

POSITIVE DEFINITE KERNELS AND BOUNDARY SPACES

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ABSTRACT. We consider a kernel based harmonic analysis of “boundary,” and boundary representations. Our setting is general: certain classes of positive definite kernels. Our theorems extend (and are motivated by) results and notions from classical harmonic analysis on the disk. Our positive definite kernels include those defined on infinite discrete sets, for example sets of vertices in electrical networks, or discrete sets which arise from sampling operations performed on positive definite kernels in a continuous setting.

Below we give a summary of main conclusions in the paper: Starting with a given positive definite kernel K we make precise generalized boundaries for K . They are measure theoretic “boundaries.” Using the theory of Gaussian processes, we show that there is always such a generalized boundary for any positive definite kernel.

1. INTRODUCTION

Our purpose is to make precise a variety of notions of “boundary” and boundary representation for general classes of positive definite kernels. And to prove theorems which allow us to carry over results and notions from classical harmonic analysis on the disk to this wider context (see [10, 11]). We stress that our positive definite kernels include those defined on infinite discrete sets, for example sets of vertices in electrical networks, or discrete sets which arise from sampling

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operations performed on positive definite kernels in a continuous setting, and with the sampling then referring to suitable discrete subsets. See, e.g., [8, 12, 16].

Below we give a summary of main conclusions in the paper: Starting with a given positive definite kernel K on $S \times S$, we introduce generalized boundaries for the set S that carries K . It is a measure theoretic “boundary” in the form of a probability space, but it is not unique. The set of measure boundaries will be denoted $\mathcal{M}(K)$. We show that there is always such a generalized boundary probability space associated to any positive definite kernel. For example, as an element in $\mathcal{M}(K)$, we can take a “measure” boundary to be the Gaussian process having K as its covariance kernel. This exists by Kolmogorov’s consistency theorem.

Definition 1.1. By a *probability space*, we mean a triple (B, \mathcal{F}, μ) where:

- B is a set,
- \mathcal{F} is a σ -algebra of subsets of B , and
- μ is a probability measure defined on \mathcal{F} , i.e., $\mu(\emptyset) = 0$, $\mu(B) = 1$, $\mu(F) \geq 0 \forall F \in \mathcal{F}$, and if $\{F_i\}_{i \in \mathbb{N}} \subset \mathcal{F}$, $F_i \cap F_j = \emptyset$, $i \neq j$ in \mathbb{N} , then $\mu(\cup_i F_i) = \sum_i \mu(F_i)$.

Definition 1.2. Let S be any set. A function $K : S \times S \rightarrow \mathbb{C}$ is *positive definite* iff (Def.)

$$\sum_i \sum_j \bar{c}_i c_j K(s_i, s_j) \geq 0, \quad (1.1)$$

for all $\{s_i\}_{i=1}^n \subset S$, and all $(c_i)_{i=1}^n \in \mathbb{C}^n$.

Definition 1.3. Fix a set S , and let $K : S \times S \rightarrow \mathbb{C}$ be a positive definite kernel.

- (1) For all $x \in S$, set

$$K_x := K(\cdot, x) : S \longrightarrow \mathbb{C} \quad (1.2)$$

as a function on S .

- (2) Let $\mathcal{H}(K)$ be the Hilbert-completion of the *span* $\{K_x : x \in S\}$, with respect to the inner product

$$\left\langle \sum c_x K_x, \sum d_y K_y \right\rangle_{\mathcal{H}(K)} := \sum \sum \bar{c}_x d_y K(x, y) \quad (1.3)$$

modulo the subspace of functions of zero $\mathcal{H}(K)$ -norm. $\mathcal{H}(K)$ is then a reproducing kernel Hilbert space (HKRS), with the reproducing property:

$$\langle K_x, \varphi \rangle_{\mathcal{H}(K)} = \varphi(x), \quad \forall x \in S, \quad \forall \varphi \in \mathcal{H}(K). \quad (1.4)$$

Note. The summations in (1.3) are all finite. Starting with finitely supported summations in (1.3), the RKHS $\mathcal{H}(k)$ is then obtained by Hilbert space completion. We use physicists’ convention, so that the inner product is conjugate linear in the first variable, and linear in the second variable.

