

(p, q) -TYPE BETA FUNCTIONS OF SECOND KIND

ALI ARAL^{1*} and VIJAY GUPTA²

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ABSTRACT. In the present article, we propose the (p, q) -variant of beta function of second kind and establish a relation between the generalized beta and gamma functions using some identities of the post-quantum calculus. As an application, we also propose the (p, q) -Baskakov–Durrmeyer operators, estimate moments and establish some direct results.

1. INTRODUCTION

The quantum calculus (q -calculus) in the field of approximation theory was discussed widely in the last two decades. Several generalizations to the q variants were recently presented in the book [3]. Further there is possibility of extension of q -calculus to post-quantum calculus, namely the (p, q) -calculus. Actually such extension of quantum calculus can not be obtained directly by substitution of q by q/p in q -calculus. But there is a link between q -calculus and (p, q) -calculus. The q calculus may be obtained by substituting $p = 1$ in (p, q) -calculus. We mentioned some previous results in this direction. Recently, Gupta [8] introduced (p, q) genuine Bernstein–Durrmeyer operators and established some direct results. (p, q) generalization of Szász–Mirakyan operators was defined in [1]. Also authors investigated a Durrmeyer type modifications of the Bernstein operators in [9]. We can also mention other papers as Bernstein operators [10], Bernstein–Stancu operators [11]. Bleimann–Butzer–Hahn operators and Szász–Mirakyan–Kantorovich

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*Corresponding author.

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operators . Besides this, we also refer to some recent related work on this topic: e.g. [5], [12] and [13].

Some basic notations of (p, q)-calculus are mentioned below:

The (p, q)-numbers are defined as

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

Obviously, it may be seen that $[n]_{p,q} = p^{n-1} [n]_{q/p}$. In The (p, q)-factorial is defined by

$$[n]_{p,q}! = \prod_{k=1}^n [k]_{p,q}, n \geq 1, [0]_{p,q}! = 1.$$

The (p, q)-binomial coefficient is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}! [k]_{p,q}!}, 0 \leq k \leq n.$$

For details see [15] and [16].

Definition 1.1. The (p, q)-power basis is defined below and it also has a link with q-power basis as

$$(x \oplus a)_{p,q}^n = (x + a)(px + qa)(p^2x + q^2a) \cdots (p^{n-1}x + q^{n-1}a).$$

$$(x \ominus a)_{p,q}^n = (x - a)(px - qa)(p^2x - q^2a) \cdots (p^{n-1}x - q^{n-1}a).$$

Definition 1.2. The (p, q)-derivative of the function f is defined as

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p - q)x}, x \neq 0$$

and $D_{p,q}f(0) = f'(0)$, provided that f is differentiable at 0. Note also that for $p = 1$, the (p, q)-derivative reduces to the q-derivative. The (p, q)-derivative fulfils the following product rules

$$D_{p,q}(f(x)g(x)) = f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x)$$

$$D_{p,q}(f(x)g(x)) = g(px)D_{p,q}f(x) + f(qx)D_{p,q}g(x).$$

The following assertions hold true:

$$D_{p,q}(x \ominus a)_{p,q}^n = [n]_{p,q} (px \ominus a)_{p,q}^{n-1}, n \geq 1$$

$$D_{p,q}(a \ominus x)_{p,q}^n = -[n]_{p,q} (a \ominus qx)_{p,q}^{n-1}, n \geq 1,$$

and $D_{p,q}(x \ominus a)_{p,q}^0 = 0$.

Definition 1.3. ([14])Let n is a nonnegative integer, we define the (p, q)-gamma function as

$$\Gamma_{p,q}(n + 1) = \frac{(p \ominus q)_{p,q}^n}{(p - q)^n} = [n]_{p,q}!, \quad 0 < q < p.$$

Proposition 1.4. *The formula of (p, q) -integration by part is given by*

$$\int_a^b f(px) D_{p,q}g(x) d_{p,q}x = f(b)g(b) - f(a)g(a) - \int_a^b g(qx) D_{p,q}f(x) d_{p,q}x \quad (1.1)$$

In the present paper, we propose the (p, q) -Baskakov–Durrmeyer operators and estimate some approximation properties, which include asymptotic formula and convergence in terms of modulus of continuity.

