NON-ISOMORPHIC $C^*$-ALGEBRAS WITH ISOMORPHIC UNITARY GROUPS

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Abstract. Dye, [Ann. of Math. (2) 61 (1955), 73–89] proved that the discrete unitary group in a factor determines the algebraic type of the factor. Afterwards, for a large class of simple unital $C^*$-algebras, Al-Rawashdeh, Booth and Giordano [J. Funct. Anal. 262 (2012), 4711–4730] proved that the algebras are $*$-isomorphic if and only if their unitary groups are isomorphic as abstract groups. In this paper, we give a counterexample in the non-simple case. Indeed, we give two $C^*$-algebras with isomorphic unitary groups but the algebras themselves are not $*$-isomorphic.

1. Introduction

In [4], H. Dye proved that two von Neumann factors not of type $I_{2n}$ are isomorphic (via a linear or a conjugate linear $*$-isomorphism) if and only if their unitary groups are isomorphic as abstract groups. Indeed, he proved the following main theorem:

Theorem 1.1 ([4], Theorem 2). Let $M$ and $N$ be factors not of type $I_{2n}$, and let $\varphi$ be a group isomorphism between their unitary groups $U(M)$ and $U(N)$. Then there exists a linear (or conjugate linear) $*$-isomorphism $\psi$ of $M$ onto $N$ which implements $\varphi$ in the following sense: for some (possible discontinuous) character $\lambda$ of $U(M)$ and all $u \in U(M)$, $\varphi(u) = \lambda(u)\psi(u)$. 

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In [3], Theorem 1], M. Broise shows that the unitary group of a factor not of type $I_n$ has no non-trivial characters. Therefore Dye’s result can be rewritten as follows:

**Theorem 1.2.** If $N$ and $M$ are two von Neumann factors not of type $I_n(n < \infty)$, then any isomorphism between their unitary groups is implemented by a linear or a conjugate linear $*$-isomorphism between the factors.

Then extending the above result to some cases of simple, unital $C^*$-algebras, the author in [1] proved that if $\varphi$ is a continuous automorphism of the unitary group of a $UHF$-algebra, then $\varphi$ is implemented by linear or conjugate linear $*$-isomorphism.

In [2], Al-Rawashdeh, Booth and Giordano generalized Dye’s approach for a large class of simple, unital $C^*$-algebras. An isomorphism of the unitary groups, induces an isomorphism of their $K$-theory. In particular, if $A$ and $B$ are both simple unital AF-algebras, both irrational rotation algebras, or both Cuntz algebras and their unitary groups are isomorphic (as abstract groups), then $A$ and $B$ are isomorphic as $C^*$-algebras. In general, they proved the following main theorems:

**Theorem 1.3** ([2], Theorem 4.10). Let $A$ and $B$ be two simple, unital AH-algebras of slow dimension growth and of real rank zero. Then $A$ and $B$ are isomorphic if and only if their unitary groups are topologically isomorphic.

**Theorem 1.4** ([2], Corollary 5.7). Let $A$ and $B$ be two unital Kirchberg algebras belonging to the $UCT$-class $\mathcal{N}$. Then $A$ and $B$ are isomorphic if and only if their unitary groups are isomorphic (as abstract groups).

In this paper, we give an example of two $C^*$-algebras whose unitary groups are isomorphic, however the algebras themselves are not $*$-isomorphic. The counterexample is given in the non-simple $C^*$-algebra $C(X)$, where $X$ is a compact set. Recall that the unitary group of $C(X)$ is the group of all continuous functions from $X$ to the unit circle $\mathbb{T}$, which is denoted by $C(X, \mathbb{T})$.

2. The Counterexample

Let us recall Milutin’s theorem which is stated as follows:

**Theorem 2.1** (Milutin). [7, p.494] If $X$ and $Y$ are two compact, metrizable spaces which are non-countable, then $C(X, \mathbb{R}) \simeq C(Y, \mathbb{R})$ as Banach spaces.

Let us recall the following results of V. Pestov in [6]. Let $\zeta$ denote the group homomorphism from $C(X, \mathbb{T})$ to the cohomotopy group $\pi^1(X)$ assigning to every mapping its homotopy class. Denote by $C^0(X, \mathbb{T})$ the kernel of $\zeta$. Let $X$ be a topological space and $\theta$ be the map of the linear space $C(X, \mathbb{R})$ to the group $C(X, \mathbb{T})$, given by $\theta(f) = \exp(2\pi if)$. The image of $C(X, \mathbb{R})$ under $\theta$ is contained in $C^0(X, \mathbb{T})$ and $\theta$ is an additive group homomorphism.
If \( x_0 \in X \), then let
\[
C(X, x_0, \mathbb{R}) = \{ f \in C(X, \mathbb{R}); f(x_0) = 0 \},
\]
\[
C(X, x_0, \mathbb{T}) = \{ f \in C(X, \mathbb{T}); f(x_0) = 1 \},
\]
\[
C^0(X, x_0, \mathbb{T}) = \{ f \in C^0(X, \mathbb{T}); f(x_0) = 1 \}.
\]

Obviously, \( \theta \) maps \( C(X, x_0, \mathbb{R}) \) to \( C^0(X, x_0, \mathbb{T}) \). Denote by \( \theta_0 \) the restriction of \( \theta \) to \( C(X, x_0, \mathbb{R}) \).