Conclusions, a summary:

- (1) For every positive definite kernel K , we define a “measure theoretic boundary space” $\mathcal{M}(K)$. Set

$$\mathcal{M}(K) := \left\{ (B, \mathcal{F}, \mu) \text{ a measure space which yields} \right. \\ \left. \text{a factorization for } K, \text{ see Definition 2.2} \right\}.$$

This set $\mathcal{M}(K)$ generalizes other notions of “boundary” used in the literature for networks, and for more general positive definite kernels, and their associated reproducing kernel Hilbert spaces (RKHSs).

- (2) For any positive definite kernel K , the corresponding $\mathcal{M}(K)$ is always non-empty. The natural Gaussian process path-space with covariance kernel K , and Wiener measure μ is in $\mathcal{M}(K)$.
- (3) Given K , let $\mathcal{H}(K)$ be the associated RKHS. Then for every $\mu \in \mathcal{M}(K)$ there is a canonical isometry W_μ mapping $\mathcal{H}(K)$ into $L^2(\mu)$. For details, see Proposition 2.8.
- (4) The isometry W_μ in (3) generally does not map onto $L^2(\mu)$.

It does however for the $\frac{1}{4}$ -Cantor example, i.e., the restriction of Hausdorff measure of dimension $\frac{1}{2}$ to the standard $\frac{1}{4}$ -Cantor set. In this case, we have a positive definite kernel on $\mathbb{D} \times \mathbb{D}$, where \mathbb{D} is the unit disk in the complex plane; and we can take the circle as boundary for \mathbb{D} . For μ , we take the corresponding $\frac{1}{4}$ -Cantor measure. But in general, for positive definite functions K , a “measure theoretic boundary space” is much “bigger” than probability spaces on the metric boundary for K .

- (5) Using the isometries from (3), we can turn $\mathcal{M}(K)$ into a partially ordered set; see Definition 3.2. Then, using Zorn’s lemma, one shows that $\mathcal{M}(K)$ always contains minimal elements. The minimal elements are not unique.
- (6) And even if μ is chosen minimal in $\mathcal{M}(K)$, the corresponding isometry W_μ still generally does not map onto $L^2(\mu)$. A case in point: the Szegő kernel, and $\mu =$ Lebesgue measure on a period interval.

Remark 1.4. The Cantor examples in (4) are special cases of affine-selfsimilarity limit (fractal) contractions. See, e.g., [3, 14].

The general role for the fractal dimension in these cases is as follows:

$$\dim_{fractal} = \frac{\ln s}{\ln d} = \log_d(s),$$

where $s =$ the number of translations in each iteration, and $d =$ the linear scale. For example, the middle-third Cantor fractal has $\dim_F = \frac{\ln 2}{\ln 3} = \log_3(2)$. The Sierpinski-gasket has $\dim_F = \frac{\ln 3}{\ln 2} < 2$. For the Sierpinski construction in \mathbb{R}^3 , we have $\dim_F = \frac{\ln 4}{\ln 2} = 2 < 3$.

2. GENERALIZED BOUNDARY SPACES FOR POSITIVE DEFINITE KERNELS

The general setup is as follows: In a general setting positive definite (p.d.) kernels K are defined on $S \times S$ where S is a prescribed set. In classical analysis such pairs (K, S) have found uses in many problems in harmonic analysis, in complex analysis, in stochastic analysis, analysis on infinite graphs, and in PDE

theory, the latter in the context of Green's functions for elliptic operators. In the complex analysis setting, S may be the disk \mathbb{D} , or the upper half-plane. For these applications, solutions typically entail consideration of boundaries, some in a natural geometric framework, and some more abstract. In some of the applications considered here, the notion of "boundary" is clear enough, for example for real or complex domains, but not for others. Take for example the case when S may instead be the set of vertices in an infinite graph.