2. (p, q) -BETA FUNCTION OF SECOND KIND

Let $m, n \in \mathbb{N}$, we define (p, q) -beta function of second kind as

$$B_{p,q}(m, n) = \int_0^\infty \frac{x^{m-1}}{(1 \oplus px)_{p,q}^{m+n}} d_{p,q}x$$

Theorem 2.1. *Let $m, n \in \mathbb{N}$. We have the following relation between (p, q) -beta and (p, q) -gamma function:*

$$B_{p,q}(m, n) = q^{[2-m(m-1)]/2} p^{-m(m+1)/2} \frac{\Gamma_{p,q}(m) \Gamma_{p,q}(n)}{\Gamma_{p,q}(m+n)}.$$

Proof. We know that

$$D_{p,q} \frac{1}{(1 \oplus x)_{p,q}^n} = -\frac{p[n]_{p,q}}{(1 \oplus px)_{p,q}^{n+1}}$$

If we choose $f(x) = x^m$ and $g(x) = -\frac{1}{p[m+n]_{p,q}(1 \oplus x)_{p,q}^{m+n}}$ and use (1.1) we have

$$\begin{aligned} B_{p,q}(m+1, n) &= \int_0^\infty \frac{x^m}{(1 \oplus px)_{p,q}^{m+n+1}} d_{p,q}x \\ &= -\frac{p^{-m}}{p[m+n]_{p,q}} \int_0^\infty (px)^m D_{p,q} \frac{1}{(1 \oplus x)_{p,q}^{m+n}} d_{p,q}x \\ &= \frac{p^{-m}}{p[m+n]_{p,q}} \int_0^\infty D_{p,q} x^m \frac{1}{(1 \oplus qx)_{p,q}^{m+n}} d_{p,q}x \\ &= \frac{p^{-m} [m]_{p,q}}{p[m+n]_{p,q}} \int_0^\infty x^{m-1} \frac{1}{(1 \oplus qx)_{p,q}^{m+n}} d_{p,q}x \\ &= \frac{p^{-m-1} [m]_{p,q}}{q^{m-1} [m+n]_{p,q}} \int_0^\infty (qx)^{m-1} \frac{1}{(1 \oplus qx)_{p,q}^{m+n}} d_{p,q}x \\ &= \frac{p^{-1} [m]_{p,q}}{(pq)^m [m+n]_{p,q}} \int_0^\infty (x)^{m-1} \frac{1}{(1 \oplus x)_{p,q}^{m+n}} d_{p,q}x \\ &= \frac{p^{-1} [m]_{p,q}}{(pq)^m [m+n]_{p,q}} B_{p,q}(m, n), \end{aligned}$$

$$B_{p,q}(1, n) = \int_0^\infty \frac{1}{(1 \oplus px)_{p,q}^{n+1}} d_{p,q}x = -\frac{1}{p[n]_{p,q}} \int_0^\infty D_{p,q} \frac{1}{(1 \oplus x)_{p,q}^n} d_{p,q}x = \frac{1}{p[n]_{p,q}}$$

and

$$\begin{aligned}
 & B_{p,q}(m, n) \\
 &= \frac{p^{-1} [m - 1]_{p,q}}{(pq)^{m-1} [m + n - 1]_{p,q}} B_{p,q}(m - 1, n) \\
 &= \frac{p^{-1} [m - 1]_{p,q}}{(pq)^{m-1} [m + n - 1]_{p,q}} \frac{p^{-1} [m - 2]_{p,q}}{(pq)^{m-2} [m + n - 2]_{p,q}} B_{p,q}(m - 2, n) \\
 &= \frac{p^{-1} [m - 1]_{p,q}}{(pq)^{m-1} [m + n - 1]_{p,q}} \frac{p^{-1} [m - 2]_{p,q}}{(pq)^{m-2} [m + n - 2]_{p,q}} \cdots \frac{p^{-1}}{pq [n + 1]_{p,q}} B_{p,q}(1, n) \\
 &= \frac{p^{-1} [m - 1]_{p,q}}{(pq)^{m-1} [m + n - 1]_{p,q}} \frac{p^{-1} [m - 2]_{p,q}}{(pq)^{m-2} [m + n - 2]_{p,q}} \cdots \frac{p^{-1}}{pq [n + 1]_{p,q}} \frac{q}{pq [n]_{p,q}} \\
 &= \frac{qp^{-m}}{(pq)^{(m-1)m/2}} \frac{\Gamma_{p,q}(m) \Gamma_{p,q}(n)}{\Gamma_{p,q}(m + n)}
 \end{aligned}$$

□

3. (p, q)-BASKAKOV–DURRMEYER OPERATORS AND MOMENTS

The (p, q)-analogue of Baskakov operators for $x \in [0, \infty)$ and $0 < q < p \leq 1$ is defined as

$$B_{n,p,q}(f; x) = \sum_{k=0}^n b_{n,k}^{p,q}(x) f\left(\frac{p^{n-1} [k]_{p,q}}{q^{k-1} [n]_{p,q}}\right), \tag{3.1}$$

where

$$b_{n,k}^{p,q}(x) = \left[\begin{matrix} n + k - 1 \\ k \end{matrix} \right]_{p,q} p^{k+n(n-1)/2} q^{k(k-1)/2} \frac{x^k}{(1 \oplus x)_{p,q}^{n+k}}.$$

In case $p = 1$, we get the q -Baskakov operators [2]. If $p = q = 1$, we get at once the well known Baskakov operators.

