{C}^n$.

We consider the following functional for n-tuples of nonnegative operators $\mathbf{P}=(P_1,\cdots,P_n)\in\mathcal{B}_+^{(n)}(H)$ as follows:

$$\sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{P}):=\left[\operatorname{tr}\left(\sum_{k=1}^{n}P_kA_k|T_k|^{2\alpha}A_k^*\right)\right]^{1/2}\times\left[\operatorname{tr}\left(\sum_{k=1}^{n}P_kA_k|T_k^*|^{2(1-\alpha)}A_k^*\right)\right]^{1/2}-\left|\operatorname{tr}\left(\sum_{k=1}^{n}P_kA_kT_kA_k^*\right)\right|,\tag{3.57}$$

where $\mathbf{T}=(T_1,\cdots,T_n)\in\mathcal{B}^{(n)}(H)$, $\mathbf{A}=(A_1,\cdots,A_n)\in\mathcal{B}_2^{(n)}(H)$ and $\alpha\in[0,1]$.

Utilizing a similar argument to the one in Theorem 3.8 we can state:

Proposition 3.12 (Dragomir, 2014, [34])**.** *Let* $\mathbf{T}=(T_1,\cdots,T_n)\in\mathcal{B}^{(n)}(H)$, $\mathbf{A}=(A_1,\cdots,A_n)\in\mathcal{B}_2^{(n)}(H)$ *and* $\alpha\in[0,1]$.

(i) For any $\mathbf{P},\ \mathbf{Q}\in\mathcal{B}_+^{(n)}(H)$

$$\sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{P}+\mathbf{Q})\geq\sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{P})+\sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{Q})\,(\geq 0),\tag{3.58}$$

namely, $\sigma_{\mathbf{A},\mathbf{T},\alpha}$ *is a superadditive functional on* $\mathcal{B}_+^{(n)}(H)$;

(ii) For any $\mathbf{P}$, $\mathbf{Q} \in \mathcal{B}_+^{(n)}(H)$ *with* $\mathbf{P} \geq \mathbf{Q}$, *namely* $P_k \geq Q_k$ *for all* $k \in \{1, \cdots, n\}$

$$\sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{P}) \geq \sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{Q}) (\geq 0), \tag{3.59}$$

namely, $\sigma_{\mathbf{A},\mathbf{B}}$ *is a monotonic nondecreasing functional on* $\mathcal{B}_+^{(n)}(H)$;

(iii) If $\mathbf{P}$, $\mathbf{Q} \in \mathcal{B}_+^{(n)}(H)$ *and there exist the constants* $M > m > 0$ *such that* $M\mathbf{Q} \geq \mathbf{P} \geq m\mathbf{Q}$ *then*

$$M\sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{Q}) \geq \sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{P}) \geq m\sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{Q}) (\geq 0). \tag{3.60}$$

If $\mathbf{P} = (p_1 1_H, \cdots, p_n 1_H)$ with $p_k \geq 0$, $k \in \{1, \cdots, n\}$ then the functional of real nonnegative weights $\mathbf{p} = (p_1, \cdots, p_n)$ defined by

$$\sigma_{\mathbf{A},\mathbf{T},\alpha}(\mathbf{p}) := \left[\operatorname{tr}\left(\sum_{k=1}^{n} p_k |A_k|^2 |T_k|^{2\alpha}\right)\right]^{1/2} \times \left[\operatorname{tr}\left(\sum_{k=1}^{n} p_k |A_k|^2 |T_k^*|^{2(1-\alpha)}\right)\right]^{1/2} - \left|\operatorname{tr}\left(\sum_{k=1}^{n} p_k |A_k|^2 T_k\right)\right| \tag{3.61}$$

has the same properties as in Theorem 3.8.

Moreover, we have the simple bounds

$$\begin{aligned}
&\max_{k\in\{1,\cdots,n\}}\{p_k\}\left(\left[\operatorname{tr}\left(\sum_{k=1}^{n} |A_k|^2 |T_k|^{2\alpha}\right)\right]^{1/2}\right. \\
&\quad\left.\times\left[\operatorname{tr}\left(\sum_{k=1}^{n} |A_k|^2 |T_k^*|^{2(1-\alpha)}\right)\right]^{1/2} - \left|\operatorname{tr}\left(\sum_{k=1}^{n} p_k |A_k|^2 T_k\right)\right|\right) \\
&\geq \left[\operatorname{tr}\left(\sum_{k=1}^{n} p_k |A_k|^2 |T_k|^{2\alpha}\right)\right]^{1/2} \left[\operatorname{tr}\left(\sum_{k=1}^{n} p_k |A_k|^2 |T_k^*|^{2(1-\alpha)}\right)\right]^{1/2} \\
&\quad - \left|\operatorname{tr}\left(\sum_{k=1}^{n} p_k |A_k|^2 T_k\right)\right| \\
&\geq \min_{k\in\{1,\cdots,n\}}\{p_k\}\left(\left[\operatorname{tr}\left(\sum_{k=1}^{n} |A_k|^2 |T_k|^{2\alpha}\right)\right]^{1/2}\right. \\
&\quad\left.\times\left[\operatorname{tr}\left(\sum_{k=1}^{n} |A_k|^2 |T_k^*|^{2(1-\alpha)}\right)\right]^{1/2} - \left|\operatorname{tr}\left(\sum_{k=1}^{n} p_k |A_k|^2 T_k\right)\right|\right).
\end{aligned} \tag{3.62}$$

3.5. **Further Inequalities for Power Series.** We have the following version of Kato's inequality for functions defined by power series:

Theorem 3.13 (Dragomir, 2014, [34])**.** *Let* $f(\lambda) := \sum_{n=1}^{\infty} \alpha_n \lambda^n$ *be a power series with complex coefficients and convergent on the open disk* $D(0, R)$, $R > 0$. *Let*

$N \in \mathcal{B}(H)$ be a normal operator. If for some $\alpha \in (0,1)$, $|N|^{2\alpha}$, $|N|^{2(1-\alpha)} \in \mathcal{B}_1(H)$ with $\operatorname{tr}\left(|N|^{2\alpha}\right)$, $\operatorname{tr}\left(|N|^{2(1-\alpha)}\right) < R$, *then*

$$\left|\operatorname{tr}\left(f\left(N\right)\right)\right|^2 \leq \operatorname{tr}\left(f_a\left(|N|^{2\alpha}\right)\right) \operatorname{tr}\left(f_a\left(|N|^{2(1-\alpha)}\right)\right). \tag{3.63}$$

Proof. Since N is a normal operator, then for any natural number $k \geq 1$ we have $\left|N^k\right|^{2\alpha} = |N|^{2\alpha k}$ and $\left|N^k\right|^{2(1-\alpha)} = |N|^{2(1-\alpha)k}$.

By the generalized triangle inequality for the modulus we have for $n \geq 2$

$$\left|\operatorname{tr}\left(\sum_{k=1}^{n} \alpha_k N^k\right)\right| = \left|\sum_{k=1}^{n} \alpha_k \operatorname{tr}\left(N^k\right)\right| \leq \sum_{k=1}^{n} |\alpha_k| \left|\operatorname{tr}\left(N^k\right)\right|. \tag{3.64}$$

If for some $\alpha \in (0,1)$ we have $|N|^{2\alpha}$, $|N|^{2(1-\alpha)} \in \mathcal{B}_1(H)$, then by Corollary 3.3 we have $N \in \mathcal{B}_1(H)$. Now, since N, $|N|^{2\alpha}$, $|N|^{2(1-\alpha)} \in \mathcal{B}_1(H)$ then any natural power of these operators belong to $\mathcal{B}_1(H)$ and by (3.25) we have

$$\left|\operatorname{tr}\left(N^k\right)\right|^2 \leq \operatorname{tr}\left(|N|^{2\alpha k}\right) \operatorname{tr}\left(|N|^{2(1-\alpha)k}\right), \tag{3.65}$$

for any natural number $k \geq 1$.

Making use of (3.65) we have

$$\sum_{k=1}^{n} |\alpha_k| \left|\operatorname{tr}\left(N^k\right)\right| \leq \sum_{k=1}^{n} |\alpha_k| \left(\operatorname{tr}\left(|N|^{2\alpha k}\right)\right)^{1/2} \left(\operatorname{tr}\left(|N|^{2(1-\alpha)k}\right)\right)^{1/2}. \tag{3.66}$$

Utilizing the weighted Cauchy–Bunyakowsky–Schwarz inequality for sums we also have

$$\begin{aligned} &\sum_{k=1}^{n} |\alpha_k| \left(\operatorname{tr}\left(|N|^{2\alpha k}\right)\right)^{1/2} \left(\operatorname{tr}\left(|N|^{2(1-\alpha)k}\right)\right)^{1/2} \qquad (3.67) \\ &\leq \left[\sum_{k=1}^{n} |\alpha_k| \left(\left(\operatorname{tr}\left(|N|^{2\alpha k}\right)\right)^{1/2}\right)^2\right]^{1/2} \\ &\quad\times \left[\sum_{k=1}^{n} |\alpha_k| \left(\left(\operatorname{tr}\left(|N|^{2(1-\alpha)k}\right)\right)^{1/2}\right)^2\right]^{1/2} \\ &= \left[\sum_{k=1}^{n} |\alpha_k| \operatorname{tr}\left(|N|^{2\alpha k}\right)\right]^{1/2} \left[\sum_{k=1}^{n} |\alpha_k| \operatorname{tr}\left(|N|^{2(1-\alpha)k}\right)\right]^{1/2}. \end{aligned}$$

Making use of (3.64), (3.66) and (3.67) we get the inequality

$$\left|\operatorname{tr}\left(\sum_{k=1}^{n} \alpha_k N^k\right)\right|^2 \leq \operatorname{tr}\left(\sum_{k=1}^{n} |\alpha_k| \, |N|^{2\alpha k}\right) \operatorname{tr}\left(\sum_{k=1}^{n} |\alpha_k| \, |N|^{2(1-\alpha)k}\right) \tag{3.68}$$

for any $n \geq 2$.

Due to the fact that $\operatorname{tr}\left(|N|^{2\alpha}\right), \operatorname{tr}\left(|N|^{2(1-\alpha)}\right) < R$ it follows by (3.25) that $\operatorname{tr}(|N|) < R$ and the operator series

$$\sum_{k=1}^{\infty} \alpha_k N^k, \quad \sum_{k=1}^{\infty} |\alpha_k| |N|^{2\alpha k} \text{ and } \sum_{k=1}^{\infty} |\alpha_k| |N|^{2(1-\alpha)k}$$

are convergent in the Banach space $\mathcal{B}_1(H)$.

Taking the limit over $n \to \infty$ in (3.68) and using the continuity of the $\operatorname{tr}(\cdot)$ on $\mathcal{B}_1(H)$ we deduce the desired result (3.63). □

Example 3.14. a) If we take in $f(\lambda) = (1 \pm \lambda)^{-1} - 1 = \mp\lambda\left((1 \pm \lambda)^{-1}\right)$, $|\lambda| < 1$ then we get from (3.63) the inequality

$$\left|\operatorname{tr}\left(N\left((1_H \pm N)^{-1}\right)\right)\right|^2 \le \operatorname{tr}\left(|N|^{2\alpha}\left(1_H - |N|^{2\alpha}\right)^{-1}\right) \operatorname{tr}\left(|N|^{2(1-\alpha)}\left(1_H - |N|^{2(1-\alpha)}\right)^{-1}\right), \tag{3.69}$$

provided that $N \in \mathcal{B}(H)$ is a normal operator and for $\alpha \in (0,1)$, $|N|^{2\alpha}$, $|N|^{2(1-\alpha)} \in \mathcal{B}_1(H)$ with $\operatorname{tr}\left(|N|^{2\alpha}\right), \operatorname{tr}\left(|N|^{2(1-\alpha)}\right) < 1$.

b) If we take in (3.63) $f(\lambda) = \exp(\lambda) - 1$, $\lambda \in \mathbb{C}$ then we get the inequality

$$|\operatorname{tr}(\exp(N) - 1_H)|^2 \le \operatorname{tr}\left(\exp\left(|N|^{2\alpha}\right) - 1_H\right) \operatorname{tr}\left(\exp\left(|N|^{2(1-\alpha)}\right) - 1_H\right), \tag{3.70}$$

provided that $N \in \mathcal{B}(H)$ is a normal operator and for $\alpha \in (0,1)$, $|N|^{2\alpha}$, $|N|^{2(1-\alpha)} \in \mathcal{B}_1(H)$.

The following result also holds:

Theorem 3.15 (Dragomir, 2014, [34])**.** *Let $f(\lambda) := \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a power series with complex coefficients and convergent on the open disk $D(0,R)$, $R > 0$. If $T \in \mathcal{B}(H)$, $A \in \mathcal{B}_2(H)$ are normal operators that double commute, i.e. $TA = AT$ and $TA^* = A^*T$ and $\operatorname{tr}\left(|A|^2 |T|^{2\alpha}\right), \operatorname{tr}\left(|A|^2 |T|^{2(1-\alpha)}\right) < R$ for some $\alpha \in [0,1]$, then*

$$\left|\operatorname{tr}\left(f\left(|A|^2 T\right)\right)\right|^2 \le \operatorname{tr}\left(f_a\left(|A|^2 |T|^{2\alpha}\right)\right) \operatorname{tr}\left(f_a\left(|A|^2 |T|^{2(1-\alpha)}\right)\right). \tag{3.71}$$

Proof. From the inequality (3.56) we have

$$\left|\operatorname{tr}\left(\sum_{k=0}^{n} \alpha_k \left|A^k\right|^2 T^k\right)\right|^2 \le \operatorname{tr}\left(\sum_{k=0}^{n} |\alpha_k| \left|A^k\right|^2 \left|T^k\right|^{2\alpha}\right) \operatorname{tr}\left(\sum_{k=0}^{n} |\alpha_k| \left|A^k\right|^2 \left|T^k\right|^{2(1-\alpha)}\right). \tag{3.72}$$

Since A and T are normal operators, then $\left|A^k\right|^2 = |A|^{2k}$, $\left|T^k\right|^{2\alpha} = |T|^{2\alpha k}$ and $\left|T^k\right|^{2(1-\alpha)} = |T|^{2(1-\alpha)k}$ for any natural number $k \ge 0$ and $\alpha \in [0,1]$.

Since T and A double commute, then is easy to see that

$$|A|^{2k} T^k = \left(|A|^2 T\right)^k, \quad |A|^{2k} |T|^{2\alpha k} = \left(|A|^2 |T|^{2\alpha}\right)^k$$

and

$$|A|^{2k}|T|^{2(1-\alpha)k}=\left(|A|^2|T|^{2(1-\alpha)}\right)^k$$

for any natural number $k\geq 0$ and $\alpha\in[0,1]$.

Therefore (3.72) is equivalent to

$$\left|\operatorname{tr}\left(\sum_{k=0}^{n}\alpha_k\left(|A|^2T\right)^k\right)\right|^2 \tag{3.73}$$
$$\leq \operatorname{tr}\left(\sum_{k=0}^{n}|\alpha_k|\left(|A|^2|T|^{2\alpha}\right)^k\right)\operatorname{tr}\left(\sum_{k=0}^{n}|\alpha_k|\left(|A|^2|T|^{2(1-\alpha)}\right)^k\right),$$

for any natural number $n\geq 1$and $\alpha\in[0,1]$.

Due to the fact that $\operatorname{tr}\left(|A|^2|T|^{2\alpha}\right),\operatorname{tr}\left(|A|^2|T|^{2(1-\alpha)}\right)<R$ it follows by (3.56) for $n=1$ that $\operatorname{tr}\left(|A|^2T\right)<R$ and the operator series

$$\sum_{k=1}^{\infty}\alpha_kN^k,\ \ \sum_{k=1}^{\infty}|\alpha_k|\,|N|^{2\alpha k}\ \text{ and }\ \sum_{k=1}^{\infty}|\alpha_k|\,|N|^{2(1-\alpha)k}$$

are convergent in the Banach space $\mathcal{B}_1(H)$.