The problem considered in the present section below is motivated by positive definite kernels arising naturally from classical frameworks, but our present emphasis will be applications when there is not already a given, or a natural boundary available at the outset. Below, we begin with a rigorous definition of "boundary" associated to a given pair (K, S) as specified above; see Definition 2.2. Starting with Definition 2.5, for different choices of "boundaries" we explore implications for a new harmonic analysis. Our first result in this framework is Proposition 2.8 below. Its corollaries are then explored in such applications as harmonic analysis, Example 2.11; the study of fractal measures for iterated function systems (IFSs), and for Gaussian processes, Corollary 2.10 and Theorem 2.1.

In Section 3 we turn to a discussion of the variety of "all" boundaries associated with a given pair (K, S) ; and when (K, S) is given, we establish the existence of minimal boundaries, Theorem 3.1.

Remark 2.1. (1) Given a positive definite kernel K on $S \times S$, there is then an associated mapping $E_S : S \rightarrow \{\text{Functions on } S\}$ given by

$$E_S(t) = K(t, \cdot), \quad (2.1)$$

where the dot "." in (2.1) indicates the independent variable; so

$$S \ni s \longrightarrow K(t, s) \in \mathbb{C}.$$

(2) We shall assume that E_S is 1-1, i.e., if $s_1, s_2 \in S$, and $K(s_1, t) = K(s_2, t)$, $\forall t \in S$, then it follows that $s_1 = s_2$. This is not a strong limiting condition on K .

(3) We shall view the Cartesian product

$$B_S := \prod_S \mathbb{C} = \mathbb{C}^S \quad (2.2)$$

as the set of all functions $S \rightarrow \mathbb{C}$.

It follows from assumption (2) that $E_S : S \rightarrow B_S$ is an injection, i.e., with E_S , we may identify S as a "subset" of B_S .

For $v \in S$, set $\pi_v : B_S \rightarrow \mathbb{C}$,

$$\pi_v(x) = x(v), \quad \forall x \in B_S; \quad (2.3)$$

i.e., π_v is the coordinate mapping at v . The topology on B_S shall be the product topology; and similarly the σ -algebra \mathcal{F}_S will be the one generated by $\{\pi_v\}_{v \in S}$, i.e., generated by the family of subsets

$$\pi_v^{-1}(M), \quad v \in S, \text{ and } M \subset \mathbb{C} \text{ a Borel set.} \quad (2.4)$$

Definition 2.2. Fix a positive definite kernel $K : S \times S \rightarrow \mathbb{C}$. Let $\mathcal{M}(K)$ be the set of all probability spaces (see Definition 1.1), so that $(B, \mathcal{F}, \mu) \in \mathcal{M}(K)$ iff (Def.) there exists an extension

$$K^B : S \times B \longrightarrow \mathbb{C}, \text{ and}$$

$$\int_B \overline{K^B(s_1, b)} K^B(s_2, b) d\mu(b) = K(s_1, s_2), \quad (2.5)$$

for all $(s_1, s_2) \in S \times S$.

Remark 2.3. In Examples 2.12-3.1, we discuss the case where

$$S = \mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$$

$$B = \partial\mathbb{D} = \{z \in \mathbb{C} \mid |z| = 1, \text{ or } z = e^{ix}, x \in (-\pi, \pi]\};$$

but in the definition of $\mathcal{M}(K)$, we allow all possible measure spaces (B, \mathcal{F}, μ) as long as the factorization (2.5) holds.