Remark 3.1. Starting with the following relations between (p, q)-calculus and q -calculus:

$$\left[\begin{matrix} n + k - 1 \\ k \end{matrix} \right]_{p,q} = p^{k(n-1)} \left[\begin{matrix} n + k - 1 \\ k \end{matrix} \right]_{q/p}$$

and

$$(x \oplus a)_{p,q}^n = p^{n(n-1)/2} (x + a)_{q/p}^n$$

and using moments of q -Baskakov operators (see [2], [3]), it can easily be verified by simple computation that

$$B_{n,p,q}(1; x) = 1, B_{n,p,q}(t; x) = x, B_{n,p,q}(t^2; x) = x^2 + \frac{p^{n-1} x}{[n]_{p,q}} \left(1 + \frac{p}{q} x\right).$$

Definition 3.2. Using (p, q) -beta function of second kind, we propose below for $x \in [0, \infty)$, $0 < q < p \leq 1$ the (p, q) analogue of Baskakov–Durrmeyer operators

$$D_n^{p,q}(f; x) = [n-1]_{p,q} \sum_{k=0}^{\infty} b_{n,k}^{p,q}(x) q^{[k(k+1)-2]/2} p^{(k+1)(k+2)/2} \int_0^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{p,q} \frac{t^k}{(1 \oplus pt)_{p,q}^{k+n}} f(p^k t) d_{p,q} t \quad (3.2)$$

where $b_{n,k}^{p,q}(x)$ is as defined in (3.1).

Lemma 3.3. For $x \in [0, \infty]$, $0 < q < p \leq 1$, we have

$$\begin{aligned} (1) \quad & D_n^{p,q}(1; x) = 1 \\ (2) \quad & D_n^{p,q}(t; x) = \frac{1}{qp^2[n-2]_{p,q}} + \frac{[2]_{p,q}}{p^2q^2[n-2]_{p,q}}x + \frac{1}{p^n}x \\ (3) \quad & D_n^{p,q}(t^2; x) = \frac{[2]_{p,q}}{q^3[n-2]_{p,q}[n-3]_{p,q}} + \left(\frac{(p^5q(q+2p)+1)[3]_{p,q}}{p^6q^4[n-2]_{p,q}[n-3]_{p,q}} + \frac{p^5q(q+2p)+1}{p^{3+n}q[n-2]_{p,q}} \right) x \\ & + \frac{q^2+pq+p^2}{p^{9+n}q^2[n-2]_{p,q}}x^2 + \frac{[3]_{p,q}}{p^{10+n}q^3[n-3]_{p,q}}x^2 + \frac{(p^{n+2}[3]_{p,q}+q[2]_{p,q}[3]_{p,q})}{p^{12+n}q^6[n-2]_{p,q}[n-3]_{p,q}}x^2 + \frac{1}{p^{7+2n}}x^2. \end{aligned}$$

Proof. Using (3.2) and Remark 3.1, we have

$$\begin{aligned} D_n^{p,q}(1; x) &= [n-1]_{p,q} \sum_{k=0}^{\infty} b_{n,k}^{p,q}(x) q^{[k(k+1)-2]/2} p^{(k+1)(k+2)/2} \\ &\quad \times \int_0^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{p,q} \frac{t^k}{(1 \oplus pt)_{p,q}^{k+n}} d_{p,q} t \\ &= [n-1]_{p,q} \sum_{k=0}^{\infty} b_{n,k}^{p,q}(x) q^{[k(k+1)-2]/2} p^{(k+1)(k+2)/2} \\ &\quad \times \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{p,q} B_{p,q}(k+1, n-1) \\ &= B_{n,p,q}(1; x) = 1. \end{aligned}$$

Next using the identity $[k + 1]_{p,q} = q^k + p[k]_{p,q}$ and applying Remark 3.1, we have

$$\begin{aligned}
 D_n^{p,q}(t; x) &= [n - 1]_{p,q} \sum_{k=0}^{\infty} b_{n,k}^{p,q}(x) q^{[k(k+1)-2]/2} p^{(k+1)(k+2)/2} \int_0^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{p,q} \frac{t^{k+1} p^k}{(1 \oplus pt)_{p,q}^{k+n}} d_{p,q} t \\
 &= [n - 1]_{p,q} \sum_{k=0}^{\infty} q^{[k(k+1)-2]/2} p^{(k+1)(k+2)/2} b_{n,k}^{p,q}(x) \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{p,q} p^k B_{p,q}(k+2, n-2) \\
 &= \sum_{k=0}^{\infty} p^{-2} q^{-k-1} b_{n,k}^{p,q}(x) \cdot \frac{[k+1]_{p,q}}{[n-2]_{p,q}} \\
 &= \frac{1}{[n-2]_{p,q} p^2} \sum_{k=0}^{\infty} q^{-k-1} b_{n,k}^{p,q}(x) (q^k + p[k]_{p,q}) \\
 &= \frac{1}{[n-2]_{p,q} q p^2} B_{n,p,q}(1; x) + \frac{[n]_{p,q}}{p^n q^2 [n-2]_{p,q}} B_{n,p,q}(t; x) \\
 &= \frac{1}{q p^2 [n-2]_{p,q}} + \frac{[n]_{p,q} x}{p^n q^2 [n-2]_{p,q}}.
 \end{aligned}$$