Taking the limit over $n\to\infty$ in (3.73) and using the continuity of the $\operatorname{tr}(\cdot)$ on $\mathcal{B}_1(H)$ we deduce the desired result (3.71). □

Example 3.16. a) If we take $f(\lambda)=(1\pm\lambda)^{-1}$, $|\lambda|<1$ then we get from (3.71) the inequality

$$\left|\operatorname{tr}\left(\left(1_H\pm|A|^2T\right)^{-1}\right)\right|^2 \tag{3.74}$$
$$\leq \operatorname{tr}\left(\left(1_H-|A|^2|T|^{2\alpha}\right)^{-1}\right)\operatorname{tr}\left(\left(1_H-|A|^2|T|^{2(1-\alpha)}\right)^{-1}\right),$$

provided that $T\in\mathcal{B}(H)$, $A\in\mathcal{B}_2(H)$ are normal operators that double commute and $\operatorname{tr}\left(|A|^2|T|^{2\alpha}\right),\operatorname{tr}\left(|A|^2|T|^{2(1-\alpha)}\right)<1$ for $\alpha\in[0,1]$.

b) If we take in (3.71) $f(\lambda)=\exp(\lambda)$, $\lambda\in\mathbb{C}$ then we get the inequality

$$\left|\operatorname{tr}\left(\exp\left(|A|^2T\right)\right)\right|^2\leq\operatorname{tr}\left(\exp\left(|A|^2|T|^{2\alpha}\right)\right)\operatorname{tr}\left(\exp\left(|A|^2|T|^{2(1-\alpha)}\right)\right), \tag{3.75}$$

provided that $T\in\mathcal{B}(H)$ and $A\in\mathcal{B}_2(H)$ are normal operators that double commute and $\alpha\in[0,1]$.

Theorem 3.17 (Dragomir, 2014, [34])**.** *Let $f(z):=\sum_{j=0}^{\infty}p_jz^j$ and $g(z):=\sum_{j=0}^{\infty}q_jz^j$ be two power series with nonnegative coefficients and convergent on the open disk $D(0,R)$, $R>0$. If $T\in\mathcal{B}(H)$, $A\in\mathcal{B}_2(H)$ are normal operators that double commute and $\operatorname{tr}\left(|A|^2|T|^{2\alpha}\right)$, $\operatorname{tr}\left(|A|^2|T|^{2(1-\alpha)}\right)<R$ for $\alpha\in[0,1]$,*

then

$$\begin{aligned} & \left[\operatorname{tr}\left(f\left(|A|^2|T|^{2\alpha}\right)+g\left(|A|^2|T|^{2\alpha}\right)\right)\right]^{1/2} \\ & \quad\times\left[\operatorname{tr}\left(f\left(|A|^2|T|^{2(1-\alpha)}\right)+g\left(|A|^2|T|^{2(1-\alpha)}\right)\right)\right]^{1/2} \\ & \quad-\left|\operatorname{tr}\left(f\left(|A|^2 T\right)+g\left(|A|^2 T\right)\right)\right| \\ & \geq\left[\operatorname{tr}\left(f\left(|A|^2|T|^{2\alpha}\right)\right)\right]^{1/2}\left[\operatorname{tr}\left(f\left(|A|^2|T|^{2(1-\alpha)}\right)\right)\right]^{1/2} \\ & \quad-\left|\operatorname{tr}\left(f\left(|A|^2 T\right)\right)\right| \\ & \quad+\left[\operatorname{tr}\left(g\left(|A|^2|T|^{2\alpha}\right)\right)\right]^{1/2}\left[\operatorname{tr}\left(g\left(|A|^2|T|^{2(1-\alpha)}\right)\right)\right]^{1/2} \\ & \quad-\left|\operatorname{tr}\left(g\left(|A|^2 T\right)\right)\right|(\geq 0) . \end{aligned} \tag{3.76}$$

Moreover, if $p_j \geq q_j$ for any $j \in \mathbb{N}$, then, with the above assumptions on T and A,

$$\begin{aligned} & \left[\operatorname{tr}\left(f\left(|A|^2|T|^{2\alpha}\right)\right)\right]^{1/2}\left[\operatorname{tr}\left(f\left(|A|^2|T|^{2(1-\alpha)}\right)\right)\right]^{1/2} \\ & \quad-\left|\operatorname{tr}\left(f\left(|A|^2 T\right)\right)\right| \\ & \geq\left[\operatorname{tr}\left(g\left(|A|^2|T|^{2\alpha}\right)\right)\right]^{1/2}\left[\operatorname{tr}\left(g\left(|A|^2|T|^{2(1-\alpha)}\right)\right)\right]^{1/2} \\ & \quad-\left|\operatorname{tr}\left(g\left(|A|^2 T\right)\right)\right|(\geq 0) . \end{aligned} \tag{3.77}$$

The proof follows in a similar way to the proof of Theorem 3.15 by making use of the superadditivity and monotonicity properties of the functional $\sigma_{\mathbf{A},\mathbf{T},\alpha}(\cdot)$. We omit the details.

Example 3.18. Now, observe that if we take

$$f(\lambda)=\sinh \lambda=\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^{2n+1}$$

and

$$g(\lambda)=\cosh \lambda=\sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n}$$

then

$$f(\lambda)+g(\lambda)=\exp \lambda=\sum_{n=0}^{\infty} \frac{1}{n!} \lambda^{n}$$

for any $\lambda \in \mathbb{C}$.

If $T\in\mathcal{B}(H)$, $A\in\mathcal{B}_2(H)$ are normal operators that double commute and $\alpha\in[0,1]$, then by (3.76)

$$\left[\operatorname{tr}\left(\exp\left(|A|^2|T|^{2\alpha}\right)\right)\right]^{1/2}\left[\operatorname{tr}\left(\exp\left(|A|^2|T|^{2(1-\alpha)}\right)\right)\right]^{1/2}-\left|\operatorname{tr}\left(\exp\left(|A|^2T\right)\right)\right| \tag{3.78}$$
$$\geq\left[\operatorname{tr}\left(\sinh\left(|A|^2|T|^{2\alpha}\right)\right)\right]^{1/2}\left[\operatorname{tr}\left(\sinh\left(|A|^2|T|^{2(1-\alpha)}\right)\right)\right]^{1/2}-\left|\operatorname{tr}\left(\sinh\left(|A|^2T\right)\right)\right|$$
$$+\left[\operatorname{tr}\left(\cosh\left(|A|^2|T|^{2\alpha}\right)\right)\right]^{1/2}\left[\operatorname{tr}\left(\cosh\left(|A|^2|T|^{2(1-\alpha)}\right)\right)\right]^{1/2}-\left|\operatorname{tr}\left(\cosh\left(|A|^2T\right)\right)\right|$$
$$(\geq 0).$$

Now, consider the series $\frac{1}{1-\lambda}=\sum_{n=0}^{\infty}\lambda^n$, $\lambda\in D(0,1)$ and $\ln\frac{1}{1-\lambda}=\sum_{n=1}^{\infty}\frac{1}{n}\lambda^n$, $\lambda\in D(0,1)$ and define $p_n=1$, $n\geq 0$, $q_0=0$, $q_n=\frac{1}{n}$, $n\geq 1$, then we observe that for any $n\geq 0$, $p_n\geq q_n$.

If $T\in\mathcal{B}(H)$, $A\in\mathcal{B}_2(H)$ are normal operators that double commute, $\alpha\in[0,1]$ and $\operatorname{tr}\left(|A|^2|T|^{2\alpha}\right)$, $\operatorname{tr}\left(|A|^2|T|^{2(1-\alpha)}\right)<1$, then by (3.77)

$$\left[\operatorname{tr}\left(\left(1_H-|A|^2|T|^{2\alpha}\right)^{-1}\right)\right]^{1/2}\left[\operatorname{tr}\left(\left(1_H-|A|^2|T|^{2(1-\alpha)}\right)^{-1}\right)\right]^{1/2} \tag{3.79}$$
$$-\left|\operatorname{tr}\left(\left(1_H-|A|^2T\right)^{-1}\right)\right|$$
$$\geq\left[\operatorname{tr}\left(\ln\left(1_H-|A|^2|T|^{2\alpha}\right)^{-1}\right)\right]^{1/2}$$
$$\times\left[\operatorname{tr}\left(\ln\left(1_H-|A|^2|T|^{2(1-\alpha)}\right)^{-1}\right)\right]^{1/2}$$
$$-\left|\operatorname{tr}\left(\ln\left(1_H-|A|^2T\right)^{-1}\right)\right|(\geq 0).$$

4. Reverses of Schwarz Inequality

4.1. Some Classical Facts.

Let $\overline{\mathbf{a}}=(a_1,\dots,a_n)$ and $\overline{\mathbf{b}}=(b_1,\dots,b_n)$ be two positive *n-tuples* with

$$0<m_1\leq a_i\leq M_1<\infty \text{ and } 0<m_2\leq b_i\leq M_2<\infty; \tag{4.1}$$

for each $i\in\{1,\dots,n\}$, and some constants m_1, m_2, M_1, M_2.

The following reverses of the Cauchy–Bunyakowsky–Schwarz inequality for positive sequences of real numbers are well known:

a) *Pólya-Szegö's inequality* [73]:

$$\frac{\sum_{k=1}^n a_k^2\sum_{k=1}^n b_k^2}{\left(\sum_{k=1}^n a_kb_k\right)^2}\leq\frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}}+\sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2.$$

b) *Shisha–Mond's inequality* [76]:

$$\frac{\sum_{k=1}^n a_k^2}{\sum_{k=1}^n a_kb_k}-\frac{\sum_{k=1}^n a_kb_k}{\sum_{k=1}^n b_k^2}\leq\left[\left(\frac{M_1}{m_2}\right)^{\frac{1}{2}}-\left(\frac{m_1}{M_2}\right)^{\frac{1}{2}}\right]^2.$$

If $\overline{\mathbf{w}} = (w_1, \dots, w_n)$ is a positive sequence, then the following weighted inequalities also hold:

c) *Cassels' inequality* [81]. If the positive real sequences $\overline{\mathbf{a}} = (a_1, \dots, a_n)$ and $\overline{\mathbf{b}} = (b_1, \dots, b_n)$ satisfy the condition

$$0 < m \leq \frac{a_k}{b_k} \leq M < \infty \text{ for each } k \in \{1, \dots, n\}, \tag{4.2}$$

then

$$\frac{\left(\sum_{k=1}^{n} w_k a_k^2\right)\left(\sum_{k=1}^{n} w_k b_k^2\right)}{\left(\sum_{k=1}^{n} w_k a_k b_k\right)^2} \leq \frac{(M+m)^2}{4mM}.$$

For other recent results providing discrete reverse inequalities, see the monograph online [21].

The following reverse of Schwarz's inequality in inner product spaces holds [22].

Theorem 4.1 (Dragomir, 2003, [22])**.** *Let A, $a \in \mathbb{C}$ and x, $y \in H$, a complex inner product space with the inner product $\langle \cdot, \cdot \rangle$. If*

$$\operatorname{Re} \langle Ay - x, x - ay \rangle \geq 0, \tag{4.3}$$

or equivalently,

$$\left\| x - \frac{a+A}{2} \cdot y \right\| \leq \frac{1}{2} |A - a| \, \|y\|, \tag{4.4}$$

holds, then

$$0 \leq \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \leq \frac{1}{4} |A - a|^2 \|y\|^4. \tag{4.5}$$

The constant $\frac{1}{4}$ is sharp in (4.5).

In 1935, G. Grüss [55] proved the following integral inequality which gives an approximation of the integral mean of the product in terms of the product of the integrals means as follows:

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \frac{1}{b-a} \int_a^b f(x) \, dx \cdot \frac{1}{b-a} \int_a^b g(x) \, dx \right| \\ &\quad \leq \frac{1}{4} (\Phi - \phi)(\Gamma - \gamma), \end{aligned} \tag{4.6}$$

where f, $g : [a, b] \to \mathbb{R}$ are integrable on $[a, b]$ and satisfy the condition

$$\phi \leq f(x) \leq \Phi, \; \gamma \leq g(x) \leq \Gamma \tag{4.7}$$

for each $x \in [a, b]$, where ϕ, Φ, γ, Γ are given real constants.

Moreover, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

In [24], in order to generalize the Grüss integral inequality in abstract structures the author has proved the following inequality in inner product spaces.

Theorem 4.2 (Dragomir, 1999, [24])**.** *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}$ $(\mathbb{K} = \mathbb{R}, \mathbb{C})$ and $e \in H$, $\|e\| = 1$. If φ, γ, Φ, Γ are real or complex numbers and x, y are vectors in H such that the conditions*

$$\operatorname{Re} \langle \Phi e - x, x - \varphi e \rangle \geq 0 \text{ and } \operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle \geq 0 \tag{4.8}$$

hold, then

$$\left|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right|\leq\frac{1}{4}\left|\Phi-\varphi\right|\left|\Gamma-\gamma\right|. \tag{4.9}$$

The constant $\frac{1}{4}$ *is best possible in the sense that it can not be replaced by a smaller constant.*

For other results of this type, see the recent monograph [27] and the references therein.

For other Grüss type results for integral and sums see the papers [1]-[3], [11]-[13], [14]-[16], [23]-[30], [43], [72], [87] and the references therein.

4.2. **Additive Reverses of Schwarz Trace Inequality.** We denote by

$$\mathcal{B}_1^+(H):=\{P:\ P\in\mathcal{B}_1(H),\ P\text{ is selfadjoint and }P\geq 0\}.$$

We obtained recently the following result [36]:

Theorem 4.3 (Dragomir, 2014, [36])**.** *For any* $A,\ C\in\mathcal{B}(H)$ *and* $P\in\mathcal{B}_1^+(H)\setminus\{0\}$

$$\begin{aligned}&\left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \qquad (4.10)\\&\leq\inf_{\lambda\in\mathbb{C}}\left\|A-\lambda\cdot 1_H\right\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)\\&\leq\inf_{\lambda\in\mathbb{C}}\left\|A-\lambda\cdot 1_H\right\|\left[\frac{\operatorname{tr}\left(P\left|C\right|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2},\end{aligned}$$

where $\|\cdot\|$ *is the operator norm.*

Proof. We observe that, for any $\lambda\in\mathbb{C}$ we have

$$\begin{aligned}&\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left[P\left(A-\lambda 1_H\right)\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)\right] \qquad (4.11)\\&=\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left[PA\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)\right]\\&\quad-\frac{\lambda}{\operatorname{tr}(P)}\operatorname{tr}\left[P\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)\right]\\&=\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}.\end{aligned}$$

Taking the modulus in (4.11) and utilizing the properties of the trace, we have

$$\begin{aligned}&\left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|\\&=\frac{1}{\operatorname{tr}(P)}\left|\operatorname{tr}\left[P(A-\lambda 1_H)\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)\right]\right|\\&=\frac{1}{\operatorname{tr}(P)}\left|\operatorname{tr}\left[(A-\lambda 1_H)\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right]\right|\\&\leq\|A-\lambda 1_H\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)\end{aligned}$$

for any $\lambda\in\mathbb{C}$.

Utilizing Schwarz's inequality we also have

$$\begin{aligned}&\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)\\&=\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}P^{1/2}\right|\right)\\&\leq\left[\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}\right|^2\right)\right]^{1/2}[\operatorname{tr}(P)]^{1/2}.\end{aligned}\tag{4.12}$$

Observe that

$$\begin{aligned}&\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}\right|^2\right)\\&=\operatorname{tr}\left(\left(\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}\right)^*\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}\right)\\&=\operatorname{tr}\left(P^{1/2}\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)^*\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}\right)\\&=\operatorname{tr}\left(\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)^*\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right)\\&=\operatorname{tr}\left(\left(C^*-\frac{\overline{\operatorname{tr}(PC)}}{\operatorname{tr}(P)}1_H\right)\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right)\\&=\operatorname{tr}\left[\left(|C|^2-\frac{\overline{\operatorname{tr}(PC)}}{\operatorname{tr}(P)}C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}C^*+\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^21_H\right)P\right]\\&=\left(\frac{\operatorname{tr}\left(|C|^2P\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right)\operatorname{tr}(P).\end{aligned}\tag{4.13}$$

By (4.12) and (4.13) we get

$$\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)\leq\left(\frac{\operatorname{tr}\left(|C|^2P\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right)^{1/2}\operatorname{tr}(P)$$

and by (4.24) we have

$$\begin{aligned}
&\left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \qquad (4.14)\\
&\quad\le \|A-\lambda\cdot 1_H\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)\\
&\quad\le \|A-\lambda\cdot 1_H\|\left(\frac{\operatorname{tr}\left(|C|^2P\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right)^{1/2}
\end{aligned}$$

for any $\lambda\in\mathbb{C}$.