Questions:

- (1) Given (1.1) what are the solutions (B, \mathcal{F}, μ) to (2.5)?
- (2) Are there extensions $K^B : S \times B \rightarrow \mathbb{C}$ such that B is a boundary with respect to the metric on S ? That is,

$$\text{dist}_K(s_1, s_2) = \|K_{s_1} - K_{s_2}\|_{\mathcal{H}} \quad (2.6)$$

and $\lim K^B(\cdot, b) = \lim_{i \rightarrow \infty} K(\cdot, s_i)$.

- (3) Find the subsets $S_0 \subset S$ such that the following sampling property holds for all $f \in C(B)$ (or for a subspace of $C(B)$):

$$f(b) = \sum_{s_i \in S_0} f(s_i) K^B(s_i, b), \quad \forall b \in B. \quad (2.7)$$

Example 2.4 (Shannon). Let BL be the space of band-limited functions on \mathbb{R} , where

$$BL = \left\{ f \in L^2(\mathbb{R}) \mid \hat{f}(\xi) = 0, \xi \in \mathbb{R} \setminus \left[-\frac{1}{2}, \frac{1}{2}\right] \right\}.$$

We have

$$f(t) = \sum_{n \in \mathbb{Z}} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)}, \quad \forall t \in \mathbb{R}, \forall f \in BL. \quad (2.8)$$

Definition 2.5. We say $(B, \mathcal{F}, \mu) \in GC$, *generalized Carleson measures*, iff (Def.) there exists a constant C_μ such that

$$\int_B |\tilde{f}(b)|^2 d\mu(b) \leq C_\mu \|f\|_{\mathcal{H}(K)}^2, \quad \forall f \in \mathcal{H}(K), \quad (2.9)$$

where \tilde{f} in (2.9) is defined via the extension

$$\tilde{f}(b) := \langle K_b^B, f \rangle_{\mathcal{H}(K)}, \quad b \in B, f \in \mathcal{H}(K). \quad (2.10)$$

Set $(GC)_1 :=$ generalized Carleson measures with $C_\mu = 1$.

Note. The case $C_\mu = 1$ is of special interest. For classical theory on Carleson measures, we refer to [2, 4, 5, 13, 15, 17].

Definition 2.6. Let $\mathcal{H}_i, i = 1, 2$ be Hilbert spaces. We say that \mathcal{H}_1 is *boundedly contained* in \mathcal{H}_2 iff (Def.) $\mathcal{H}_1 \subset \mathcal{H}_2$ (as a subset), and if the inclusion map $\mathcal{H}_1 \rightarrow \mathcal{H}_2, h \mapsto h$, is bounded. That is, there exists $C < \infty$ such that for all $h \in \mathcal{H}_1$,

$$\|h\|_{\mathcal{H}_2} \leq C \|h\|_{\mathcal{H}_1}. \quad (2.11)$$

Remark 2.7. Note that if (B, \mathcal{F}, μ) is a measure space, $K : S \times S \rightarrow \mathbb{C}$ is a positive definite kernel, then $(B, \mathcal{F}, \mu) \in GC$ if and only if $\mathcal{H}(K)$ is boundedly contained in $L^2(B, \mathcal{F}, \mu)$; see (2.9).

We stress that with the inclusion $\mathcal{H}(K) \subset L^2(\mu)$ we can make the implicit identification $f \sim \tilde{f}$ where

$$\tilde{f}(b) = \langle \tilde{K}_b, f \rangle_{\mathcal{H}(K)}, \quad \forall f \in \mathcal{H}(K), b \in B; \quad (2.12)$$

and (2.12) is to be understood for a.a. b w.r.t. (\mathcal{F}, μ) .