Further using the identity $[k + 2]_{p,q} = q^{k+1} + p q^k + p^2 [k]_{p,q}$ and by Remark 3.1, we get

$$\begin{aligned}
 D_n^{p,q}(t^2; x) &= [n - 1]_{p,q} \sum_{k=0}^{\infty} b_{n,k}^{p,q}(x) q^{[k(k+1)-2]/2} p^{(k+1)(k+2)/2} \\
 &\quad \times \int_0^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{p,q} \frac{t^{k+2} p^{2k}}{(1 \oplus pt)_{p,q}^{k+n}} d_{p,q} t \\
 &= [n - 1]_{p,q} \sum_{k=0}^{\infty} q^{[k(k+1)-2]/2} p^{(k+1)(k+2)/2} b_{n,k}^{p,q}(x) \\
 &\quad \times \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{p,q} p^{2k} \cdot B_{p,q}(k+3, n-3) \\
 &= \sum_{k=0}^{\infty} q^{-(2k+3)} p^{-5} b_{n,k}^{p,q}(x) \cdot \frac{[k+2]_{p,q} [k+1]_{p,q}}{[n-2]_{p,q} [n-3]_{p,q}} \\
 &= \sum_{k=0}^{\infty} \frac{b_{n,k}^{p,q}(x) p^{-5} \cdot q^{-(2k+3)} (p^3 [k]_{p,q}^2 + q^k (p[2]_{p,q} + p^2) [k]_{p,q} + q^{2k} [2]_{p,q})}{[n-2]_{p,q} [n-3]_{p,q}} \\
 &= \frac{1}{[n-2]_{p,q} [n-3]_{p,q}} \sum_{k=0}^{\infty} b_{n,k}^{p,q}(x) \left[\left(\frac{p^{n-1}}{q^{k-1}} [k]_{p,q} \right)^2 \frac{p^{-7-2n}}{q^5} \right. \\
 &\quad \left. + ([2]_{p,q} + p) \left(\frac{p^{n-1}}{q^{k-1}} [k]_{p,q} \right) \frac{p^{-3-n}}{q^4} + q^{-3} [2]_{p,q} \right]
 \end{aligned}$$