Taking the infimum over $\lambda\in\mathbb{C}$ in (4.14) we get the desired result (4.23). □

We also have [36]:

Corollary 4.4. *Let α, $\beta\in\mathbb{C}$ and $A\in B(H)$ such that*

$$\left\|A-\frac{\alpha+\beta}{2}\cdot 1_H\right\|\le\frac{1}{2}|\beta-\alpha|.$$

For any $C\in\mathcal{B}(H)$ and $P\in\mathcal{B}_1^+(H)\setminus\{0\}$

$$\begin{aligned}
&\left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \qquad (4.15)\\
&\quad\le\frac{1}{2}|\beta-\alpha|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)\\
&\quad\le\frac{1}{2}|\beta-\alpha|\left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2}.
\end{aligned}$$

In particular, if $C\in\mathcal{B}(H)$ is such that

$$\left\|C-\frac{\alpha+\beta}{2}\cdot 1_H\right\|\le\frac{1}{2}|\beta-\alpha|,$$

then

$$\begin{aligned}
0&\le\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2 \qquad (4.16)\\
&\le\frac{1}{2}|\beta-\alpha|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)\\
&\le\frac{1}{2}|\beta-\alpha|\left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2}\le\frac{1}{4}|\beta-\alpha|^2.
\end{aligned}$$

Also

$$\begin{aligned}&\left|\frac{\operatorname{tr}(PC^2)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^2\right| \\ &\qquad\le\frac{1}{2}|\beta-\alpha|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\ &\qquad\le\frac{1}{2}|\beta-\alpha|\left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2}\le\frac{1}{4}|\beta-\alpha|^2.\end{aligned}\tag{4.17}$$

For other related results see [36].

In order to simplify writing, we use the following notation

$$\mathcal{B}_+(H):=\{P\in\mathcal{B}(H),\ P\text{ is selfadjoint and }P\ge 0\}.$$

The following result holds:

Theorem 4.5 (Dragomir, 2014, [38])**.** *Let, either* $P\in\mathcal{B}_+(H)$, $A,\ B\in\mathcal{B}_2(H)$ *or* $P\in\mathcal{B}_1^+(H)$, $A,\ B\in\mathcal{B}(H)$ *and* $\gamma,\ \Gamma\in\mathbb{C}$. *Then*

(i) We have

$$\begin{aligned}0&\le\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right)-|\operatorname{tr}(PB^*A)|^2 \\ &=\operatorname{Re}\left[\left(\Gamma\operatorname{tr}\left(P|B|^2\right)-\operatorname{tr}(PB^*A)\right)\left(\operatorname{tr}(PA^*B)-\overline{\gamma}\operatorname{tr}\left(P|B|^2\right)\right)\right] \\ &\quad-\operatorname{tr}\left(P|B|^2\right)\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}B^*\right)(\Gamma B-A)\right]\right) \\ &\le\frac{1}{4}|\Gamma-\gamma|^2\left[\operatorname{tr}\left(P|B|^2\right)\right]^2 \\ &\quad-\operatorname{tr}\left(P|B|^2\right)\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}B^*\right)(\Gamma B-A)\right]\right).\end{aligned}\tag{4.18}$$

(ii) If

$$\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}B^*\right)(\Gamma B-A)\right]\right)\ge 0\tag{4.19}$$

or, equivalently

$$\operatorname{tr}\left(P\left|A-\frac{\gamma+\Gamma}{2}B\right|^2\right)\le\frac{1}{4}|\Gamma-\gamma|^2\operatorname{tr}\left(P|B|^2\right),\tag{4.20}$$

then

$$\begin{aligned}0&\le\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right)-|\operatorname{tr}(PB^*A)|^2 \\ &\le\operatorname{Re}\left[\left(\Gamma\operatorname{tr}\left(P|B|^2\right)-\operatorname{tr}(PB^*A)\right)\left(\operatorname{tr}(PA^*B)-\overline{\gamma}\operatorname{tr}\left(P|B|^2\right)\right)\right] \\ &\le\frac{1}{4}|\Gamma-\gamma|^2\left[\operatorname{tr}\left(P|B|^2\right)\right]^2\end{aligned}\tag{4.21}$$

and

$$\begin{aligned} 0 &\leq \operatorname{tr}\left(P\left|A\right|^{2}\right)\operatorname{tr}\left(P\left|B\right|^{2}\right)-\left|\operatorname{tr}\left(PB^{*}A\right)\right|^{2} \\ &\leq \frac{1}{4}\left|\Gamma-\gamma\right|^{2}\left[\operatorname{tr}\left(P\left|B\right|^{2}\right)\right]^{2} \\ &\quad-\operatorname{tr}\left(P\left|B\right|^{2}\right)\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^{*}-\overline{\gamma}B^{*}\right)\left(\Gamma B-A\right)\right]\right) \\ &\leq \frac{1}{4}\left|\Gamma-\gamma\right|^{2}\left[\operatorname{tr}\left(P\left|B\right|^{2}\right)\right]^{2}. \end{aligned} \tag{4.22}$$

Proof. Observe that, by the trace properties, we have

$$\begin{aligned} I_1 &:= \operatorname{Re}\left[\left(\Gamma\operatorname{tr}\left(P\left|B\right|^{2}\right)-\operatorname{tr}\left(PB^{*}A\right)\right)\left(\operatorname{tr}\left(PA^{*}B\right)-\overline{\gamma}\operatorname{tr}\left(P\left|B\right|^{2}\right)\right)\right] \\ &= \operatorname{Re}\left[\left(\Gamma\operatorname{tr}\left(P\left|B\right|^{2}\right)-\operatorname{tr}\left(PB^{*}A\right)\right)\left(\overline{\operatorname{tr}\left(PB^{*}A\right)}-\overline{\gamma}\operatorname{tr}\left(P\left|B\right|^{2}\right)\right)\right] \\ &= \operatorname{Re}\left[\Gamma\operatorname{tr}\left(P\left|B\right|^{2}\right)\overline{\operatorname{tr}\left(PB^{*}A\right)}+\overline{\gamma}\operatorname{tr}\left(PB^{*}A\right)\operatorname{tr}\left(P\left|B\right|^{2}\right)\right. \\ &\quad\left.-\left|\operatorname{tr}\left(PB^{*}A\right)\right|^{2}-\Gamma\overline{\gamma}\left[\operatorname{tr}\left(P\left|B\right|^{2}\right)\right]^{2}\right. \\ &= \operatorname{tr}\left(P\left|B\right|^{2}\right)\operatorname{Re}\left[\Gamma\overline{\operatorname{tr}\left(PB^{*}A\right)}+\overline{\gamma}\operatorname{tr}\left(PB^{*}A\right)\right] \\ &\quad-\left|\operatorname{tr}\left(PB^{*}A\right)\right|^{2}-\left[\operatorname{tr}\left(P\left|B\right|^{2}\right)\right]^{2}\operatorname{Re}\left(\Gamma\overline{\gamma}\right) \end{aligned} \tag{4.23}$$

and

$$\begin{aligned} I_2 &:= \operatorname{tr}\left(P\left|B\right|^{2}\right)\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^{*}-\overline{\gamma}B^{*}\right)\left(\Gamma B-A\right)\right]\right) \\ &= \operatorname{tr}\left(P\left|B\right|^{2}\right)\operatorname{Re}\left[\operatorname{tr}\left(\Gamma PA^{*}B+\overline{\gamma}PB^{*}A-\overline{\gamma}\Gamma PB^{*}B-PA^{*}A\right)\right] \\ &= \operatorname{tr}\left(P\left|B\right|^{2}\right)\operatorname{Re}\left[\Gamma\operatorname{tr}\left(PA^{*}B\right)+\overline{\gamma}\operatorname{tr}\left(PB^{*}A\right)\right] \\ &\quad-\overline{\gamma}\Gamma\operatorname{tr}\left(P\left|B\right|^{2}\right)-\operatorname{tr}\left(P\left|A\right|^{2}\right)] \\ &= \operatorname{tr}\left(P\left|B\right|^{2}\right)\operatorname{Re}\left[\Gamma\overline{\operatorname{tr}\left(PB^{*}A\right)}+\overline{\gamma}\operatorname{tr}\left(PB^{*}A\right)\right] \\ &\quad-\left[\operatorname{tr}\left(P\left|B\right|^{2}\right)\right]^{2}\operatorname{Re}\left(\overline{\gamma}\Gamma\right)-\operatorname{tr}\left(P\left|B\right|^{2}\right)\operatorname{tr}\left(P\left|A\right|^{2}\right), \end{aligned}$$

for P a selfadjoint operator with $P \geq 0$, A, $B \in \mathcal{B}_2(H)$ and γ, $\Gamma \in \mathbb{C}$.
Then we have

$$I_1 - I_2 = \operatorname{tr}\left(P\left|B\right|^{2}\right)\operatorname{tr}\left(P\left|A\right|^{2}\right)-\left|\operatorname{tr}\left(PB^{*}A\right)\right|^{2},$$

which proves the equality in (4.18).
Utilizing the elementary inequality for complex numbers

$$\operatorname{Re}\left(u\overline{v}\right) \leq \frac{1}{4}\left|u+v\right|^{2},\ u,v \in \mathbb{C},$$

we have

$$\begin{aligned}
&\operatorname{Re}\left[\left(\Gamma \operatorname{tr}\left(P|B|^2\right)-\operatorname{tr}\left(P B^* A\right)\right)\left(\operatorname{tr}\left(P A^* B\right)-\overline{\gamma} \operatorname{tr}\left(P|B|^2\right)\right)\right] \\
&=\operatorname{Re}\left[\left(\Gamma \operatorname{tr}\left(P|B|^2\right)-\operatorname{tr}\left(P B^* A\right)\right)\left(\overline{\operatorname{tr}\left(P B^* A\right)-\gamma \operatorname{tr}\left(P|B|^2\right)}\right)\right] \\
&\leq \frac{1}{4}\left[\Gamma \operatorname{tr}\left(P|B|^2\right)-\operatorname{tr}\left(P B^* A\right)+\operatorname{tr}\left(P B^* A\right)-\gamma \operatorname{tr}\left(P|B|^2\right)\right]^2 \\
&=\frac{1}{4}|\Gamma-\gamma|^2\left[\operatorname{tr}\left(P|B|^2\right)\right]^2,
\end{aligned}$$

which proves the last inequality in (4.18).

We have the equalities

$$\begin{aligned}
&\frac{1}{4}|\Gamma-\gamma|^2 P|B|^2-P\left|A-\frac{\gamma+\Gamma}{2} B\right|^2 \qquad (4.24)\\
&\quad=P\left[\frac{1}{4}|\Gamma-\gamma|^2|B|^2-\left|A-\frac{\gamma+\Gamma}{2} B\right|^2\right] \\
&\quad=P\left[\frac{1}{4}|\Gamma-\gamma|^2|B|^2-\left(A-\frac{\gamma+\Gamma}{2} B\right)^*\left(A-\frac{\gamma+\Gamma}{2} B\right)\right] \\
&\quad=P\left[\frac{1}{4}|\Gamma-\gamma|^2|B|^2\right. \\
&\qquad\left.-|A|^2+\overline{\frac{\gamma+\Gamma}{2}} B^* A+\frac{\gamma+\Gamma}{2} A^* B-\left|\frac{\gamma+\Gamma}{2}\right|^2|B|^2\right] \\
&\quad=P\left[-|A|^2+\overline{\frac{\gamma+\Gamma}{2}} B^* A+\frac{\gamma+\Gamma}{2} A^* B\right. \\
&\qquad\left.+\left(\frac{1}{4}|\Gamma-\gamma|^2-\left|\frac{\gamma+\Gamma}{2}\right|^2\right)|B|^2\right] \\
&\quad=P\left[-|A|^2+\overline{\frac{\gamma+\Gamma}{2}} B^* A+\frac{\gamma+\Gamma}{2} A^* B-\operatorname{Re}(\Gamma \overline{\gamma})|B|^2\right]
\end{aligned}$$

for any bounded operators A, B, P and the complex numbers γ, $\Gamma \in \mathbb{C}$.

Let P be a selfadjoint operator with $P \geq 0$, A, $B \in \mathcal{B}_2(H)$ and γ, $\Gamma \in \mathbb{C}$. Taking the trace in (4.24) we get

$$\begin{aligned}
&\frac{1}{4}|\Gamma-\gamma|^2 \operatorname{tr}\left(P|B|^2\right)-\operatorname{tr}\left(P\left|A-\frac{\gamma+\Gamma}{2} B\right|^2\right) \qquad (4.25)\\
&=-\operatorname{tr}\left(P|A|^2\right)-\operatorname{Re}(\Gamma \overline{\gamma}) \operatorname{tr}\left(P|B|^2\right)+\overline{\frac{\gamma+\Gamma}{2}} \operatorname{tr}\left(P B^* A\right)+\frac{\gamma+\Gamma}{2} \operatorname{tr}\left(P A^* B\right) \\
&=-\operatorname{tr}\left(P|A|^2\right)-\operatorname{Re}(\Gamma \overline{\gamma}) \operatorname{tr}\left(P|B|^2\right)+\overline{\frac{\gamma+\Gamma}{2}} \operatorname{tr}\left(P B^* A\right)+\frac{\gamma+\Gamma}{2} \overline{\operatorname{tr}\left(P B^* A\right)}
\end{aligned}$$

$$= -\operatorname{tr}\left(P\left|A\right|^2\right) - \operatorname{Re}\left(\Gamma\overline{\gamma}\right)\operatorname{tr}\left(P\left|B\right|^2\right) + \frac{\overline{\gamma+\Gamma}}{2}\operatorname{tr}\left(PB^*A\right) + \overline{\frac{\overline{\gamma+\Gamma}}{2}\operatorname{tr}\left(PB^*A\right)}$$

$$= -\operatorname{tr}\left(P\left|A\right|^2\right) - \operatorname{Re}\left(\Gamma\overline{\gamma}\right)\operatorname{tr}\left(P\left|B\right|^2\right) + 2\operatorname{Re}\left[\frac{\overline{\gamma+\Gamma}}{2}\operatorname{tr}\left(PB^*A\right)\right]$$

$$= -\operatorname{tr}\left(P\left|A\right|^2\right) - \operatorname{Re}\left(\Gamma\overline{\gamma}\right)\operatorname{tr}\left(P\left|B\right|^2\right) + \operatorname{Re}\left[\overline{\gamma}\operatorname{tr}\left(PB^*A\right)\right] + \operatorname{Re}\left[\overline{\Gamma}\operatorname{tr}\left(PB^*A\right)\right]$$

$$= -\operatorname{tr}\left(P\left|A\right|^2\right) - \operatorname{Re}\left(\Gamma\overline{\gamma}\right)\operatorname{tr}\left(P\left|B\right|^2\right) + \operatorname{Re}\left[\overline{\gamma}\operatorname{tr}\left(PB^*A\right)\right] + \operatorname{Re}\left[\overline{\overline{\Gamma}\operatorname{tr}\left(PB^*A\right)}\right]$$

$$= -\operatorname{tr}\left(P\left|A\right|^2\right) - \operatorname{Re}\left(\Gamma\overline{\gamma}\right)\operatorname{tr}\left(P\left|B\right|^2\right) + \operatorname{Re}\left[\overline{\gamma}\operatorname{tr}\left(PB^*A\right)\right] + \operatorname{Re}\left[\Gamma\overline{\operatorname{tr}\left(PB^*A\right)}\right].$$

Utilizing the equality for I_2 above, we conclude that (4.19) holds if and only if (4.20) holds, and the inequalities (4.21) and (4.22) thus follow from (4.18).

The case $P \in \mathcal{B}_1^+(H)$, $A, B \in \mathcal{B}(H)$ goes likewise and the details are omitted. □

For two given operators $T, U \in B(H)$ and two given scalars $\alpha, \beta \in \mathbb{C}$ consider the transform

$$\mathcal{C}_{\alpha,\beta}(T,U) = (T^* - \bar{\alpha}U^*)(\beta U - T).$$

This transform generalizes the transform

$$\mathcal{C}_{\alpha,\beta}(T) := (T^* - \bar{\alpha}1_H)(\beta 1_H - T) = \mathcal{C}_{\alpha,\beta}(T, 1_H),$$

where 1_H is the identity operator, which has been introduced in [32] in order to provide some generalizations of the well known Kantorovich inequality for operators in Hilbert spaces.

We recall that a bounded linear operator T on the complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$ is called *accretive* if $\operatorname{Re}\langle Ty, y\rangle \geq 0$ for any $y \in H$.