In [9], we showed that for all positive definite kernel $K(s, t), (s, t) \in S \times S$, we have $\mathcal{M}(K) \neq \emptyset$. Moreover,

Proposition 2.8. *Fix a positive definite kernel $K : S \times S \rightarrow \mathbb{C}$, then*

$$\mathcal{M}(K) \subset (GC)_1. \quad (2.13)$$

If $(B, \mathcal{F}, \mu) \in \mathcal{M}(K)$, then the mapping

$$\mathcal{H}(K) \ni K(s, \cdot) \xrightarrow{\text{on } s} K^B(s, \cdot) \in L^2(B, \mu) \xrightarrow{\text{on } B} \quad (2.14)$$

extends by linearity and closure to an isometry (see Definition 2.5)

$$W_B : \mathcal{H}(K) \longrightarrow L^2(B, \mu), \quad f \longrightarrow \tilde{f}.$$

However, W_B is generally not onto $L^2(B, \mu)$.

More specifically, we have

$$\left\| \sum_j c_j K(s_j, \cdot) \right\|_{\mathcal{H}(K)}^2 = \left\| \sum_j c_j K^B(s_j, \cdot) \right\|_{L^2(B, \mu)}^2, \quad (2.15)$$

or equivalently,

$$\sum_{j_1} \sum_{j_2} \overline{c_{j_1}} c_{j_2} K(s_{j_1}, s_{j_2}) = \int_B \left| \sum_j c_j K^B(s_j, b) \right|^2 d\mu(b) \quad (2.16)$$

for all finite sums, where $\{s_j\}, \{c_j\} \subset \mathbb{C}^n, \forall n \in \mathbb{N}$.

Proof. Suppose $(B, \mathcal{F}, \mu) \in \mathcal{M}(K)$, i.e., assume (B, \mathcal{F}, μ) is a measure space such that (2.5) holds. Set $K^B = \tilde{K}$, refer to the extension $\tilde{K} : S \times B \rightarrow \mathbb{C}$ introduced in (2.6).

We claim that then (2.9) holds for all $f \in \mathcal{H}(K)$. Here \tilde{f} is defined via \tilde{K} ; see (2.10):

$$\tilde{f}(b) := \langle \tilde{K}_b, f \rangle_{\mathcal{H}(K)}, \quad \forall f \in \mathcal{H}(K), \forall b \in B.$$

Claim: $f \mapsto \tilde{f}$ is isometric from $\mathcal{H}(K)$ into $L^2(\mu)$, i.e.,

$$\|\tilde{f}\|_{L^2(B, \mu)} = \|f\|_{\mathcal{H}(K)}, \quad \forall f \in \mathcal{H}(K). \quad (2.17)$$

Proof of (2.17). It is enough to consider the case where $f = \sum_i c_i K_{s_i}$ (finite sum), see (1.1); so that $\tilde{f} = \sum_i c_i \tilde{K}_{s_i}$ on B , and

$$\begin{aligned} \|\tilde{f}\|_{L^2(B,\mu)}^2 &= \sum_i \sum_j \bar{c}_i c_j \langle \tilde{K}_{s_i} \tilde{K}_{s_j} \rangle_{L^2(B,\mu)} \\ &= \sum_i \sum_j \bar{c}_i c_j \int_B \overline{\tilde{K}_{s_i}(b)} \tilde{K}_{s_j}(b) d\mu(b) \\ &= \sum_i \sum_j \bar{c}_i c_j K(s_i, s_j) \quad (\text{see (2.5), use } \mu \in \mathcal{M}(K)) \\ &= \|f\|_{\mathcal{H}(K)}^2, \quad \text{by (1.1) and the defn. of } \mathcal{H}(K). \end{aligned}$$

□

Corollary 2.9. *Suppose $\mathcal{H}(K) \ni f \xrightarrow{W_B} \tilde{f} \in L^2(B, \mu)$ is bounded, i.e., that μ is a Carleson measure, then the adjoint operator*

$$W_B^* : L^2(B, \mu) \longrightarrow \mathcal{H}(K)$$

is given by

$$W_B^*(F)(s) = \int_B \overline{\tilde{K}(s, b)} F(b) d\mu(b), \quad \forall F \in L^2(B, \mu). \quad (2.18)$$