$$\begin{aligned}
&= \frac{q^{-3}[2]_{p,q}}{[n-2]_{p,q}[n-3]_{p,q}} B_{n,p,q}(1; x) + \frac{p^{-3-n}}{q^4} \frac{([2]_{p,q} + p)[n]_{p,q}}{[n-2]_{p,q}[n-3]_{p,q}} B_{n,p,q}(t; x) \\
&\quad + \frac{p^{-7-2n}}{q^5} \frac{[n]_{p,q}^2}{[n-2]_{p,q}[n-3]_{p,q}} B_{n,p,q}(t^2; x) \\
&= \frac{[2]_{p,q}}{q^3[n-2]_{p,q}[n-3]_{p,q}} + \frac{([2]_{p,q} + p)[n]_{p,q}}{p^{3+n}q^4[n-2]_{p,q}[n-3]_{p,q}} x \\
&\quad + \frac{[n]_{p,q}^2}{p^{7+2n}q^5[n-2]_{p,q}[n-3]_{p,q}} \left(x^2 + \frac{p^{n-1}x}{[n]_{p,q}} \left(1 + \frac{p}{q}x \right) \right) \\
&= \frac{[2]_{p,q}}{q^3[n-2]_{p,q}[n-3]_{p,q}} + \frac{([2]_{p,q} + p)[n]_{p,q}}{p^{3+n}q^4[n-2]_{p,q}[n-3]_{p,q}} x \\
&\quad + \frac{[n]_{p,q}^2}{p^{7+2n}q^5[n-2]_{p,q}[n-3]_{p,q}} \left(x^2 + \frac{p^{n-1}x}{[n]_{p,q}} \left(1 + \frac{p}{q}x \right) \right). \\
&= \frac{[2]_{p,q}}{q^3[n-2]_{p,q}[n-3]_{p,q}} + \frac{([2]_{p,q} + p)[n]_{p,q}}{p^{3+n}q^4[n-2]_{p,q}[n-3]_{p,q}} x \\
&\quad + \frac{[n]_{p,q}^2}{p^{7+2n}q^5[n-2]_{p,q}[n-3]_{p,q}} x^2 + \frac{[n]_{p,q}}{p^{8+n}q^5[n-2]_{p,q}[n-3]_{p,q}} x + \\
&\quad \times \frac{[n]_{p,q}}{p^{7+n}q^6[n-2]_{p,q}[n-3]_{p,q}} x^2. \\
&= \frac{[2]_{p,q}}{q^3[n-2]_{p,q}[n-3]_{p,q}} + \frac{([2]_{p,q} + p)[n]_{p,q}}{p^{3+n}q^4[n-2]_{p,q}[n-3]_{p,q}} x \\
&\quad + \frac{1}{p^{7+2n}} x^2 + \frac{[2]_{p,q}}{p^{9+n}q^2[n-2]_{p,q}} x^2 + \frac{[3]_{p,q}}{p^{10+n}q^3[n-3]_{p,q}} x^2 \\
&\quad + \frac{[2]_{p,q}[3]_{p,q}}{p^{12+n}q^5[n-2]_{p,q}[n-3]_{p,q}} x^2 \\
&\quad + \frac{[n]_{p,q}}{p^{8+n}q^5[n-2]_{p,q}[n-3]_{p,q}} x + \frac{[n]_{p,q}}{p^{7+n}q^6[n-2]_{p,q}[n-3]_{p,q}} x^2 \\
&= \frac{[2]_{p,q}}{q^3[n-2]_{p,q}[n-3]_{p,q}} + \left(\frac{(p^5q(q+2p)+1)[3]_{p,q}}{p^6q^4[n-2]_{p,q}[n-3]_{p,q}} + \frac{p^5q(q+2p)+1}{p^{3+n}q[n-2]_{p,q}} \right) x \\
&\quad + \frac{q^2 + pq + p^2}{p^{9+n}q^2[n-2]_{p,q}} x^2 + \frac{[3]_{p,q}}{p^{10+n}q^3[n-3]_{p,q}} x^2 \\
&\quad + \frac{(p^{n+2}[3]_{p,q} + q[2]_{p,q}[3]_{p,q})}{p^{12+n}q^6[n-2]_{p,q}[n-3]_{p,q}} x^2 + \frac{1}{p^{7+2n}} x^2.
\end{aligned}$$

□

4. WEIGHTED APPROXIMATION

We consider the following class of functions:

Let $H_{x^2} [0, \infty)$ be the set of all functions f defined on $[0, \infty)$ satisfying the condition $|f(x)| \leq M_f(1+x^2)$, where M_f is a constant depending only on f . By $C_{x^2} [0, \infty)$, we denote the subspace of all continuous functions belonging to $H_{x^2} [0, \infty)$. Also, let $C_{x^2}^* [0, \infty)$ be the subspace of all functions $f \in C_{x^2} [0, \infty)$, for which $\lim_{|x| \rightarrow \infty} \frac{f(x)}{1+x^2}$ is finite. The norm on $C_{x^2}^* [0, \infty)$ is $\|f\|_{x^2} = \sup_{x \in [0, \infty)} \frac{|f(x)|}{1+x^2}$.

Now we shall discuss the weighted approximation theorem, where the approximation formula holds true on the interval $[0, \infty)$.

Theorem 4.1. *Let $p = p_n$ and $q = q_n$ satisfies $0 < q_n < p_n \leq 1$ and for n sufficiently large $p_n \rightarrow 1, q_n \rightarrow 1$ and $q_n^n \rightarrow 1$ and $p_n^n \rightarrow 1$. For each $f \in C_{x^2}^* [0, \infty)$, we have*

$$\lim_{n \rightarrow \infty} \|D_n^{p_n, q_n}(f) - f\|_{x^2} = 0.$$

Proof. Using the Theorem in [7] we see that it is sufficient to verify the following three conditions

$$\lim_{n \rightarrow \infty} \|D_n^{p_n, q_n}(t^\nu, x) - x^\nu\|_{x^2} = 0, \quad \nu = 0, 1, 2. \tag{4.1}$$

Since $D_n^{p_n, q_n}(1, x) = 1$ the first condition of (4.1) is fulfilled for $\nu = 0$.