Utilizing the following identity

$$\begin{aligned}\operatorname{Re}\langle\mathcal{C}_{\alpha,\beta}(T,U)x,x\rangle &= \operatorname{Re}\langle\mathcal{C}_{\beta,\alpha}(T,U)x,x\rangle \\ &= \frac{1}{4}\left|\beta-\alpha\right|^2\left\|Ux\right\|^2 - \left\|Tx - \frac{\alpha+\beta}{2}\cdot Ux\right\|^2 \\ &= \frac{1}{4}\left|\beta-\alpha\right|^2\left\langle\left|U\right|^2x,x\right\rangle - \left\langle\left|T - \frac{\alpha+\beta}{2}\cdot U\right|^2x,x\right\rangle\end{aligned} \tag{4.26}$$

that holds for any scalars α, β and any vector $x \in H$, we can give a simple characterization result that is useful in the following:

Lemma 4.6. *For $\alpha, \beta \in \mathbb{C}$ and $T, U \in B(H)$ the following statements are equivalent:*

(i) *The transform $\mathcal{C}_{\alpha,\beta}(T,U)$ (or, equivalently, $\mathcal{C}_{\beta,\alpha}(T,U)$) is accretive;*

(ii) *We have the norm inequality*

$$\left\|Tx - \frac{\alpha+\beta}{2}\cdot Ux\right\| \leq \frac{1}{2}\left|\beta-\alpha\right|\left\|Ux\right\| \tag{4.27}$$

for any $x \in H$;

(iii) *We have the following inequality in the operator order*

$$\left|T-\frac{\alpha+\beta}{2}\cdot U\right|^2 \leq \frac{1}{4}\left|\beta-\alpha\right|^2 \left|U\right|^2.$$

As a consequence of the above lemma we can state:

Corollary 4.7. *Let α, $\beta \in \mathbb{C}$ and T, $U \in B(H)$. If $\mathcal{C}_{\alpha,\beta}(T,U)$ is accretive, then*

$$\left\|T-\frac{\alpha+\beta}{2}\cdot U\right\| \leq \frac{1}{2}\left|\beta-\alpha\right| \left\|U\right\|. \tag{4.28}$$

Remark 4.8. In order to give examples of linear operators T, $U \in B(H)$ and numbers α, $\beta \in \mathbb{C}$ such that the transform $\mathcal{C}_{\alpha,\beta}(T,U)$ is accretive, it suffices to select two bounded linear operator S and V and the complex numbers z, w ($w \neq 0$) with the property that $\|Sx - zVx\| \leq |w| \|Vx\|$ for any $x \in H$, and, by choosing $T = S$, $U = V$, $\alpha = \frac{1}{2}(z+w)$ and $\beta = \frac{1}{2}(z-w)$ we observe that T and U satisfy (4.27), i.e., $\mathcal{C}_{\alpha,\beta}(T,U)$ is accretive.

Corollary 4.9. *Let, either $P \in \mathcal{B}_+(H)$, A, $B \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, A, $B \in \mathcal{B}(H)$ and γ, $\Gamma \in \mathbb{C}$. If the transform $\mathcal{C}_{\gamma,\Gamma}(A,B)$ is accretive, then we have the inequalities (4.21) and (4.22).*

The case of selfadjoint operators is as follows.

Corollary 4.10. *Let P, A, B be selfadjoint operators with either $P \in \mathcal{B}_+(H)$, A, $B \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, A, $B \in \mathcal{B}(H)$ and m, $M \in \mathbb{R}$ with $M > m$. If $(A-mB)(MB-A) \geq 0$, then*

$$\begin{aligned} 0 &\leq \operatorname{tr}\left(PA^2\right)\operatorname{tr}\left(PB^2\right) - \left[\operatorname{tr}(PBA)\right]^2 \\ &\leq \left[\left(M\operatorname{tr}\left(PB^2\right) - \operatorname{tr}(PBA)\right)\left(\operatorname{tr}(PAB) - m\operatorname{tr}\left(PB^2\right)\right)\right] \\ &\leq \frac{1}{4}(M-m)^2\left[\operatorname{tr}\left(PB^2\right)\right]^2 \end{aligned} \tag{4.29}$$

and

$$\begin{aligned} 0 &\leq \operatorname{tr}\left(PA^2\right)\operatorname{tr}\left(PB^2\right) - \left[\operatorname{tr}(PBA)\right]^2 \\ &\leq \frac{1}{4}(M-m)^2\left[\operatorname{tr}\left(PB^2\right)\right]^2 - \operatorname{tr}\left(PB^2\right)\operatorname{tr}\left[P(A-mB)(MB-A)\right] \\ &\leq \frac{1}{4}(M-m)^2\left[\operatorname{tr}\left(PB^2\right)\right]^2. \end{aligned} \tag{4.30}$$

We also have the following result:

Theorem 4.11 (Dragomir, 2014, [38])**.** *Let, either $P \in \mathcal{B}_+(H)$, A, $B \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, A, $B \in \mathcal{B}(H)$ and $\lambda \in \mathbb{C}$.*

(i) We have

$$\begin{aligned} 0 &\leq \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|A|^2\right) - \left|\operatorname{tr}(PB^*A)\right|^2 \\ &= \operatorname{tr}\left(P\left|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}A - \lambda B\right|^2\right) - \left|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\lambda - \operatorname{tr}(PB^*A)\right|^2. \end{aligned} \tag{4.31}$$

(ii) If there is $r > 0$ such that

$$\operatorname{tr}\left(P\left|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}A-\lambda B\right|^2\right)\le r^2\left[\operatorname{tr}\left(P|B|^2\right)\right],$$

then we have the reverse of Schwarz inequality

$$\begin{aligned}0 &\le \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|A|^2\right)-|\operatorname{tr}(PB^*A)|^2 \\ &\le r^2\left[\operatorname{tr}\left(P|B|^2\right)\right]-\left|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\lambda-\operatorname{tr}(PB^*A)\right|^2 \\ &\le r^2\left[\operatorname{tr}\left(P|B|^2\right)\right].\end{aligned} \tag{4.32}$$

Proof. Using the properties of trace, we have for $P \ge 0$, A, $B \in \mathcal{B}_2(H)$ and $\lambda \in \mathbb{C}$ that

$$\begin{aligned}J_1 &:= \operatorname{tr}\left(P\left|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}A-\lambda B\right|^2\right) \\ &= \operatorname{tr}\left(P\left(\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}A-\lambda B\right)^*\left(\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}A-\lambda B\right)\right) \\ &= \operatorname{tr}\left(P\left[\operatorname{tr}\left(P|B|^2\right)|A|^2+|\lambda|^2|B|^2\right.\right. \\ &\quad \left.\left.-\overline{\lambda}\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}B^*A-\lambda\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}A^*B\right]\right) \\ &= \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|A|^2\right)+|\lambda|^2\operatorname{tr}\left(P|B|^2\right) \\ &\quad -\overline{\lambda}\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\operatorname{tr}(PB^*A)-\lambda\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\operatorname{tr}(PA^*B) \\ &= \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|A|^2\right)+|\lambda|^2\operatorname{tr}\left(P|B|^2\right) \\ &\quad -\overline{\lambda}\operatorname{tr}(PB^*A)\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}-\overline{\overline{\lambda}\operatorname{tr}(PB^*A)}\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2} \\ &= \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|A|^2\right)+|\lambda|^2\operatorname{tr}\left(P|B|^2\right) \\ &\quad -2\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\operatorname{Re}\left(\overline{\lambda}\operatorname{tr}(PB^*A)\right)\end{aligned}$$

and

$$\begin{aligned}J_2 &:= \left|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\lambda-\operatorname{tr}(PB^*A)\right|^2 \\ &= \left(\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\lambda-\operatorname{tr}(PB^*A)\right)\overline{\left(\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\lambda-\operatorname{tr}(PB^*A)\right)} \\ &= \operatorname{tr}\left(P|B|^2\right)|\lambda|^2-2\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\operatorname{Re}\left(\overline{\lambda}\operatorname{tr}(PB^*A)\right)+|\operatorname{tr}(PB^*A)|^2.\end{aligned}$$

Therefore

$$\begin{aligned}&J_1-J_2 \\ &= \operatorname{tr}\left(P\left|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}A-\lambda B\right|^2\right)-\left|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\lambda-\operatorname{tr}(PB^*A)\right|^2\end{aligned}$$

and the equality (4.31) is proved.

The inequality (4.32) follows from (4.31).

The other case is similar. □

Corollary 4.12. *Let, either $P \in \mathcal{B}_+(H)$, C, $D \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, C, $D \in \mathcal{B}(H)$ and δ, $\Delta \in \mathbb{C}$.*

If

$$\operatorname{Re}\left(\operatorname{tr}\left[P\left(C^* - \overline{\delta}D^*\right)(\Delta D - C)\right]\right) \geq 0 \tag{4.33}$$

or, equivalently

$$\operatorname{tr}\left(P\left|C - \frac{\delta + \Delta}{2}D\right|^2\right) \leq \frac{1}{4}|\Delta - \delta|^2 \operatorname{tr}\left(P|D|^2\right), \tag{4.34}$$

then

$$\begin{aligned} 0 &\leq \operatorname{tr}\left(P|C|^2\right)\operatorname{tr}\left(P|D|^2\right) - |\operatorname{tr}(PD^*C)|^2 \\ &\leq \frac{1}{4}|\Delta - \delta|^2\left[\operatorname{tr}\left(P|D|^2\right)\right]^2 - \left|\frac{\delta + \Delta}{2}\operatorname{tr}\left(P|D|^2\right) - \operatorname{tr}(PD^*C)\right|^2 \\ &\leq \frac{1}{4}|\Delta - \delta|^2\left[\operatorname{tr}\left(P|D|^2\right)\right]^2. \end{aligned} \tag{4.35}$$

Proof. The equivalence of the inequalities (4.33) and (4.34) follows from Theorem 4.5 (ii).

If we write the inequality (4.34) for $C = A$ and $D = B$, we have

$$\operatorname{tr}\left(P\left|A - \frac{\delta + \Delta}{2}B\right|^2\right) \leq \frac{1}{4}|\Delta - \delta|^2 \operatorname{tr}\left(P|B|^2\right).$$

If we multiply this inequality by $\operatorname{tr}\left(P|B|^2\right) \geq 0$ we get

$$\begin{aligned} &\operatorname{tr}\left(P\left|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2} A - \frac{\delta + \Delta}{2}\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2} B\right|^2\right) \\ &\leq \frac{1}{4}|\Delta - \delta|^2 \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|B|^2\right). \end{aligned} \tag{4.36}$$

Let

$$\lambda = \frac{\delta + \Delta}{2}\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2} \text{ and } r = \frac{1}{2}|\Delta - \delta|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}.$$

Then by (4.36) we have

$$\operatorname{tr}\left(P\left|\left[\operatorname{tr}\left(P|B|^2\right)\right]^{1/2} A - \lambda B\right|^2\right) \leq r^2 \operatorname{tr}\left(P|B|^2\right),$$

and by (4.32) we get

$$\begin{aligned} 0 &\leq \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|A|^2\right) - |\operatorname{tr}(PB^*A)|^2 \\ &\leq \frac{1}{4}|\Delta - \delta|^2\left[\operatorname{tr}\left(P|B|^2\right)\right]^2 - \left|\frac{\delta + \Delta}{2}\operatorname{tr}\left(P|B|^2\right) - \operatorname{tr}(PB^*A)\right|^2 \\ &\leq \frac{1}{4}|\Delta - \delta|^2\left[\operatorname{tr}\left(P|B|^2\right)\right]^2, \end{aligned}$$

and the inequality (4.35) is proved. □

Corollary 4.13. *Let, either $P \in \mathcal{B}_+(H)$, C, $D \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, C, $D \in \mathcal{B}(H)$ and δ, $\Delta \in \mathbb{C}$. If the transform $\mathcal{C}_{\delta,\Delta}(C,D)$ is accretive, then we have the inequalities (4.35).*

The case of selfadjoint operators is as follows.

Corollary 4.14. *Let P, C, D be selfadjoint operators with either $P \in \mathcal{B}_+(H)$, C, $D \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, C, $D \in \mathcal{B}(H)$ and n, $N \in \mathbb{R}$ with $N > n$. If $(C - nD)(ND - C) \geq 0$, then*

$$\begin{aligned} 0 &\leq \operatorname{tr}\left(PC^2\right)\operatorname{tr}\left(PD^2\right) - [\operatorname{tr}(PDC)]^2 \qquad (4.37) \\ &\leq \frac{1}{4}(N-n)^2\left[\operatorname{tr}\left(PD^2\right)\right]^2 - \left(\frac{n+N}{2}\operatorname{tr}\left(PD^2\right) - \operatorname{tr}(PDC)\right)^2 \\ &\leq \frac{1}{4}(N-n)^2\left[\operatorname{tr}\left(PD^2\right)\right]^2. \end{aligned}$$

4.3. **Trace Inequalities of Grüss Type.** Let P be a selfadjoint operator with $P \geq 0$. The functional $\langle\cdot,\cdot\rangle_{2,P}$ defined by

$$\langle A, B\rangle_{2,P} := \operatorname{tr}(PB^*A) = \operatorname{tr}(APB^*) = \operatorname{tr}(B^*AP)$$

is a *nonnegative Hermitian form* on $\mathcal{B}_2(H)$.

For a pair of complex numbers (α,β) and $P \in \mathcal{B}_+(H)$, in order to simplify the notations, we say that the pair of operators $(U,V) \in \mathcal{B}_2(H) \times \mathcal{B}_2(H)$ has the *trace P-(α,β)-property* if

$$\operatorname{Re}\left(\operatorname{tr}\left[P\left(U^* - \overline{\alpha}V^*\right)(\beta V - U)\right]\right) \geq 0$$

or, equivalently

$$\operatorname{tr}\left(P\left|U - \frac{\alpha+\beta}{2}V\right|^2\right) \leq \frac{1}{4}|\beta-\alpha|^2\operatorname{tr}\left(P|V|^2\right).$$

The above definitions can be also considered in the case when $P \in \mathcal{B}_1^+(H)$ and A, $B \in \mathcal{B}(H)$.

Theorem 4.15 (Dragomir, 2014, [38])**.** *Let, either $P \in \mathcal{B}_+(H)$, A, B, $C \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, A, B, $C \in \mathcal{B}(H)$ and λ, Γ, δ, $\Delta \in \mathbb{C}$. If (A,C) has the trace P-(λ,Γ)-property and (B,C) has the trace P-(δ,Δ)-property, then*

$$\begin{aligned} &\left|\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right) - \operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)\right| \qquad (4.38) \\ &\leq \operatorname{tr}\left(P|C|^2\right)\left[\frac{1}{4}|\Gamma-\gamma||\Delta-\delta|\operatorname{tr}\left(P|C|^2\right)\right. \\ &\quad - \left[\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^* - \overline{\gamma}C^*\right)(\Gamma C - A)\right]\right)\right]^{1/2} \\ &\quad \left.\times\left[\operatorname{Re}\left(\operatorname{tr}\left[P\left(B^* - \overline{\delta}C^*\right)(\Delta C - B)\right]\right)\right]^{1/2}\right] \\ &\leq \frac{1}{4}|\Gamma-\gamma||\Delta-\delta|\left[\operatorname{tr}\left(P|C|^2\right)\right]^2. \end{aligned}$$

Proof. We prove in the case that $P \in \mathcal{B}_+(H)$ and $A,\ B,\ C \in \mathcal{B}_2(H)$.

Making use of the Schwarz inequality for the nonnegative hermitian form $\langle \cdot, \cdot \rangle_{2,P}$ we have

$$\left| \langle A, B \rangle_{2,P} \right|^2 \le \langle A, A \rangle_{2,P} \langle B, B \rangle_{2,P}$$

for any $A,\ B \in \mathcal{B}_2(H)$.

Let $C \in \mathcal{B}_2(H)$, $C \ne 0$. Define the mapping $[\cdot, \cdot]_{2,P,C} : \mathcal{B}_2(H) \times \mathcal{B}_2(H) \to \mathbb{C}$ by

$$[A, B]_{2,P,C} := \langle A, B \rangle_{2,P} \|C\|_{2,P}^2 - \langle A, C \rangle_{2,P} \langle C, B \rangle_{2,P}.$$

Observe that $[\cdot, \cdot]_{2,P,C}$ is a nonnegative Hermitian form on $\mathcal{B}_2(H)$ and by Schwarz inequality we also have

$$\begin{aligned} &\left| \langle A, B \rangle_{2,P} \|C\|_{2,P}^2 - \langle A, C \rangle_{2,P} \langle C, B \rangle_{2,P} \right|^2 \\ &\quad \le \left[\|A\|_{2,P}^2 \|C\|_{2,P}^2 - \left| \langle A, C \rangle_{2,P} \right|^2 \right] \left[\|B\|_{2,P}^2 \|C\|_{2,P}^2 - \left| \langle B, C \rangle_{2,P} \right|^2 \right] \end{aligned}$$

for any $A,\ B \in \mathcal{B}_2(H)$, namely

$$\begin{aligned} &\left| \operatorname{tr}(PB^*A) \operatorname{tr}\left(P|C|^2\right) - \operatorname{tr}(PC^*A) \operatorname{tr}(PB^*C) \right|^2 \\ &\quad \le \left[\operatorname{tr}\left(P|A|^2\right) \operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PC^*A)|^2 \right] \\ &\qquad \times \left[\operatorname{tr}\left(P|B|^2\right) \operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PC^*B)|^2 \right], \end{aligned} \tag{4.39}$$

where for the last term we used the equality $\left| \langle B, C \rangle_{2,P} \right|^2 = \left| \langle C, B \rangle_{2,P} \right|^2$.