Proof. For all $F \in L^2(B, \mu)$, and all $s \in S$, we have

$$\begin{aligned} \langle K_s, W_B^* F \rangle_{\mathcal{H}(K)} &= (W_B^* F)(s) \quad (\text{reprod prop., and } W_B^* F \in \mathcal{H}(K)) \\ &= \langle W_B K_s, F \rangle_{L^2(\mu)} \quad (\text{by duality}) \\ &= \int_B \overline{\tilde{K}(s, b)} F(b) d\mu(b) \end{aligned}$$

which is the desired conclusion (2.18). □

We now turn to the Gaussian measure boundary:

Corollary 2.10. *Suppose $K : S \times S \rightarrow \mathbb{C}$ is a given positive definite kernel, and that there is a measure space (\mathcal{F}, μ) where \mathcal{F} is a σ -algebra of subsets of S , such that the RKHS $\mathcal{H}(K)$ satisfies $\mathcal{H}(K) \subset L^2(S, \mathcal{F}, \mu)$ (isometric inclusion), then $(S, \mathcal{F}, \mu) \in \mathcal{M}(K)$ iff*

$$K(s, t) = \int_S \overline{K(s, x)} K(t, x) d\mu(x), \quad \forall (s, t) \in S \times S. \quad (2.19)$$

Example 2.11. The condition in (2.19) is satisfied for Bargmann's Hilbert space \mathcal{H} of entire analytic functions on \mathbb{C} (see [1, 6]) subject to

$$\begin{aligned} \|f\|_{\mathcal{H}}^2 &= \frac{1}{2\pi} \iint_{\mathbb{R}^2} |f(x + iy)|^2 e^{-\frac{x^2+y^2}{2}} dx dy \\ &= \frac{1}{2\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-\frac{|z|^2}{2}} dx dy < \infty. \end{aligned} \quad (2.20)$$

The following kernel (Bargmann's kernel) is positive definite on $\mathbb{C} \times \mathbb{C}$:

$$K(z, w) = \exp\left(\frac{\bar{z}w}{2} - \frac{|z|^2 + |w|^2}{4}\right). \quad (2.21)$$

It is known that K in (2.21) satisfies (2.19) with respect to the measure μ on \mathbb{C} , given by

$$d\mu(z) = dA(z) = \frac{1}{2\pi} dx dy. \quad (2.22)$$

Theorem 2.1. *Let (K, S) be a positive definite kernel such that the associated mapping $E_S : S \rightarrow B_S$ is 1-1 (see (2.1)).*

Then there is a probability space $(B_S, \mathcal{F}_S, \mu_S)$ which satisfies the condition (1.1) in Definition 1.2.

Proof. This argument is essentially the Kolmogorov inductive limit construction. For every $n \in \mathbb{N}$, $\forall \{s_1, \dots, s_n\} \subset S$, we associate a measure $\mu^{\{s_1, \dots, s_n\}}$ on B_S as follows:

Let $\mu^{\{s_1, \dots, s_n\}}$ be the measure on B_S which has $(\pi_{s_1}, \dots, \pi_{s_n})$ as an n vector valued random variable with Gaussian the specific distribution: mean zero, and joint covariance matrix $\{K(s_i, s_j)\}_{i,j=1}^n$. By a standard argument, one checks that then $\mu^{\{s_1, \dots, s_n\}}$ is a consistent system of measures on B_S ; and (by Kolmogorov) that there is a unique probability measure μ_S on the measure space (B_S, \mathcal{F}_S) such that, for all (s_1, \dots, s_n) , the marginal distribution of μ_S coincides with $\mu^{\{s_1, \dots, s_n\}}$. \square

Example 2.12 (W_B is onto). Let

$$V = \mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\} \quad (2.23)$$

$$B = \partial\mathbb{D} = \{z \in \mathbb{C} \mid |z| = 1, \text{ or } z = e^{ix}, x \in (-\pi, \pi]\}. \quad (2.24)$$