We can write for $n > 3$

$$\begin{aligned} \|D_n^{p_n, q_n}(t, x) - x\|_{x^2} &\leq \frac{1}{q_n p_n^2 [n-2]_{p_n, q_n}} \\ &\quad + \left(\frac{[2]_{p_n, q}}{p_n^n q_n^2 [n-2]_{p_n, q_n}} + \frac{1}{p_n^n} - 1 \right) \sup_{x \in [0, \infty)} \frac{x}{1+x^2} \end{aligned}$$

and

$$\begin{aligned} &\|D_n^{p_n, q_n}(t^2, x) - x^2\|_{x^2} \\ &\leq \left(\frac{q_n^2 + p_n q_n + p_n^2}{p_n^{9+n} q_n^2 [n-2]_{p_n, q_n}} + \frac{[3]_{p_n, q_n}}{p_n^{10+n} q_n^3 [n-3]_{p_n, q}} + \frac{(p_n^{n+2} [3]_{p_n, q_n} + q_n [2]_{p_n, q_n} [3]_{p_n, q_n})}{p_n^{12+n} q_n^6 [n-2]_{p_n, q_n} [n-3]_{p_n, q_n}} \right) \\ &\quad \times \sup_{x \in [0, \infty)} \frac{x^2}{1+x^2} \\ &\quad + \left(\frac{1}{p_n^{7+2n}} - 1 \right) \sup_{x \in [0, \infty)} \frac{x^2}{1+x^2} \\ &\quad + \left(\frac{(p_n^5 q_n (q_n + 2p_n) + 1) [3]_{p_n, q_n}}{p_n^6 q_n^4 [n-2]_{p_n, q_n} [n-3]_{p_n, q_n}} + \frac{p_n^5 q_n (q_n + 2p_n) + 1}{p_n^{3+n} q_n [n-2]_{p_n, q_n}} \right) \sup_{x \in [0, \infty)} \frac{x}{1+x^2} \\ &\quad + \frac{[2]_{p_n, q_n}}{q_n^3 [n-2]_{p_n, q_n} [n-3]_{p_n, q_n}} \end{aligned}$$

which implies that

$$\lim_{n \rightarrow \infty} \|D_n^{p_n, q_n}(t, x) - x\|_{x^2} = 0$$

and

$$\lim_{n \rightarrow \infty} \|D_n^{p_n, q_n}(t^2, x) - x^2\|_{x^2} = 0.$$

Thus the proof is completed. □

We give the following theorem to approximate all functions in $C_{x^2}[0, \infty)$.

Theorem 4.2. *Let $p = p_n$ and $q = q_n$ satisfies $0 < q_n < p_n \leq 1$ and for n sufficiently large $p_n \rightarrow 1$, $q_n \rightarrow 1$ and $q_n^n \rightarrow 1$ and $p_n^n \rightarrow 1$. For each $f \in C_{x^2}^*[0, \infty)$, we have*

$$\lim_{n \rightarrow \infty} \sup_{x \in [0, \infty)} \frac{|D_n^{p_n, q_n}(f, x) - f(x)|}{(1 + x^2)^{1+\alpha}} = 0.$$

Proof. For any fixed $x_0 > 0$,

$$\begin{aligned} & \sup_{x \in [0, \infty)} \frac{|D_n^{p_n, q_n}(f, x) - f(x)|}{(1 + x^2)^{1+\alpha}} \\ & \leq \sup_{x \leq x_0} \frac{|D_n^{p_n, q_n}(f, x) - f(x)|}{(1 + x^2)^{1+\alpha}} + \sup_{x \geq x_0} \frac{|D_n^{p_n, q_n}(f, x) - f(x)|}{(1 + x^2)^{1+\alpha}} \\ & \leq \|D_n^{p_n, q_n}(f) - f\|_{C[0, x_0]} + \|f\|_{x^2} \sup_{x \geq x_0} \frac{|D_n^{p_n, q_n}(1 + t^2, x)|}{(1 + x^2)^{1+\alpha}} \\ & \quad + \sup_{x \geq x_0} \frac{|f(x)|}{(1 + x^2)^{1+\alpha}}. \end{aligned}$$

The first term of the above inequality tends to zero from well known Korovkin's theorem. By Lemma 3.3 for any fixed $x_0 > 0$ it is easily seen that $\sup_{x \geq x_0} \frac{|D_n^{p_n, q_n}(1 + t^2, x)|}{(1 + x^2)^{1+\alpha}}$ tends to zero as $n \rightarrow \infty$. We can choose $x_0 > 0$ so large that the last part of above inequality can be made small enough. □

Remark 4.3. For $q \in (0, 1)$ and $p \in (q, 1]$ it is seen that $\lim_{n \rightarrow \infty} [n]_{p, q} = 1/(q - p)$. In order to consider convergence of (p, q) Baskakov operators we assume $p = (p_n)$ and $q = (q_n)$ such that $0 < q_n < p_n \leq 1$ and for n sufficiently large $p_n \rightarrow 1$, $q_n \rightarrow 1$ and $p_n^n \rightarrow 1$ and $q_n^n \rightarrow 1$.