Since (A, C) has the trace P-(λ, Γ)-property and (B, C) has the trace P-(δ, Δ) -property, then by (4.22) we have

$$\begin{aligned} 0 &\le \operatorname{tr}\left(P|A|^2\right) \operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PC^*A)|^2 \\ &\le \operatorname{tr}\left(P|C|^2\right) \\ &\quad \times \left[\frac{1}{4} |\Gamma - \gamma|^2 \left[\operatorname{tr}\left(P|C|^2\right)\right] - \operatorname{Re}\left(\operatorname{tr}\left[P\left(A^* - \overline{\gamma} C^*\right)(\Gamma C - A)\right]\right) \right] \end{aligned} \tag{4.40}$$

and

$$\begin{aligned} 0 &\le \operatorname{tr}\left(P|B|^2\right) \operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PC^*B)|^2 \\ &\le \operatorname{tr}\left(P|C|^2\right) \\ &\quad \times \left[\frac{1}{4} |\Delta - \delta|^2 \left[\operatorname{tr}\left(P|C|^2\right)\right] - \operatorname{Re}\left(\operatorname{tr}\left[P\left(B^* - \overline{\delta} C^*\right)(\Delta C - B)\right]\right) \right]. \end{aligned} \tag{4.41}$$

If we multiply (4.40) with (4.41) and use (4.39), then we get

$$\begin{aligned}
&\left|\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right)-\operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)\right|^2 \qquad (4.42)\\
&\quad\le\left[\operatorname{tr}\left(P|C|^2\right)\right]^2\\
&\qquad\times\left[\frac{1}{4}|\Gamma-\gamma|^2\left[\operatorname{tr}\left(P|C|^2\right)\right]-\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}C^*\right)(\Gamma C-A)\right]\right)\right]\\
&\qquad\times\left[\frac{1}{4}|\Delta-\delta|^2\left[\operatorname{tr}\left(P|C|^2\right)\right]-\operatorname{Re}\left(\operatorname{tr}\left[P\left(B^*-\overline{\delta}C^*\right)(\Delta C-B)\right]\right)\right].
\end{aligned}$$

Utilizing the elementary inequality for positive numbers m, n, p, q

$$\left(m^2-n^2\right)\left(p^2-q^2\right)\le(mp-nq)^2,$$

we can state that

$$\begin{aligned}
&\left[\frac{1}{4}|\Gamma-\gamma|^2\left[\operatorname{tr}\left(P|C|^2\right)\right]-\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}C^*\right)(\Gamma C-A)\right]\right)\right] \qquad (4.43)\\
&\qquad\times\left[\frac{1}{4}|\Delta-\delta|^2\left[\operatorname{tr}\left(P|C|^2\right)\right]-\operatorname{Re}\left(\operatorname{tr}\left[P\left(B^*-\overline{\delta}C^*\right)(\Delta C-B)\right]\right)\right]\\
&\quad\le\left(\frac{1}{4}|\Gamma-\gamma||\Delta-\delta|\left[\operatorname{tr}\left(P|C|^2\right)\right]\right.\\
&\qquad-\left[\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}C^*\right)(\Gamma C-A)\right]\right)\right]^{1/2}\\
&\qquad\left.\times\left[\operatorname{Re}\left(\operatorname{tr}\left[P\left(B^*-\overline{\delta}C^*\right)(\Delta C-B)\right]\right)\right]^{1/2}\right)^2
\end{aligned}$$

with the term in the right hand side in the brackets being nonnegative.

Making use of (4.42) and (4.43) we then get

$$\begin{aligned}
&\left|\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right)-\operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)\right|^2 \qquad (4.44)\\
&\quad\le\left[\operatorname{tr}\left(P|C|^2\right)\right]^2\left(\frac{1}{4}|\Gamma-\gamma||\Delta-\delta|\left[\operatorname{tr}\left(P|C|^2\right)\right]\right.\\
&\qquad-\left[\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}C^*\right)(\Gamma C-A)\right]\right)\right]^{1/2}\\
&\qquad\left.\times\left[\operatorname{Re}\left(\operatorname{tr}\left[P\left(B^*-\overline{\delta}C^*\right)(\Delta C-B)\right]\right)\right]^{1/2}\right)^2.
\end{aligned}$$

Taking the square root in (4.44) we obtain the desired result (4.38). □

Corollary 4.16. *Let, either $P\in\mathcal{B}_+(H)$, A, B, $C\in\mathcal{B}_2(H)$ or $P\in\mathcal{B}_1^+(H)$, A, B, $C\in\mathcal{B}(H)$ and λ, Γ, δ, $\Delta\in\mathbb{C}$. If the transforms $\mathcal{C}_{\lambda,\Gamma}(A,C)$ and $\mathcal{C}_{\delta,\Delta}(B,C)$ are accretive, then the inequality (4.38) is valid.*

We have:

Corollary 4.17. *Let P, A, B, C be selfadjoint operators with either $P\in\mathcal{B}_+(H)$, A, B, $C\in\mathcal{B}_2(H)$ or $P\in\mathcal{B}_1^+(H)$, A, B, $C\in\mathcal{B}(H)$ and m, M, n, $N\in\mathbb{R}$ with $M>m$ and $N>n$. If $(A-mC)(MC-A)\ge 0$ and $(B-nC)(NC-B)\ge 0$*

then

$$\begin{aligned}
&\left|\operatorname{tr}(PBA)\operatorname{tr}\left(PC^2\right)-\operatorname{tr}(PCA)\operatorname{tr}(PBC)\right| \qquad (4.45)\\
&\leq \operatorname{tr}\left(PC^2\right)\left[\frac{1}{4}(M-m)(N-n)\operatorname{tr}\left(PC^2\right)\right.\\
&\quad -\left[\operatorname{Re}\left(\operatorname{tr}(A-mC)(MC-A)\right)\right]^{1/2}\\
&\quad \left.\times\left[\operatorname{Re}\left(\operatorname{tr}\left[P(B-nC)(NC-B)\right]\right)\right]^{1/2}\right]\\
&\leq \frac{1}{4}(M-m)(N-n)\left[\operatorname{tr}\left(PC^2\right)\right]^2.
\end{aligned}$$

Finally, we have:

Theorem 4.18 (Dragomir, 2014, [38])**.** *With the assumptions of Theorem 4.15*

$$\begin{aligned}
&\left|\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right)-\operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)\right| \qquad (4.46)\\
&\leq \operatorname{tr}\left(P|C|^2\right)\left[\frac{1}{4}|\Gamma-\gamma||\Delta-\delta|\operatorname{tr}\left(P|C|^2\right)\right.\\
&\quad -\left|\frac{\Gamma+\gamma}{2}\operatorname{tr}\left(P|C|^2\right)-\operatorname{tr}(PC^*A)\right|\\
&\quad \left.\times\left|\frac{\delta+\Delta}{2}\operatorname{tr}\left(P|C|^2\right)-\operatorname{tr}(PC^*B)\right|\right]\\
&\leq \frac{1}{4}|\Gamma-\gamma||\Delta-\delta|\left[\operatorname{tr}\left(P|C|^2\right)\right]^2.
\end{aligned}$$

If the transforms $\mathcal{C}_{\lambda,\Gamma}(A,C)$ and $\mathcal{C}_{\delta,\Delta}(B,C)$ are accretive, then the inequality (4.46) also holds.

The proof is similar to the one for Theorem 4.15 via the Corollary 4.12 and the details are omitted.

Corollary 4.19. *With the assumptions of Corollary 4.17*

$$\begin{aligned}
&\left|\operatorname{tr}(PBA)\operatorname{tr}\left(PC^2\right)-\operatorname{tr}(PCA)\operatorname{tr}(PBC)\right| \qquad (4.47)\\
&\leq \operatorname{tr}\left(PC^2\right)\left[\frac{1}{4}(M-m)(N-n)\operatorname{tr}\left(PC^2\right)\right.\\
&\quad -\left|\frac{M+m}{2}\operatorname{tr}\left(PC^2\right)-\operatorname{tr}(PCA)\right|\\
&\quad \left.\times\left|\frac{n+N}{2}\operatorname{tr}\left(PC^2\right)-\operatorname{tr}(PCB)\right|\right]\\
&\leq \frac{1}{4}(M-m)(N-n)\left[\operatorname{tr}\left(PC^2\right)\right]^2.
\end{aligned}$$

4.4. **Some Examples in the Case of $P\in\mathcal{B}_1(H)$.** Utilizing the above results in the case when $P\in\mathcal{B}_1^+(H)$, $A\in\mathcal{B}(H)$ and $B=1_H$ we can also state the following inequalities that complement the earlier results obtained in [36]:

Proposition 4.20 (Dragomir, 2014, [38]). *Let $P \in \mathcal{B}_1^+(H)$, $A \in \mathcal{B}(H)$ and γ, $\Gamma \in \mathbb{C}$.*

(i) We have

$$0 \le \frac{\operatorname{tr}\left(P|A|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2 \tag{4.48}$$
$$= \operatorname{Re}\left[\left(\Gamma - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA^*)}{\operatorname{tr}(P)} - \overline{\gamma}\right)\right]$$
$$- \frac{1}{\operatorname{tr}(P)}\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^* - \overline{\gamma}1_H\right)\left(\Gamma 1_H - A\right)\right]\right)$$
$$\le \frac{1}{4}|\Gamma - \gamma|^2 - \frac{1}{\operatorname{tr}(P)}\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^* - \overline{\gamma}1_H\right)\left(\Gamma 1_H - A\right)\right]\right).$$

(ii) If

$$\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^* - \overline{\gamma}1_H\right)\left(\Gamma 1_H - A\right)\right]\right) \ge 0 \tag{4.49}$$

or, equivalently

$$\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(P\left|A - \frac{\gamma + \Gamma}{2}1_H\right|^2\right) \le \frac{1}{4}|\Gamma - \gamma|^2, \tag{4.50}$$

and we say for simplicity that A has the trace P-(λ, Γ)-property, then

$$0 \le \frac{\operatorname{tr}\left(P|A|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2 \tag{4.51}$$
$$\le \operatorname{Re}\left[\left(\Gamma - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA^*)}{\operatorname{tr}(P)} - \overline{\gamma}\right)\right] \le \frac{1}{4}|\Gamma - \gamma|^2$$

and

$$0 \le \frac{\operatorname{tr}\left(P|A|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2 \tag{4.52}$$
$$\le \frac{1}{4}|\Gamma - \gamma|^2 - \frac{1}{\operatorname{tr}(P)}\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^* - \overline{\gamma}1_H\right)\left(\Gamma 1_H - A\right)\right]\right) \le \frac{1}{4}|\Gamma - \gamma|^2.$$

(iii) If the transform $\mathcal{C}_{\lambda,\Gamma}(A) := \left(A^* - \overline{\gamma}1_H\right)\left(\Gamma 1_H - A\right)$ is accretive, then the inequalities (4.51) and (4.52) also hold.

Corollary 4.21. *Let $P \in \mathcal{B}_1^+(H)$, A be a selfadjoint operator and m, $M \in \mathbb{R}$ with $M > m$.*

(i) If $\left(A - m1_H\right)\left(M1_H - A\right) \ge 0$, then

$$0 \le \frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)} - \left[\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right]^2 \tag{4.53}$$
$$\le \left[\left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right)\right] \le \frac{1}{4}(M - m)^2$$

and

$$0 \leq \frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} - \left[\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right]^2 \tag{4.54}$$
$$\leq \frac{1}{4}(M-m)^2 - \frac{1}{\operatorname{tr}(P)}\operatorname{tr}[P(A-mB)(MB-A)] \leq \frac{1}{4}(M-m)^2.$$

(ii) If $m1_H \leq A \leq M1_H$, then (4.53) and (4.54) also hold.

We have the following reverse of Schwarz inequality as well:

Proposition 4.22 (Dragomir, 2014, [38]). *Let $P \in \mathcal{B}_1^+(H)$, $A \in \mathcal{B}(H)$ and γ, $\Gamma \in \mathbb{C}$.*

(i) If A has the trace P-(λ, Γ)-property, then

$$0 \leq \frac{\operatorname{tr}\left(P|A|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2 \tag{4.55}$$
$$\leq \frac{1}{4}|\Gamma-\gamma|^2 - \left|\frac{\Gamma+\gamma}{2} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2 \leq \frac{1}{4}|\Gamma-\gamma|^2.$$

(ii) If the transform $\mathcal{C}_{\lambda,\Gamma}(A) := (A^* - \overline{\gamma}1_H)(\Gamma 1_H - A)$ is accretive, then the inequality (4.55) also holds.

Corollary 4.23. *Let $P \in \mathcal{B}_1^+(H)$, A be a selfadjoint operator and m, $M \in \mathbb{R}$ with $M > m$.*

(i) If $(A - m1_H)(M1_H - A) \geq 0$, then

$$0 \leq \frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} - \left[\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right]^2 \tag{4.56}$$
$$\leq \frac{1}{4}(M-m)^2 - \left|\frac{m+M}{2} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2 \leq \frac{1}{4}(M-m)^2.$$

(ii) If $m1_H \leq A \leq M1_H$, then (4.56) also holds.

Finally, we have the following Grüss type inequality as well:

Proposition 4.24 (Dragomir, 2014, [38]). *Let $P \in \mathcal{B}_1^+(H)$, A, $B \in \mathcal{B}(H)$ and λ, Γ, δ, $\Delta \in \mathbb{C}$.*

(i) If A has the trace P-(λ, Γ)-property and B has the trace P-(δ, Δ)-property, then

$$\left|\frac{\operatorname{tr}(PB^*A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB^*)}{\operatorname{tr}(P)}\right| \tag{4.57}$$
$$\leq \left[\frac{1}{4}|\Gamma-\gamma||\Delta-\delta|\right.$$
$$- \frac{1}{\operatorname{tr}(P)}\left[\operatorname{Re}\left(\operatorname{tr}\left[P(A^*-\overline{\gamma}1_H)(\Gamma 1_H - A)\right]\right)\right]^{1/2}$$
$$\left.\times \frac{1}{\operatorname{tr}(P)}\left[\operatorname{Re}\left(\operatorname{tr}\left[P\left(B^*-\overline{\delta}1_H\right)(\Delta 1_H - B)\right]\right)\right]^{1/2}\right] \leq \frac{1}{4}|\Gamma-\gamma||\Delta-\delta|$$

and

$$\left|\frac{\operatorname{tr}(PB^{*}A)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB^{*})}{\operatorname{tr}(P)}\right| \le \frac{1}{4}|\Gamma-\gamma||\Delta-\delta|-\left|\frac{\Gamma+\gamma}{2}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|\left|\frac{\delta+\Delta}{2}-\frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}\right| \le \frac{1}{4}|\Gamma-\gamma||\Delta-\delta|. \tag{4.58}$$

(ii) If the transforms $\mathcal{C}_{\lambda,\Gamma}(A)$ and $\mathcal{C}_{\delta,\Delta}(B)$ are accretive then (4.57) and (4.58) also hold.

The case of selfadjoint operators is as follows:

Corollary 4.25. *Let P, A, B be selfadjoint operators with $P\in\mathcal{B}_1^{+}(H)$, A, $B\in\mathcal{B}(H)$ and m, M, n, $N\in\mathbb{R}$ with $M>m$ and $N>n$.*

(i) If $(A-m1_H)(M1_H-A)\ge 0$ and $(B-n1_H)(N1_H-B)\ge 0$ then

$$\left|\frac{\operatorname{tr}(PBA)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}\right| \le \left[\frac{1}{4}(M-m)(N-n) - \frac{1}{\operatorname{tr}(P)}\left[\operatorname{Re}\left(\operatorname{tr}(A-m1_H)(M1_H-A)\right)\right]^{1/2} \times\frac{1}{\operatorname{tr}(P)}\left[\operatorname{Re}\left(\operatorname{tr}\left[P(B-n1_H)(N1_H-B)\right]\right)\right]^{1/2}\right] \le \frac{1}{4}(M-m)(N-n) \tag{4.59}$$

and

$$\left|\frac{\operatorname{tr}(PBA)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}\right| \le \frac{1}{4}(M-m)(N-n)-\left|\frac{m+M}{2}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|\left|\frac{n+N}{2}-\frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}\right| \le \frac{1}{4}(M-m)(N-n). \tag{4.60}$$

(ii) If $m1_H\le A\le M1_H$ and $n1_H\le B\le N1_H$ then (4.59) and (4.60) also hold.