Set

$$K(z, w) = \prod_{l=0}^{\infty} \left(1 + (\bar{z}w)^{4^l}\right), \quad (z, w) \in \mathbb{D} \times \mathbb{D},$$

and

$$K^B(z, x) = \prod_{l=0}^{\infty} \left(1 + (\bar{z}e^{i2\pi x})^{4^l}\right), \quad (z, x) \in \mathbb{D} \times B.$$

Then (2.5) holds for the case when $\mu_{\frac{1}{4}}$ = the $\frac{1}{4}$ -Cantor measure on B ; see [7].

Proof. (Sketch) Set

$$\begin{aligned} \Lambda_4 &= \left\{ \sum_{i=0}^n b_i 4^i \mid b_i \in \{0, 1\}, n \in \mathbb{N} \right\} \\ &= \{0, 1, 4, 5, 16, 17, 20, 21, 64, 65, \dots\} \end{aligned} \quad (2.25)$$

then

$$\prod_{l=0}^{\infty} \left(1 + t^{4^l}\right) = \sum_{\lambda \in \Lambda} t^\lambda, \quad |t| < 1. \quad (2.26)$$

The desired conclusion

$$K(z, w) = \int_{C_{\frac{1}{4}}} \overline{K_{C_{\frac{1}{4}}}(z, x)} K_{C_{\frac{1}{4}}}(w, x) d\mu_{\frac{1}{4}}(x) \quad (2.27)$$

follows from the fact that $\{e_\lambda \mid \lambda \in \Lambda_4\}$ is an ONB in $L^2(C_{1/4}, \mu_{1/4})$ by [10]. \square

3. BOUNDARY THEORY

We now turn to the details regarding boundary theory. To connect it to the classical theory of kernel spaces of analytic functions on the disk, we begin with an example, and we then turn to the case of the most general positive definite kernels; but not necessarily restricting the domain of the kernels to be considered.

Example 3.1 (W_B is not onto). Let

$$\begin{aligned} K(z, w) &= \frac{1}{1 - \bar{z}w}, \text{ (Szegő kernel)} \\ K^B(z, x) &= \frac{1}{1 - \bar{z}e^{i2\pi x}}, \text{ and} \\ \mu &= \text{restriction of Lebesgue measure to } [0, 1]. \end{aligned} \quad (3.1)$$

Let \mathbb{H}_2 be the Hardy space on \mathbb{D} . Then

$$W_B : \mathbb{H}_2 \longrightarrow L^2([0, 1], \mu_{Leb})$$

is isometric, but not onto. Indeed,

$$W_B(\mathbb{H}_2) = \overline{\text{span}}^{L^2(0,1)} \{e_n(x) \mid n \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}\}.$$

Returning to the general case, we show below that there is always a minimal element in $\mathcal{M}(K)$; see Definition 2.2.

Definition 3.2. Suppose $(B_i, \mathcal{F}_i, \mu_i) \in \mathcal{M}(K)$, $i = 1, 2$. We say that

$$(B_1, \mathcal{F}_1, \mu_1) \leq (B_2, \mathcal{F}_2, \mu_2) \quad (3.2)$$

if $\exists \varphi : B_2 \longrightarrow B_1$, s.t.

$$\mu_2 \circ \varphi^{-1} = \mu_1, \text{ and} \quad (3.3)$$

$$\varphi^{-1}(\mathcal{F}_1) = \mathcal{F}_2. \quad (3.4)$$

Lemma 3.3. $\mathcal{M}(K)$ has minimal elements.