5. QUANTITATIVE APPROXIMATION

Let $C_B[0, \infty)$ denote the space of all real valued continuous and bounded functions on $[0, \infty)$. In this space we consider the norm

$$\|f\|_{C_B} = \sup_{x \in [0, \infty)} |f(x)|.$$

Now we give the first and second order modulus of continuity of function $f \in C_B$ (see [4], [6]) The first modulus of continuity is defined as

$$\omega_1(f; \delta) = \sup_{\substack{x, u, v \geq 0 \\ |u - v| \leq \delta}} |f(x + u) - f(x + v)|$$

and the second order modulus of continuity is defined

$$\omega_2(f; \delta) = \sup_{\substack{x, u, v \geq 0 \\ |u-v| \leq \delta}} |f(x + 2u) - 2f(x + u + v) + f(x + 2v)|, \quad \delta \geq 0.$$

We will use the Steklov mean function for $f \in C_B$

$$f_h(x) = \frac{4}{h^2} \int_0^{\frac{h}{2}} \int_0^{\frac{h}{2}} [2f(x + u + v) - f(x + 2(u + v))] dudv. \quad (5.1)$$

Since $f_h \in C_B$ we can write

$$f_h(x) - f(x) = \frac{4}{h^2} \int_0^{\frac{h}{2}} \int_0^{\frac{h}{2}} [2f(x + u + v) - f(x + 2(u + v)) - f(x)] dudv.$$

It is obvious that

$$|f_h(x) - f(x)| \leq \omega_2(f; h)$$

and

$$\|f_h - f\|_{C_B} \leq \omega_2(f; h). \quad (5.2)$$

If f is continuous, then $f'_h \in C_B$ and

$$f'_h(x) = \frac{4}{h^2} \left[2 \int_0^{\frac{h}{2}} \left(f\left(x + v + \frac{h}{2}\right) - f(x + v) \right) dv - \frac{1}{2} \int_0^{\frac{h}{2}} (f(x + h + 2v) - f(x + v)) dv \right].$$

Thus we have

$$\|f'_h\|_{C_B} \leq \frac{5}{h} \omega_1(f; h). \quad (5.3)$$

Similarly $f''_h \in C_B$ and

$$\|f''_h\|_{C_B} \leq \frac{9}{h^2} \omega_2(f; h). \quad (5.4)$$

Theorem 5.1. *Let $q \in (0, 1)$ and $p \in (q, 1]$. The operator $D_n^{p,q}$ maps space C_B into C_B and*

$$\|D_n^{p,q}(f)\|_{C_B} \leq \|f\|_{C_B}.$$

Proof. Let $q \in (0, 1)$ and $p \in (q, 1]$. From Lemma 3.3. we have

$$\begin{aligned}
|D_n^{p,q}(f; x)| &\leq [n-1]_{p,q} \sum_{k=0}^{\infty} b_{n,k}^{p,q}(x) q^{[k(k+1)-2]/2} p^{(k+1)(k+2)/2} \\
&\quad \int_0^{\infty} \left[\begin{matrix} n+k-1 \\ k \end{matrix} \right]_{p,q} \frac{t^k}{(1 \oplus pt)_{p,q}^{k+n}} |f(p^k t)| d_{p,q} t \\
&\leq \sup_{x \in [0, \infty)} |f(x)| [n-1]_{p,q} \sum_{k=0}^{\infty} b_{n,k}^{p,q}(x) q^{[k(k+1)-2]/2} p^{(k+1)(k+2)/2} \\
&\quad \int_0^{\infty} \left[\begin{matrix} n+k-1 \\ k \end{matrix} \right]_{p,q} \frac{t^k}{(1 \oplus pt)_{p,q}^{k+n}} d_{p,q} t \\
&= \sup_{x \in [0, \infty)} |f(x)| D_n^{p,q}(1; x) = \|f\|_{C_B}.
\end{aligned}$$

□

We are going to study the degree of approximation in terms of $\omega_1(f; \delta)$ and $\omega_2(f; \delta)$, first and second order modulus of continuity.