5. Cassels Type Inequalities

5.1. General Inequalities.

We have the following result:

Theorem 5.1 (Dragomir, 2014, [39])**.** *Let, either $P\in\mathcal{B}_{+}(H)$, A, $B\in\mathcal{B}_2(H)$ or $P\in\mathcal{B}_1^{+}(H)$, A, $B\in\mathcal{B}(H)$ and γ, $\Gamma\in\mathbb{C}$ with $\operatorname{Re}(\Gamma\overline{\gamma})=\operatorname{Re}(\Gamma)\operatorname{Re}(\gamma)+\operatorname{Im}(\Gamma)\operatorname{Im}(\gamma)>0$.*

(i) If (A,B) *satisfies the* P-(γ,Γ)*-trace property, then*

$$\begin{aligned}&\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right) \quad (5.1)\\ &\leq \frac{1}{4}\cdot\frac{\left[\operatorname{Re}(\gamma+\Gamma)\operatorname{Retr}(PB^*A)+\operatorname{Im}(\gamma+\Gamma)\operatorname{Imtr}(PB^*A)\right]^2}{\operatorname{Re}(\Gamma)\operatorname{Re}(\gamma)+\operatorname{Im}(\Gamma)\operatorname{Im}(\gamma)}\\ &\leq \frac{1}{4}\cdot\frac{|\gamma+\Gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})}\left|\operatorname{tr}(PB^*A)\right|^2.\end{aligned}$$

(ii) If the transform $\mathcal{C}_{\gamma,\Gamma}(A,B)$ *is accretive, then the inequality (5.1) also holds.*

Proof. (i) If (A,B) satisfies the P-(γ,Γ)-trace property, then, on utilizing the calculations above, we have

$$\begin{aligned}0 &\leq \frac{1}{4}|\Gamma-\gamma|^2\operatorname{tr}\left(P|B|^2\right)-\operatorname{tr}\left(P\left|A-\frac{\gamma+\Gamma}{2}B\right|^2\right)\\ &= -\operatorname{tr}\left(P|A|^2\right)-\operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right)\\ &\quad+\operatorname{Re}\left[\overline{\gamma}\operatorname{tr}(PB^*A)\right]+\operatorname{Re}\left[\Gamma\overline{\operatorname{tr}(PB^*A)}\right]\\ &= -\operatorname{tr}\left(P|A|^2\right)-\operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right)\\ &\quad+\operatorname{Re}\left[\overline{\gamma}\operatorname{tr}(PB^*A)\right]+\operatorname{Re}\left[\overline{\Gamma\overline{\operatorname{tr}(PB^*A)}}\right]\\ &= -\operatorname{tr}\left(P|A|^2\right)-\operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right)\\ &\quad+\operatorname{Re}\left[\overline{\gamma}\operatorname{tr}(PB^*A)\right]+\operatorname{Re}\left[\overline{\Gamma}\operatorname{tr}(PB^*A)\right]\\ &= -\operatorname{tr}\left(P|A|^2\right)-\operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right)+\operatorname{Re}\left[\left(\overline{\gamma}+\overline{\Gamma}\right)\operatorname{tr}(PB^*A)\right],\end{aligned}$$

which implies that

$$\begin{aligned}&\operatorname{tr}\left(P|A|^2\right)+\operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right) \quad (5.2)\\ &\leq \operatorname{Re}\left[\left(\overline{\gamma}+\overline{\Gamma}\right)\operatorname{tr}(PB^*A)\right]\\ &= \operatorname{Re}(\gamma+\Gamma)\operatorname{Retr}(PB^*A)+\operatorname{Im}(\gamma+\Gamma)\operatorname{Imtr}(PB^*A).\end{aligned}$$

Making use of the elementary inequality

$$2\sqrt{pq}\leq p+q,\ \ p,q\geq 0,$$

we also have

$$2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right)}\leq\operatorname{tr}\left(P|A|^2\right)+\operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right). \quad (5.3)$$

Utilizing (5.2) and (5.3) we get

$$\begin{aligned}&\sqrt{\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right)} \quad (5.4)\\ &\leq \frac{\operatorname{Re}(\gamma+\Gamma)\operatorname{Retr}(PB^*A)+\operatorname{Im}(\gamma+\Gamma)\operatorname{Imtr}(PB^*A)}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\end{aligned}$$

that is equivalent with the first inequality in (5.1).

The second inequality in (5.1) is obvious by Schwarz inequality

$$(ab+cd)^2\leq\left(a^2+c^2\right)\left(b^2+d^2\right),\ \ a,b,c,d\in\mathbb{R}.$$

The (ii) is obvious from (i). □

Remark 5.2. We observe that the inequality between the first and last term in (5.1) is equivalent to

$$0 \le \operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right) - |\operatorname{tr}(PB^*A)|^2 \le \frac{1}{4}\cdot\frac{|\gamma-\Gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})}|\operatorname{tr}(PB^*A)|^2. \quad (5.5)$$

Corollary 5.3. *Let, either* $P \in \mathcal{B}_+(H)$, $A \in \mathcal{B}_2(H)$ *or* $P \in \mathcal{B}_1^+(H)$, $A \in \mathcal{B}(H)$ *and* γ, $\Gamma \in \mathbb{C}$ *with* $\operatorname{Re}(\Gamma\overline{\gamma}) = \operatorname{Re}(\Gamma)\operatorname{Re}(\gamma) + \operatorname{Im}(\Gamma)\operatorname{Im}(\gamma) > 0$.

(i) If A satisfies the P-(γ,Γ)-trace property, namely

$$\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^* - \overline{\gamma}1_H\right)(\Gamma 1_H - A)\right]\right) \ge 0 \quad (5.6)$$

or, equivalently

$$\operatorname{tr}\left(P\left|A - \frac{\gamma+\Gamma}{2}1_H\right|^2\right) \le \frac{1}{4}|\Gamma-\gamma|^2\operatorname{tr}(P), \quad (5.7)$$

then

$$\begin{aligned}\frac{\operatorname{tr}\left(P|A|^2\right)}{\operatorname{tr}(P)} & \quad (5.8)\\ &\le \frac{1}{4}\cdot\frac{\left[\operatorname{Re}(\gamma+\Gamma)\frac{\operatorname{Re}\operatorname{tr}(PA)}{\operatorname{tr}(P)} + \operatorname{Im}(\gamma+\Gamma)\frac{\operatorname{Im}\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right]^2}{\operatorname{Re}(\Gamma)\operatorname{Re}(\gamma)+\operatorname{Im}(\Gamma)\operatorname{Im}(\gamma)}\\ &\le \frac{1}{4}\cdot\frac{|\gamma+\Gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})}\left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2.\end{aligned}$$

(ii) If the transform $\mathcal{C}_{\gamma,\Gamma}(A)$ is accretive, then the inequality (5.1) also holds.

(iii) We have

$$0 \le \frac{\operatorname{tr}\left(P|A|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2 \le \frac{1}{4}\cdot\frac{|\gamma-\Gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})}\left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|^2. \quad (5.9)$$

Remark 5.4. The case of selfadjoint operators is as follows.

Let A, B be selfadjoint operators and either $P \in \mathcal{B}_+(H)$, A, $B \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, A, $B \in \mathcal{B}(H)$ and m, $M \in \mathbb{R}$ with $mM > 0$.

(i) If (A, B) satisfies the P-(m, M)-trace property, then

$$\operatorname{tr}\left(PA^2\right)\operatorname{tr}\left(PB^2\right) \le \frac{(m+M)^2}{4mM}\left[\operatorname{tr}(PBA)\right]^2 \quad (5.10)$$

or, equivalently

$$0 \le \operatorname{tr}\left(PA^2\right)\operatorname{tr}\left(PB^2\right) - \left[\operatorname{tr}(PBA)\right]^2 \le \frac{(m-M)^2}{4mM}\left[\operatorname{tr}(PBA)\right]^2. \quad (5.11)$$

(ii) If the transform $\mathcal{C}_{m,M}(A, B)$ is accretive, then the inequality (5.10) also holds.

(iii) If $(A - mB)(MB - A) \ge 0$, then (5.10) is valid.

5.2. **Trace Inequalities of Grüss Type.** We have the following Grüss type inequality:

Theorem 5.5 (Dragomir, 2014, [39])**.** *Let, either* $P \in \mathcal{B}_+(H)$, $A, B, C \in \mathcal{B}_2(H)$ *or* $P \in \mathcal{B}_1^+(H)$, $A, B, C \in \mathcal{B}(H)$ *with* $P|A|^2$, $P|B|^2$, $P|C|^2 \neq 0$ *and* $\lambda, \Gamma, \delta, \Delta \in \mathbb{C}$ *with* $\operatorname{Re}(\Gamma\overline{\gamma})$, $\operatorname{Re}(\Delta\overline{\delta}) > 0$. *If* (A, C) *has the trace* P*-*(λ, Γ)*-property and* (B, C) *has the trace* P*-*(δ, Δ)*-property, then*

$$\left| \frac{\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)} - 1 \right| \leq \frac{1}{4} \cdot \frac{|\gamma - \Gamma||\delta - \Delta|}{\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})\, Re\left(\Delta\overline{\delta}\right)}}. \tag{5.12}$$

Proof. We prove in the case that $P \in \mathcal{B}_+(H)$ and $A, B, C \in \mathcal{B}_2(H)$.

Making use of the Schwarz inequality for the nonnegative hermitian form $\langle\cdot,\cdot\rangle_{2,P}$ we have

$$\left|\langle A, B\rangle_{2,P}\right|^2 \leq \langle A, A\rangle_{2,P}\langle B, B\rangle_{2,P}$$

for any $A, B \in \mathcal{B}_2(H)$.

Let $C \in \mathcal{B}_2(H)$, $C \neq 0$. Define the mapping $[\cdot,\cdot]_{2,P,C} : \mathcal{B}_2(H) \times \mathcal{B}_2(H) \to \mathbb{C}$ by

$$[A, B]_{2,P,C} := \langle A, B\rangle_{2,P}\|C\|_{2,P}^2 - \langle A, C\rangle_{2,P}\langle C, B\rangle_{2,P}.$$

Observe that $[\cdot,\cdot]_{2,P,C}$ is a nonnegative Hermitian form on $\mathcal{B}_2(H)$ and by Schwarz inequality we also have

$$\begin{aligned}&\left|\langle A, B\rangle_{2,P}\|C\|_{2,P}^2 - \langle A, C\rangle_{2,P}\langle C, B\rangle_{2,P}\right|^2 \\ &\quad\leq \left[\|A\|_{2,P}^2\|C\|_{2,P}^2 - \left|\langle A, C\rangle_{2,P}\right|^2\right]\left[\|B\|_{2,P}^2\|C\|_{2,P}^2 - \left|\langle B, C\rangle_{2,P}\right|^2\right]\end{aligned}$$

for any $A, B \in \mathcal{B}_2(H)$, namely

$$\begin{aligned}&\left|\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right) - \operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)\right|^2 \\ &\quad\leq \left[\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PC^*A)|^2\right] \\ &\qquad\times\left[\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PB^*C)|^2\right],\end{aligned} \tag{5.13}$$

where for the last term we used the equality $\left|\langle B, C\rangle_{2,P}\right|^2 = \left|\langle C, B\rangle_{2,P}\right|^2$.

Since (A, C) has the trace P-(λ, Γ)-property and (B, C) has the trace P-(δ, Δ) -property, then by (5.5) we have

$$0 \leq \operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PC^*A)|^2 \leq \frac{1}{4} \cdot \frac{|\gamma - \Gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})}|\operatorname{tr}(PC^*A)|^2 \tag{5.14}$$

and

$$0 \leq \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PB^*C)|^2 \leq \frac{1}{4} \cdot \frac{|\delta - \Delta|^2}{\operatorname{Re}\left(\Delta\overline{\delta}\right)}|\operatorname{tr}(PB^*C)|^2. \tag{5.15}$$

If we multiply the inequalities (5.14) and (5.15) we get

$$\left[\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right)-|\operatorname{tr}(PC^*A)|^2\right] \times\left[\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right)-|\operatorname{tr}(PB^*C)|^2\right] \leq \frac{1}{16}\cdot\frac{|\gamma-\Gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})}\frac{|\delta-\Delta|^2}{\operatorname{Re}\left(\Delta\overline{\delta}\right)}|\operatorname{tr}(PC^*A)|^2|\operatorname{tr}(PB^*C)|^2. \tag{5.16}$$

If we use (5.13) and (5.16) we get

$$\left|\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right)-\operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)\right|^2 \leq \frac{1}{16}\cdot\frac{|\gamma-\Gamma|^2}{\operatorname{Re}(\Gamma\overline{\gamma})}\frac{|\delta-\Delta|^2}{\operatorname{Re}\left(\Delta\overline{\delta}\right)}|\operatorname{tr}(PC^*A)|^2|\operatorname{tr}(PB^*C)|^2. \tag{5.17}$$

Since P, A, B, $C\neq 0$ then by (5.14) and (5.15) we get $\operatorname{tr}(PC^*A)\neq 0$ and $\operatorname{tr}(PB^*C)\neq 0$.
Now, if we take the square root in (5.17) and divide by $|\operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)|$ we obtain the desired result (5.12). □

Corollary 5.6. *Let, either* $P\in\mathcal{B}_+(H)$, A, $B\in\mathcal{B}_2$ *or* $P\in\mathcal{B}_1^+(H)$, A, $B\in\mathcal{B}(H)$ *with* $P|A|^2$, $P|B|^2\neq 0$ *and* λ, Γ, δ, $\Delta\in\mathbb{C}$ *with* $\operatorname{Re}(\Gamma\overline{\gamma})$, $\operatorname{Re}\left(\Delta\overline{\delta}\right)>0$. *If* A *has the trace* P*-*(λ,Γ)*-property and* B *has the trace* P*-*(δ,Δ)*-property, then*

$$\left|\frac{\operatorname{tr}(PB^*A)\operatorname{tr}(P)}{\operatorname{tr}(PA)\operatorname{tr}(PB^*)}-1\right|\leq\frac{1}{4}\cdot\frac{|\gamma-\Gamma||\delta-\Delta|}{\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})\operatorname{Re}\left(\Delta\overline{\delta}\right)}}. \tag{5.18}$$

The case of selfadjoint operators is useful for applications.

Remark 5.7. Assume that A, B, C are selfadjoint operators. If, either $P\in\mathcal{B}_+(H)$, A, B, $C\in\mathcal{B}_2(H)$ or $P\in\mathcal{B}_1^+(H)$, A, B, $C\in\mathcal{B}(H)$ with PA^2, PB^2, $PC^2\neq 0$ and m, M, n, $N\in\mathbb{R}$ with mM, $nN>0$. If (A,C) has the trace P*-*(m,M)*-property and* (B,C) has the trace P*-*(n,N)*-property, then*

$$\left|\frac{\operatorname{tr}(PBA)\operatorname{tr}(PC^2)}{\operatorname{tr}(PCA)\operatorname{tr}(PBC)}-1\right|\leq\frac{1}{4}\cdot\frac{(M-m)(N-n)}{\sqrt{mnMN}}. \tag{5.19}$$

If A has the trace P*-*(k,K)*-property and* B has the trace P*-*(l,L)*-property, then*

$$\left|\frac{\operatorname{tr}(PBA)\operatorname{tr}(P)}{\operatorname{tr}(PA)\operatorname{tr}(PB)}-1\right|\leq\frac{1}{4}\cdot\frac{(K-k)(L-l)}{\sqrt{klKL}}, \tag{5.20}$$

where kK, $lL>0$.