Proof. If (3.3)-(3.4) hold, then

$$L^2(B_1, \mu_1) \ni f \xrightarrow{W_{21}} f \circ \varphi \in L^2(B_2, \mu_2)$$

is isometric, i.e.,

$$\int_{B_2} \underbrace{|f \circ \varphi|^2}_{W_{21}f} d\mu_2 = \int_{B_1} |f|^2 d\mu_1, \quad (3.5)$$

and

$$W_{B_2} = W_{21}W_{B_1} \text{ on } \mathcal{H}(K), \quad (3.6)$$

i.e., the diagram commutes:

$$\begin{array}{ccc}
 & & L^2(B_1, \mu_1) \\
 & \nearrow^{W_{B_1}} & \downarrow W_{21} \\
 \mathcal{H}(K) & \xrightarrow{W_{B_2}} & L^2(B_2, \mu_2)
 \end{array}$$

We can then use Zorn's lemma to prove that $\forall K$, $\mathcal{M}(K)$ has minimal elements (B, \mathcal{F}, μ) . (See the proof of Theorem 3.1 below.) But even if (B, \mathcal{F}, μ) is minimal, $W_B : \mathcal{H}(K) \rightarrow L^2(\mu)$ may *not* be onto. \square

In the next result, we shall refer to the partial order " \leq " from (3.2) when considering minimal elements in $\mathcal{M}(K)$. And, in referring to $\mathcal{M}(K)$, we have in mind a fixed positive definite function $K : S \times S \rightarrow \mathbb{C}$, specified at the outset; see Definitions 1.2 and 2.2.

Theorem 3.1. *Let (K, S) be a fixed positive definite kernel, and let $\mathcal{M}(K)$ be the corresponding boundary space from Definition 2.2.*

Then, for every $(X, \lambda) \in \mathcal{M}(K)$, there is a $(M, \nu) \in \mathcal{M}(K)$ such that

$$(M, \nu) \leq (X, \lambda), \quad (3.7)$$

and (M, ν) is minimal in the following sense: Suppose $(B, \mu) \in \mathcal{M}(K)$ and

$$(B, \mu) \leq (M, \nu), \quad (3.8)$$

then it follows that $(B, \mu) \simeq (M, \nu)$, i.e., we also have $(M, \nu) \leq (B, \mu)$.

Proof. We shall use Zorn's lemma, and the argument from Lemma 3.3.

Let $\mathcal{L} = \{(B, \mu)\}$ be a linearly ordered subset of $\mathcal{M}(K)$ s.t.

$$(B, \mu) \leq (X, \lambda), \quad \forall (B, \mu) \in \mathcal{L}; \quad (3.9)$$

and such that, for every pair (B_i, μ_i) , $i = 1, 2$, in \mathcal{L} , one of the following two cases must hold:

$$(B_1, \mu_1) \leq (B_2, \mu_2), \text{ or } (B_2, \mu_2) \leq (B_1, \mu_1). \quad (3.10)$$

To apply Zorn's lemma, we must show that there is a $(B_{\mathcal{L}}, \mu_{\mathcal{L}}) \in \mathcal{M}(K)$ such that

$$(B_{\mathcal{L}}, \mu_{\mathcal{L}}) \leq (B, \mu), \quad \forall (B, \mu) \in \mathcal{L}. \quad (3.11)$$

Now, using (3.9)-(3.10), we conclude that the measure spaces $\{(B, \mu)\}_{\mathcal{L}}$ have an inductive limit, i.e., the existence of:

$$\mu_{\mathcal{L}} := \operatorname{ind} \lim_{B \xrightarrow{\mathcal{L}} B_{\mathcal{L}}} \mu_B. \quad (3.12)$$

In other words, we may apply Kolmogorov's consistency to the family \mathcal{L} of measure spaces in order to justify the inductive limit construction in (3.12).

We have proved that every linearly ordered subset \mathcal{L} (as specified) has a "lower bound" in the sense of (3.11). Hence Zorn's lemma applies, and the desired conclusion follows, i.e., there is a pair $(M, \nu) \in \mathcal{M}(K)$ which satisfies the condition (3.8) from the theorem. \square

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