Theorem 5.2. *Let $q \in (0, 1)$ and $p \in (q, 1]$. If $f \in C_B$, then*

$$\begin{aligned}
&|D_n^{p,q}(f; x) - f(x)| \\
&\leq 5\omega_1\left(f; \frac{1}{\sqrt{[n-2]_{p,q}}}\right) \\
&\quad \times \left(\frac{1}{qp^2 \sqrt{[n-2]_{p,q}}} + \frac{[2]_{p,q}}{p^2 q^2 \sqrt{[n-2]_{p,q}}} x + \left(\frac{1}{p^n} - 1\right) \sqrt{[n-2]_{p,q}} x \right) \\
&\quad + \frac{9}{2} \omega_2\left(f; \frac{1}{\sqrt{[n-2]_{p,q}}}\right) \left(\frac{p^{7+2n} - 2p^{7+n} - 1}{p^{7+2n}} \right) [n-2]_{p,q} x^2 \\
&\quad + \frac{q^2 + pq + p^2 - 2p^{8+n} - 2qp^{7+n}}{p^{9+n} q^2} x^2 \\
&\quad + \frac{[3]_{p,q} [n-2]_{p,q}}{p^{10+n} q^3 [n-3]_{p,q}} x^2 + \frac{(p^{n+2} [3]_{p,q} + q [2]_{p,q} [3]_{p,q})}{p^{12+n} q^6 [n-3]_{p,q}} x^2 \\
&\quad + \left(\frac{(p^5 q (q+2p) + 1) [3]_{p,q}}{p^6 q^4 [n-3]_{p,q}} + \frac{p^5 q (q+2p) + 1}{p^{3+n} q} - \frac{2}{qp^2} \right) x + \frac{[2]_{p,q}}{q^3 [n-3]_{p,q}}
\end{aligned}$$

Proof. We use the Stiecklov function f_h defined by (5.1). For $x \geq 0$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
|D_n^{p,q}(f; x) - f(x)| &\leq D_n^{p,q}(|f - f_h|; x) + |D_n^{p,q}(f_h - f_h(x); x)| \\
&\quad + |f_h(x) - f(x)|.
\end{aligned}$$

By (5.2) we can write

$$D_n^{p,q}(|f - f_h|; x) \leq \|D_n^{p,q}(f - f_h)\|_{C_B} \leq \|f - f_h\|_{C_B} \leq \omega_2(f; h).$$

Since $D_n^{p,q}$ is a linear positive operator we get

$$|D_n^{p,q}(f_h - f_h(x); x)| \leq |f'_h(x)| D_n^{p,q}(t - x; x) + \frac{1}{2} \|f''\|_{C_B} D_n^{p,q}((t - x)^2; x).$$

By Lemma 3.3, (5.3) and (5.4) we have

$$\begin{aligned} & |D_n^{p,q}(f_h - f_h(x); x)| \\ & \leq \frac{5}{h} \omega_1(f; h) \left(\frac{1}{qp^2[n - 2]_{p,q}} + \frac{[2]_{p,q}}{p^2q^2[n - 2]_{p,q}}x + \left(\frac{1}{p^n} - 1\right)x \right) \\ & \quad + \frac{9}{2h^2} \omega_2(f; h) D_n^{p,q}((t - x)^2; x), \end{aligned}$$

where

$$\begin{aligned} & D_n^{p,q}((t - x)^2; x) \\ & = \left(\frac{p^{7+2n} - 2p^{7+n} - 1}{p^{7+2n}} \right) x^2 + \frac{q^2 + pq + p^2 - 2p^{8+n} - 2qp^{7+n}}{p^{9+n}q^2[n - 2]_{p,q}} x^2 \\ & \quad + \frac{[3]_{p,q}}{p^{10+n}q^3[n - 3]_{p,q}} x^2 + \frac{(p^{n+2}[3]_{p,q} + q[2]_{p,q}[3]_{p,q})}{p^{12+n}q^6[n - 2]_{p,q}[n - 3]_{p,q}} x^2 \\ & \quad + \left(\frac{(p^5q(q + 2p) + 1)[3]_{p,q}}{p^6q^4[n - 2]_{p,q}[n - 3]_{p,q}} + \frac{p^5q(q + 2p) + 1}{p^{3+n}q[n - 2]_{p,q}} - \frac{2}{qp^2[n - 2]_{p,q}} \right) x \\ & \quad + \frac{[2]_{p,q}}{q^3[n - 2]_{p,q}[n - 3]_{p,q}} \end{aligned}$$

for $x \geq 0, h > 0$. Setting $h = \sqrt{\frac{1}{[n-2]_{p,q}}}$, we have desired result. □

Remark 5.3. From Theorem 5.2 we can say that that the order of approximation of $D_n^{p,q}(f; x)$ to $f(x)$ is at least as good as the order of approximation to $f(x)$ by classical Baskakov–Durrmeyer operators for any $x \in [0, \infty)$ as a depending on selection of q_n and p_n . If we choose p and q as in Remark 4.3, we have an approximation process with the aid of operator (3.2).

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¹DEPARTMENT OF MATHEMATICS, KIRIKKALE UNIVERSITY, TR-71450, YAHSIHAN KIRIKKALE, TURKEY.

E-mail address: aliaral73@yahoo.com

²DEPARTMENT OF MATHEMATICS, NETAJI SUBHAS INSTITUTE OF TECHNOLOGY, SECTOR 3 DWARKA, NEW DELHI-110078, INDIA.

E-mail address: vijaygupta2001@hotmail.com