We observe that, if $0<k1_H\leq A\leq K1_H$ and $0<l1_H\leq B\leq L1_H$, then by (5.21)

$$|\operatorname{tr}(PBA)\operatorname{tr}(P)-\operatorname{tr}(PA)\operatorname{tr}(PB)|\leq\frac{1}{4}\cdot\frac{(K-k)(L-l)}{\sqrt{klKL}}\operatorname{tr}(PA)\operatorname{tr}(PB) \tag{5.21}$$

or, equivalently

$$\left|\frac{\operatorname{tr}(PBA)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}\right|\leq\frac{1}{4}\cdot\frac{(K-k)(L-l)}{\sqrt{klKL}}\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}. \tag{5.22}$$

5.3. **Applications for Convex Functions.** In the paper [37] we obtained amongst other the following reverse of the Jensen trace inequality:

Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m,M]$ for some scalars m, M with $m<M$. If f is a continuously differentiable convex function on $[m,M]$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$, then we have

$$\begin{aligned} 0 &\leq \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)} - f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \quad (5.23) \\ &\leq \frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \cdot \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)} \\ &\leq \begin{cases} \frac{1}{2}[f'(M)-f'(m)] \frac{\operatorname{tr}\left(P\left|A-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)} \\ \\ \frac{1}{2}(M-m) \frac{\operatorname{tr}\left(P\left|f'(A)-\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)} \end{cases} \\ &\leq \begin{cases} \frac{1}{2}[f'(M)-f'(m)] \left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\ \\ \frac{1}{2}(M-m) \left[\frac{\operatorname{tr}\left(P[f'(A)]^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \end{cases} \\ &\leq \frac{1}{4}[f'(M)-f'(m)](M-m). \end{aligned}$$

Let $\mathcal{M}_n(\mathbb{C})$ be the space of all square matrices of order n with complex elements and $A \in \mathcal{M}_n(\mathbb{C})$ be a Hermitian matrix such that $Sp(A) \subseteq [m,M]$ for some scalars m, M with $m<M$. If f is a continuously differentiable convex function on $[m,M]$, then by taking $P = I_n$ in (5.23) we get

$$\begin{aligned} 0 &\leq \frac{\operatorname{tr}(f(A))}{n} - f\left(\frac{\operatorname{tr}(A)}{n}\right) \quad (5.24) \\ &\leq \frac{\operatorname{tr}(f'(A)A)}{n} - \frac{\operatorname{tr}(A)}{n} \cdot \frac{\operatorname{tr}(f'(A))}{n} \\ &\leq \begin{cases} \frac{1}{2}[f'(M)-f'(m)] \frac{\operatorname{tr}\left(\left|A-\frac{\operatorname{tr}(A)}{n}1_H\right|\right)}{n} \\ \\ \frac{1}{2}(M-m) \frac{\operatorname{tr}\left(\left|f'(A)-\frac{\operatorname{tr}(f'(A))}{n}1_H\right|\right)}{n} \end{cases} \\ &\leq \begin{cases} \frac{1}{2}[f'(M)-f'(m)] \left[\frac{\operatorname{tr}\left(A^2\right)}{n} - \left(\frac{\operatorname{tr}(A)}{n}\right)^2\right]^{1/2} \\ \\ \frac{1}{2}(M-m) \left[\frac{\operatorname{tr}\left([f'(A)]^2\right)}{n} - \left(\frac{\operatorname{tr}(f'(A))}{n}\right)^2\right]^{1/2} \end{cases} \\ &\leq \frac{1}{4}[f'(M)-f'(m)](M-m). \end{aligned}$$

The following reverse inequality also holds:

Proposition 5.8 (Dragomir, 2014, [39]). *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $0 < m < M$. If f is a continuously differentiable convex function on $[m, M]$ with $f'(m) > 0$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$, then*

$$\begin{aligned} 0 &\leq \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)} - f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \\ &\leq \frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \cdot \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)} \\ &\leq \frac{1}{4} \cdot \frac{(M-m)[f'(M) - f'(m)]}{\sqrt{mMf'(m)f'(M)}} \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}. \end{aligned} \tag{5.25}$$

The proof follows by the inequality (5.22) and the details are omitted.

Let $A \in \mathcal{M}_n(\mathbb{C})$ be a Hermitian matrix such that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuously differentiable convex function on $[m, M]$ with $f'(m) > 0$ then by taking $P = I_n$ in (5.25) we get

$$\begin{aligned} 0 &\leq \frac{\operatorname{tr}(f(A))}{n} - f\left(\frac{\operatorname{tr}(A)}{n}\right) \\ &\leq \frac{\operatorname{tr}(f'(A)A)}{n} - \frac{\operatorname{tr}(A)}{n} \cdot \frac{\operatorname{tr}(f'(A))}{n} \\ &\leq \frac{1}{4} \cdot \frac{(M-m)[f'(M) - f'(m)]}{\sqrt{mMf'(m)f'(M)}} \frac{\operatorname{tr}(A)}{n} \frac{\operatorname{tr}(f'(A))}{n}. \end{aligned} \tag{5.26}$$

We consider the power function $f : (0, \infty) \to (0, \infty)$, $f(t) = t^r$ with $t \in \mathbb{R} \setminus \{0\}$. For $r \in (-\infty, 0) \cup [1, \infty)$, f is convex while for $r \in (0, 1)$, f is concave.

Let $r \geq 1$ and A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $0 < m < M$. If $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, then

$$\begin{aligned} 0 &\leq \frac{\operatorname{tr}(PA^r)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^r \\ &\leq r\left[\frac{\operatorname{tr}(PA^r)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \cdot \frac{\operatorname{tr}(PA^{r-1})}{\operatorname{tr}(P)}\right] \\ &\leq \frac{1}{4} r \frac{(M-m)(M^{r-1} - m^{r-1})}{m^{r/2}M^{r/2}} \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PA^{r-1})}{\operatorname{tr}(P)}. \end{aligned} \tag{5.27}$$

If we take the first and last term in (5.27) we get the inequality:

$$\begin{aligned} 0 &\leq \frac{\operatorname{tr}(P)\operatorname{tr}(PA^r)}{\operatorname{tr}(PA)\operatorname{tr}(PA^{r-1})} - \frac{\operatorname{tr}(P)[\operatorname{tr}(PA)]^{r-1}}{\operatorname{tr}(PA^{p-1})[\operatorname{tr}(P)]^{r-1}} \\ &\leq \frac{1}{4} r \frac{(M-m)(M^{r-1} - m^{r-1})}{m^{r/2}M^{r/2}}. \end{aligned} \tag{5.28}$$

Consider the convex function $f : \mathbb{R} \to (0, \infty)$, $f(t) = \exp t$ and let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $0 < m < M$. If $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, then using (5.25) we have

$$0 \leq \frac{\operatorname{tr}(P \exp A)}{\operatorname{tr}(P)} - \exp\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \leq \frac{\operatorname{tr}(PA \exp A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \cdot \frac{\operatorname{tr}(P \exp A)}{\operatorname{tr}(P)} \leq \frac{1}{4} \cdot \frac{(M-m)(\exp M - \exp m)}{\sqrt{mM \exp(m+M)}} \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(P \exp A)}{\operatorname{tr}(P)}. \tag{5.29}$$

If we take the first and last term in (5.29) we get the inequality:

$$0 \leq \frac{\operatorname{tr}(P)}{\operatorname{tr}(PA)} - \frac{[\operatorname{tr}(P)]^2 \exp\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)}{\operatorname{tr}(PA)\operatorname{tr}(P \exp A)} \leq \frac{1}{4} \cdot \frac{(M-m)(\exp M - \exp m)}{\sqrt{mM \exp(m+M)}}. \tag{5.30}$$

6. Shisha–Mond Type Trace Inequalities

6.1. **General Results.** We have the following result:

Theorem 6.1 (Dragomir, 2014, [40])**.** *Let, either $P \in \mathcal{B}_+(H)$, A, $B \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, A, $B \in \mathcal{B}(H)$ and γ, $\Gamma \in \mathbb{C}$ with $\Gamma + \gamma \neq 0$.*

(i) If (A, B) satisfies the P-(γ, Γ)-trace property, then

$$\sqrt{\operatorname{tr}(P|A|^2)\operatorname{tr}(P|B|^2)} \leq \frac{\operatorname{Re}(\gamma+\Gamma)\operatorname{Retr}(PB^*A) + \operatorname{Im}(\gamma+\Gamma)\operatorname{Imtr}(PB^*A)}{|\Gamma+\gamma|} + \frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\operatorname{tr}(P|B|^2) \leq |\operatorname{tr}(PB^*A)| + \frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\operatorname{tr}(P|B|^2). \tag{6.1}$$

(ii) If the transform $\mathcal{C}_{\gamma,\Gamma}(A, B)$ is accretive, then the inequality (6.1) also holds.

Proof. (i) If (A, B) satisfies the P-(γ, Γ)-trace property, then

$$\operatorname{tr}\left(P\left|A - \frac{\gamma+\Gamma}{2}B\right|^2\right) \leq \frac{1}{4}|\Gamma-\gamma|^2 \operatorname{tr}(P|B|^2)$$

that is equivalent to

$$\operatorname{tr}(P|A|^2) - \operatorname{Re}\left[\left(\overline{\gamma}+\overline{\Gamma}\right)\operatorname{tr}(PB^*A)\right] + \frac{1}{4}|\Gamma+\gamma|^2\operatorname{tr}(P|B|^2) \leq \frac{1}{4}|\Gamma-\gamma|^2\operatorname{tr}(P|B|^2),$$

which implies that

$$\operatorname{tr}\left(P|A|^2\right)+\frac{1}{4}|\Gamma+\gamma|^2\operatorname{tr}\left(P|B|^2\right) \le \operatorname{Re}\left[\left(\overline{\gamma}+\overline{\Gamma}\right)\operatorname{tr}\left(PB^*A\right)\right]+\frac{1}{4}|\Gamma-\gamma|^2\operatorname{tr}\left(P|B|^2\right). \tag{6.2}$$

Making use of the elementary inequality

$$2\sqrt{pq}\le p+q,\ \ p,q\ge 0,$$

we also have

$$|\Gamma+\gamma|\left[\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\le \operatorname{tr}\left(P|A|^2\right)+\frac{1}{4}|\Gamma+\gamma|^2\operatorname{tr}\left(P|B|^2\right). \tag{6.3}$$

Utilizing (6.2) and (6.3) we get

$$|\Gamma+\gamma|\left[\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right)\right]^{1/2} \le \operatorname{Re}\left[\left(\overline{\gamma}+\overline{\Gamma}\right)\operatorname{tr}\left(PB^*A\right)\right]+\frac{1}{4}|\Gamma-\gamma|^2\operatorname{tr}\left(P|B|^2\right). \tag{6.4}$$

Dividing by $|\Gamma+\gamma|>0$ and observing that

$$\operatorname{Re}\left[\left(\overline{\gamma}+\overline{\Gamma}\right)\operatorname{tr}\left(PB^*A\right)\right]=\operatorname{Re}\left(\gamma+\Gamma\right)\operatorname{Retr}\left(PB^*A\right)+\operatorname{Im}\left(\gamma+\Gamma\right)\operatorname{Imtr}\left(PB^*A\right)$$

we get the first inequality in (6.1).

The second inequality in (6.1) is obvious by Schwarz inequality

$$(ab+cd)^2\le\left(a^2+c^2\right)\left(b^2+d^2\right),\ \ a,b,c,d\in\mathbb{R}.$$

The (ii) is obvious from (i). □

Remark 6.2. We observe that the inequality between the first and last term in (6.1) is equivalent to

$$0\le\sqrt{\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right)}-\left|\operatorname{tr}\left(PB^*A\right)\right|\le\frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\operatorname{tr}\left(P|B|^2\right). \tag{6.6}$$

Corollary 6.3. *Let, either $P\in\mathcal{B}_+(H)$, $A\in\mathcal{B}_2(H)$ or $P\in\mathcal{B}_1^+(H)$, $A\in\mathcal{B}(H)$ and $\gamma,\Gamma\in\mathbb{C}$ with $\gamma+\Gamma\neq 0$.*

(i) If A satisfies the P-(γ,Γ)-trace property, namely

$$\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}1_H\right)\left(\Gamma 1_H-A\right)\right]\right)\ge 0 \tag{6.7}$$

or, equivalently

$$\operatorname{tr}\left(P\left|A-\frac{\gamma+\Gamma}{2}1_H\right|^2\right)\le\frac{1}{4}|\Gamma-\gamma|^2\operatorname{tr}\left(P\right), \tag{6.8}$$

then

$$\sqrt{\frac{\operatorname{tr}\left(P|A|^2\right)}{\operatorname{tr}(P)}} \le \frac{\operatorname{Re}(\gamma+\Gamma)\frac{\operatorname{Retr}(PA)}{\operatorname{tr}(P)} + \operatorname{Im}(\gamma+\Gamma)\frac{\operatorname{Imtr}(PA)}{\operatorname{tr}(P)}}{|\Gamma+\gamma|} + \frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|} \le \left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right| + \frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}. \tag{6.9}$$

(ii) If the transform $\mathcal{C}_{\gamma,\Gamma}(A)$ is accretive, then the inequality (6.1) also holds.

(iii) We have

$$0 \le \sqrt{\frac{\operatorname{tr}\left(P|A|^2\right)}{\operatorname{tr}(P)}} - \left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right| \le \frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}. \tag{6.10}$$

Remark 6.4. The case of selfadjoint operators is as follows.

Let A, B be selfadjoint operators and either $P \in \mathcal{B}_+(H)$, A, $B \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, A, $B \in \mathcal{B}(H)$ and m, $M \in \mathbb{R}$ with $m+M \neq 0$.

(i) If (A, B) satisfies the P-(m, M)-trace property, then

$$\sqrt{\operatorname{tr}(PA^2)\operatorname{tr}(PB^2)} \le \operatorname{Retr}(PBA) + \frac{(M-m)^2}{4|M+m|}\operatorname{tr}\left(PB^2\right) \le |\operatorname{tr}(PBA)| + \frac{(M-m)^2}{4|M+m|}\operatorname{tr}\left(PB^2\right) \tag{6.11}$$

and

$$0 \le \sqrt{\operatorname{tr}(PA^2)\operatorname{tr}(PB^2)} - \operatorname{Retr}(PBA) \le \frac{(M-m)^2}{4|M+m|}\operatorname{tr}\left(PB^2\right).$$

(ii) If the transform $\mathcal{C}_{m,M}(A, B)$ is accretive, then the inequality (6.11) also holds.

(iii) If $(A - mB)(MB - A) \ge 0$, then (6.11) is valid.

Corollary 6.5. *Let A, B be selfadjoint operators and either $P \in \mathcal{B}_+(H)$, A, $B \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, A, $B \in \mathcal{B}(H)$ and m, $M \in \mathbb{R}$ with $m+M \neq 0$.*

(i) If (A, B) satisfies the P-(m, M)-trace property, then

$$\left(\sqrt{\operatorname{tr}(PA^2)} + \sqrt{\operatorname{tr}(PB^2)}\right)^2 - \operatorname{tr}\left(P(A+B)^2\right) \le \frac{(M-m)^2}{4|M+m|}\operatorname{tr}\left(PB^2\right). \tag{6.12}$$

Proof. Observe that

$$\left(\sqrt{\operatorname{tr}(PA^2)} + \sqrt{\operatorname{tr}(PB^2)}\right)^2 - \operatorname{tr}\left(P(A+B)^2\right) = 2\left(\sqrt{\operatorname{tr}(PA^2)\operatorname{tr}(PB^2)} - \operatorname{Retr}(PBA)\right).$$

Utilizing (6.11) we deduce (6.12). □

6.2. **Trace Inequalities of Grüss Type.** We have the following Grüss type inequality:

Theorem 6.6 (Dragomir, 2014, [40])**.** *Let, either $P \in \mathcal{B}_+(H)$, $A, B, C \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, $A, B, C \in \mathcal{B}(H)$ with $P|A|^2, P|B|^2, P|C|^2 \neq 0$ and $\lambda, \Gamma, \delta, \Delta \in \mathbb{C}$ with $\gamma + \Gamma \neq 0$, $\delta + \Delta \neq 0$. If (A, C) has the trace P-(λ, Γ)-property and (B, C) has the trace P-(δ, Δ)-property, then*

$$\left| \frac{\operatorname{tr}(PB^*A)}{\operatorname{tr}\left(P|C|^2\right)} - \frac{\operatorname{tr}(PC^*A)}{\operatorname{tr}\left(P|C|^2\right)} \frac{\operatorname{tr}(PB^*C)}{\operatorname{tr}\left(P|C|^2\right)} \right|^2 \tag{6.13}$$
$$\leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \frac{|\Delta - \delta|^2}{|\Delta + \delta|} \sqrt{\frac{\operatorname{tr}\left(P|A|^2\right) \operatorname{tr}\left(P|B|^2\right)}{\left[\operatorname{tr}\left(P|C|^2\right)\right]^2}}.$$

Proof. We prove in the case that $P \in \mathcal{B}_+(H)$ and $A, B, C \in \mathcal{B}_2(H)$.

Making use of the Schwarz inequality for the nonnegative hermitian form $\langle \cdot, \cdot \rangle_{2,P}$ we have

$$\left| \langle A, B \rangle_{2,P} \right|^2 \leq \langle A, A \rangle_{2,P} \langle B, B \rangle_{2,P}$$

for any $A, B \in \mathcal{B}_2(H)$.

Let $C \in \mathcal{B}_2(H)$, $C \neq 0$. Define the mapping $[\cdot, \cdot]_{2,P,C} : \mathcal{B}_2(H) \times \mathcal{B}_2(H) \to \mathbb{C}$ by

$$[A, B]_{2,P,C} := \langle A, B \rangle_{2,P} \|C\|_{2,P}^2 - \langle A, C \rangle_{2,P} \langle C, B \rangle_{2,P}.$$

Observe that $[\cdot, \cdot]_{2,P,C}$ is a nonnegative Hermitian form on $\mathcal{B}_2(H)$ and by Schwarz inequality we also have

$$\left| \langle A, B \rangle_{2,P} \|C\|_{2,P}^2 - \langle A, C \rangle_{2,P} \langle C, B \rangle_{2,P} \right|^2$$
$$\leq \left[\|A\|_{2,P}^2 \|C\|_{2,P}^2 - \left| \langle A, C \rangle_{2,P} \right|^2 \right] \left[\|B\|_{2,P}^2 \|C\|_{2,P}^2 - \left| \langle B, C \rangle_{2,P} \right|^2 \right]$$

for any $A, B \in \mathcal{B}_2(H)$, namely

$$\left| \operatorname{tr}(PB^*A) \operatorname{tr}\left(P|C|^2\right) - \operatorname{tr}(PC^*A) \operatorname{tr}(PB^*C) \right|^2 \tag{6.14}$$
$$\leq \left[\operatorname{tr}\left(P|A|^2\right) \operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PC^*A)|^2 \right]$$
$$\times \left[\operatorname{tr}\left(P|B|^2\right) \operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PB^*C)|^2 \right],$$

where for the last term we used the equality $\left| \langle B, C \rangle_{2,P} \right|^2 = \left| \langle C, B \rangle_{2,P} \right|^2$.

Since (A, C) has the trace P-(λ, Γ)-property and (B, C) has the trace P-(δ, Δ)-property, then by (6.6) we have

$$0 \leq \sqrt{\operatorname{tr}\left(P|A|^2\right) \operatorname{tr}\left(P|C|^2\right)} - |\operatorname{tr}(PC^*A)| \leq \frac{1}{4} \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \operatorname{tr}\left(P|C|^2\right)$$

and

$$0 \leq \sqrt{\operatorname{tr}\left(P|B|^2\right) \operatorname{tr}\left(P|C|^2\right)} - |\operatorname{tr}(PC^*B)| \leq \frac{1}{4} \frac{|\Delta - \delta|^2}{|\Delta + \delta|} \operatorname{tr}\left(P|C|^2\right),$$

which imply

$$0\le \operatorname{tr}\left(P\left|A\right|^{2}\right)\operatorname{tr}\left(P\left|C\right|^{2}\right)-\left|\operatorname{tr}\left(PC^{*}A\right)\right|^{2} \tag{6.15}$$
$$\le \frac{1}{4}\frac{\left|\Gamma-\gamma\right|^{2}}{\left|\Gamma+\gamma\right|}\operatorname{tr}\left(P\left|C\right|^{2}\right)\left(\sqrt{\operatorname{tr}\left(P\left|A\right|^{2}\right)\operatorname{tr}\left(P\left|C\right|^{2}\right)}+\left|\operatorname{tr}\left(PC^{*}A\right)\right|\right)$$
$$\le \frac{1}{2}\frac{\left|\Gamma-\gamma\right|^{2}}{\left|\Gamma+\gamma\right|}\operatorname{tr}\left(P\left|C\right|^{2}\right)\sqrt{\operatorname{tr}\left(P\left|A\right|^{2}\right)\operatorname{tr}\left(P\left|C\right|^{2}\right)}$$

and

$$0\le \operatorname{tr}\left(P\left|B\right|^{2}\right)\operatorname{tr}\left(P\left|C\right|^{2}\right)-\left|\operatorname{tr}\left(PB^{*}C\right)\right|^{2} \tag{6.16}$$
$$\le \frac{1}{4}\frac{\left|\Delta-\delta\right|^{2}}{\left|\Delta+\delta\right|}\operatorname{tr}\left(P\left|C\right|^{2}\right)\left(\sqrt{\operatorname{tr}\left(P\left|B\right|^{2}\right)\operatorname{tr}\left(P\left|C\right|^{2}\right)}+\left|\operatorname{tr}\left(PC^{*}B\right)\right|\right)$$
$$\le \frac{1}{2}\frac{\left|\Delta-\delta\right|^{2}}{\left|\Delta+\delta\right|}\operatorname{tr}\left(P\left|C\right|^{2}\right)\sqrt{\operatorname{tr}\left(P\left|B\right|^{2}\right)\operatorname{tr}\left(P\left|C\right|^{2}\right)}.$$

If we multiply the inequalities (6.15) and (6.16) we get

$$\left[\operatorname{tr}\left(P\left|A\right|^{2}\right)\operatorname{tr}\left(P\left|C\right|^{2}\right)-\left|\operatorname{tr}\left(PC^{*}A\right)\right|^{2}\right] \tag{6.17}$$
$$\times\left[\operatorname{tr}\left(P\left|B\right|^{2}\right)\operatorname{tr}\left(P\left|C\right|^{2}\right)-\left|\operatorname{tr}\left(PB^{*}C\right)\right|^{2}\right]$$
$$\le \frac{1}{4}\cdot\frac{\left|\Gamma-\gamma\right|^{2}}{\left|\Gamma+\gamma\right|}\frac{\left|\Delta-\delta\right|^{2}}{\left|\Delta+\delta\right|}\operatorname{tr}\left(P\left|C\right|^{2}\right)\sqrt{\operatorname{tr}\left(P\left|A\right|^{2}\right)\operatorname{tr}\left(P\left|C\right|^{2}\right)}$$
$$\times\operatorname{tr}\left(P\left|C\right|^{2}\right)\sqrt{\operatorname{tr}\left(P\left|B\right|^{2}\right)\operatorname{tr}\left(P\left|C\right|^{2}\right)}.$$

If we use (6.14) and (6.17) we get

$$\left|\operatorname{tr}\left(PB^{*}A\right)\operatorname{tr}\left(P\left|C\right|^{2}\right)-\operatorname{tr}\left(PC^{*}A\right)\operatorname{tr}\left(PB^{*}C\right)\right|^{2} \tag{6.18}$$
$$\le \frac{1}{4}\cdot\frac{\left|\Gamma-\gamma\right|^{2}}{\left|\Gamma+\gamma\right|}\frac{\left|\Delta-\delta\right|^{2}}{\left|\Delta+\delta\right|}\operatorname{tr}\left(P\left|C\right|^{2}\right)\sqrt{\operatorname{tr}\left(P\left|A\right|^{2}\right)\operatorname{tr}\left(P\left|C\right|^{2}\right)}$$
$$\times\operatorname{tr}\left(P\left|C\right|^{2}\right)\sqrt{\operatorname{tr}\left(P\left|B\right|^{2}\right)\operatorname{tr}\left(P\left|C\right|^{2}\right)}.$$

Since $P\left|C\right|^{2}\neq 0$ then by (6.18) we get the desired result (6.13). □

Corollary 6.7. *Let, either* $P\in\mathcal{B}_{+}(H)$, A, $B\in\mathcal{B}_{2}$ *or* $P\in\mathcal{B}_{1}^{+}(H)$, A, $B\in\mathcal{B}(H)$ *with* $P\left|A\right|^{2}$, $P\left|B\right|^{2}\neq 0$ *and* λ, Γ, δ, $\Delta\in\mathbb{C}$ *with* $\gamma+\Gamma\neq 0$, $\delta+\Delta\neq 0$. *If* A *has the trace* P-(λ,Γ)*-property and* B *has the trace* P-(δ,Δ)*-property, then*

$$\left|\frac{\operatorname{tr}\left(PB^{*}A\right)}{\operatorname{tr}\left(P\right)}-\frac{\operatorname{tr}\left(PA\right)}{\operatorname{tr}\left(P\right)}\frac{\operatorname{tr}\left(PB^{*}\right)}{\operatorname{tr}\left(P\right)}\right|^{2} \tag{6.19}$$
$$\le \frac{1}{4}\cdot\frac{\left|\Gamma-\gamma\right|^{2}}{\left|\Gamma+\gamma\right|}\frac{\left|\Delta-\delta\right|^{2}}{\left|\Delta+\delta\right|}\sqrt{\frac{\operatorname{tr}\left(P\left|A\right|^{2}\right)\operatorname{tr}\left(P\left|B\right|^{2}\right)}{\left[\operatorname{tr}\left(P\right)\right]^{2}}}.$$

The case of selfadjoint operators is useful for applications.

Remark 6.8. Assume that A, B, C are selfadjoint operators. If, either $P\in\mathcal{B}_{+}(H)$, A, B, $C\in\mathcal{B}_{2}(H)$ or $P\in\mathcal{B}_{1}^{+}(H)$, A, B, $C\in\mathcal{B}(H)$ with PA^{2}, PB^{2},

$PC^2 \neq 0$ and $m,\ M,\ n,\ N \in \mathbb{R}$ with $m+M,\ n+N \neq 0$. If (A,C) has the trace P-(m,M)*-property and* (B,C) has the trace P-(n,N)*-property, then*

$$\left|\frac{\operatorname{tr}(PBA)}{\operatorname{tr}(PC^2)}-\frac{\operatorname{tr}(PCA)}{\operatorname{tr}(PC^2)}\frac{\operatorname{tr}(PBC)}{\operatorname{tr}(PC^2)}\right|^2 \leq \frac{1}{4}\cdot\frac{(M-m)^2}{|M+m|}\frac{(N-n)^2}{|N+n|}\sqrt{\frac{\operatorname{tr}(PA^2)\operatorname{tr}(PB^2)}{[\operatorname{tr}(PC^2)]^2}}. \tag{6.20}$$

If A has the trace P-(k,K)*-property and* B has the trace P-(l,L)*-property, then*

$$\left|\frac{\operatorname{tr}(PBA)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}\right|^2 \leq \frac{1}{4}\cdot\frac{(K-k)^2}{|K+k|}\frac{(L-l)^2}{|L+l|}\sqrt{\frac{\operatorname{tr}(PA^2)\operatorname{tr}(PB^2)}{[\operatorname{tr}(P)]^2}}, \tag{6.21}$$

where $k+K,\ l+L \neq 0$.

6.3. **Applications for Convex Functions.** In the paper [37] we obtained amongst other the following reverse of the Jensen trace inequality:

Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m,M]$ for some scalars $m,\ M$ with $m<M$. If f is a continuously differentiable convex function on $[m,M]$ and $P \in \mathcal{B}_1(H)\setminus\{0\},\ P \geq 0$, then we have

$$\begin{aligned}
0 &\leq \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)}-f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \\
&\leq \frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\cdot\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)} \\
&\leq \begin{cases} \frac{1}{2}\left[f'(M)-f'(m)\right]\frac{\operatorname{tr}\left(P\left|A-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)} \\ \\ \frac{1}{2}(M-m)\frac{\operatorname{tr}\left(P\left|f'(A)-\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)} \end{cases} \\
&\leq \begin{cases} \frac{1}{2}\left[f'(M)-f'(m)\right]\left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\ \\ \frac{1}{2}(M-m)\left[\frac{\operatorname{tr}\left(P[f'(A)]^2\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \end{cases} \\
&\leq \frac{1}{4}\left[f'(M)-f'(m)\right](M-m).
\end{aligned} \tag{6.22}$$

Let $\mathcal{M}_n(\mathbb{C})$ be the space of all square matrices of order n with complex elements and $A \in \mathcal{M}_n(\mathbb{C})$ be a Hermitian matrix such that $Sp(A) \subseteq [m,M]$ for some scalars $m,\ M$ with $m<M$. If f is a continuously differentiable convex function

on $[m,M]$, then by taking $P=I_n$ in (6.22) we get

$$0\leq\frac{\operatorname{tr}(f(A))}{n}-f\left(\frac{\operatorname{tr}(A)}{n}\right)\tag{6.23}$$
$$\leq\frac{\operatorname{tr}(f'(A)A)}{n}-\frac{\operatorname{tr}(A)}{n}\cdot\frac{\operatorname{tr}(f'(A))}{n}$$
$$\leq\begin{cases}\frac{1}{2}[f'(M)-f'(m)]\frac{\operatorname{tr}\left(\left|A-\frac{\operatorname{tr}(A)}{n}1_H\right|\right)}{n}\\ \\ \frac{1}{2}(M-m)\frac{\operatorname{tr}\left(\left|f'(A)-\frac{\operatorname{tr}(f'(A))}{n}1_H\right|\right)}{n}\end{cases}$$
$$\leq\begin{cases}\frac{1}{2}[f'(M)-f'(m)]\left[\frac{\operatorname{tr}(A^2)}{n}-\left(\frac{\operatorname{tr}(A)}{n}\right)^2\right]^{1/2}\\ \\ \frac{1}{2}(M-m)\left[\frac{\operatorname{tr}([f'(A)]^2)}{n}-\left(\frac{\operatorname{tr}(f'(A))}{n}\right)^2\right]^{1/2}\end{cases}$$
$$\leq\frac{1}{4}[f'(M)-f'(m)](M-m).$$

The following reverse inequality also holds:

Proposition 6.9 (Dragomir, 2014, [40])**.** *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A)\subseteq[m,M]$ for some scalars m, M with $m+M\neq0$. If f is a continuously differentiable convex function on $[m,M]$ with $f'(m)+f'(M)\neq0$ and $P\in\mathcal{B}_1(H)\setminus\{0\}$, $P\geq0$, then*

$$0\leq\frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)}-f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\tag{6.24}$$
$$\leq\frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\cdot\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}$$
$$\leq\frac{1}{2}\cdot\frac{|M-m||f'(M)-f'(m)|}{\sqrt{|m+M|}\sqrt{|f'(m)+f'(M)|}}\sqrt[4]{\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(P[f'(A)]^2\right)}{\operatorname{tr}(P)}}.$$

The proof follows by the inequality (6.21) and the details are omitted.

Let $A\in\mathcal{M}_n(\mathbb{C})$ be a Hermitian matrix such that $Sp(A)\subseteq[m,M]$ for some scalars m, M with $m+M\neq0$. If f is a continuously differentiable convex function on $[m,M]$ with $f'(m)+f'(M)\neq0$ then by taking $P=I_n$ in (6.24) we get

$$0\leq\frac{\operatorname{tr}(f(A))}{n}-f\left(\frac{\operatorname{tr}(A)}{n}\right)\tag{6.25}$$
$$\leq\frac{\operatorname{tr}(f'(A)A)}{n}-\frac{\operatorname{tr}(A)}{n}\cdot\frac{\operatorname{tr}(f'(A))}{n}$$
$$\leq\frac{1}{2}\cdot\frac{|M-m||f'(M)-f'(m)|}{\sqrt{|m+M|}\sqrt{|f'(m)+f'(M)|}}\sqrt[4]{\frac{\operatorname{tr}(A^2)}{n}\frac{\operatorname{tr}\left([f'(A)]^2\right)}{n}}.$$

We consider the power function $f:(0,\infty)\to(0,\infty)$, $f(t)=t^r$ with $t\in\mathbb{R}\setminus\{0\}$. For $r\in(-\infty,0)\cup[1,\infty)$, f is convex while for $r\in(0,1)$, f is concave.

Let $r\geq 1$ and A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A)\subseteq[m,M]$ for some scalars m, M with $0<m<M$. If $P\in\mathcal{B}_1^+(H)\setminus\{0\}$, then

$$\begin{aligned} 0 &\leq \frac{\operatorname{tr}(PA^r)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^r \qquad (6.26)\\ &\leq r\left[\frac{\operatorname{tr}(PA^r)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\cdot\frac{\operatorname{tr}(PA^{r-1})}{\operatorname{tr}(P)}\right]\\ &\leq \frac{1}{2}r\frac{(M-m)(M^{r-1}-m^{r-1})}{(m+M)^{1/2}(m^{r-1}+M^{r-1})^{1/2}}\sqrt[4]{\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PA^{2(p-1)})}{\operatorname{tr}(P)}}. \end{aligned}$$

Consider the convex function $f:\mathbb{R}\to(0,\infty)$, $f(t)=\exp t$ and let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A)\subseteq[m,M]$ for some scalars m, M with $m<M$. If $P\in\mathcal{B}_1^+(H)\setminus\{0\}$, then using (6.24) we have

$$\begin{aligned} 0 &\leq \frac{\operatorname{tr}(P\exp A)}{\operatorname{tr}(P)}-\exp\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \qquad (6.27)\\ &\leq \frac{\operatorname{tr}(PA\exp A)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\cdot\frac{\operatorname{tr}(P\exp A)}{\operatorname{tr}(P)}\\ &\leq \frac{1}{2}\frac{|M-m|(\exp(M)-\exp(m))}{\sqrt{|m+M|}\sqrt{\exp m+\exp M}}\sqrt[4]{\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\exp(2A))}{\operatorname{tr}(P)}}. \end{aligned}$$

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