**Proposition 2.2** ([6], Pro.13). Let \( X \) be a path-connected space and let \( x_0 \in X \). Then the map \( \theta_0 : C(X, x_0, \mathbb{R}) \to C^0(X, x_0, \mathbb{T}) \) is an algebraic isomorphism.

For every element \( x_0 \in X \), the groups \( C^0(X, \mathbb{T}) \) and \( C^0(X, x_0, \mathbb{T}) \oplus \mathbb{T} \) are isomorphic under the mapping \( f \mapsto (f.f(x_0)^{-1}, f(x_0)) \). Similarly, the groups \( C(X, x_0, \mathbb{R}) \oplus \mathbb{R} \) and \( C(X, \mathbb{R}) \) under the mapping \( f \mapsto (f - f(x_0), f(x_0)) \), (see [6], Lemma 7).

Consider the following short exact sequence:
\[
0 \to C^0(X, \mathbb{T}) \xrightarrow{\iota} C(X, \mathbb{T}) \xrightarrow{\zeta} \pi^1(X) \to 0.
\]
If \( X \) is compact, then \( C(X, \mathbb{T}) \) splits, i.e. \( C(X, \mathbb{T}) = C^0(X, \mathbb{T}) \oplus \pi^1(X) \). Now let us prove the following lemma:

**Lemma 2.3.** Let \( X \) and \( Y \) be two compact spaces. If \( C(Y, \mathbb{R}) \) and \( C(X, \mathbb{R}) \) are isomorphic as Banach spaces, then there is an isomorphism between \( C(Y, \mathbb{R}) \) and \( C(X, \mathbb{R}) \) which sends 1 (as a constant function) to itself and hence sends all constant functions to constants.

**Proof.** Let \( \psi \) denote the isomorphism from \( C(Y, \mathbb{R}) \) onto \( C(X, \mathbb{R}) \). If \( x_0 \in X \), and \( k \in \mathbb{R}\setminus\{-1\} \), then we define
\[
\varphi_k : C(X, \mathbb{R}) \to C(X, \mathbb{R})
\]
\[
g \mapsto g + kg(x_0).
\]
It is clear that \( \varphi_k \) is a linear map and \( \varphi_k(1) = 1 + k \).

The map \( \varphi_k \) is surjective: If \( h \in C(X, \mathbb{R}) \), then \( h - \frac{k}{k+1}h(x_0) \in C(X, \mathbb{R}) \) and
\[
\varphi_k(h - \frac{k}{k+1}h(x_0)) = h + kh(x_0) - \frac{k}{k+1}h(x_0)\varphi_k(1) = h.
\]
Now to show that \( \varphi_k \) is injective, let \( g \in \ker(\varphi_k) \). Then for every \( x \in X \), \( g(x) + kg(x_0) = 0 \) and in particular, \((k + 1)g(x_0) = 0\), therefore \( g = 0 \), hence \( \varphi_k \) is a bijective.

Let \( \psi(1) = f \). As \( f \) is a non-zero function which belongs to \( C(X, \mathbb{R}) \), there exists \( x_0 \in X \) such that \( |f(x_0)| = \|f\|_{\infty} \). Let \( k = 2\text{sign}(f(x_0)) \). Then for all \( x \in X \),
\[
\varphi_k(f)(x) = f(x) + kf(x_0)
\]
\[
= f(x) + 2\text{sign}(f(x_0)).f(x_0)
\]
\[
= f(x) + 2|f(x_0)| > 0.
\]
The map $\psi_1 = \varphi_k \circ \psi$ is an isomorphism from $C(Y, \mathbb{R})$ onto $C(X, \mathbb{R})$ with $\psi_1(1) > 0$. Then define $\Phi : C(Y, \mathbb{R}) \to C(X, \mathbb{R})$ by $g \mapsto \frac{1}{\psi_1(1)} \psi_1(g)$ and hence the lemma is checked. \qed

Finally, let us introduce the following main counterexample:

**Example 2.4.** Consider $X = [0, 1]$ and $Y = [0, 1] \times [0, 1]$ as subspaces of the usual topology of $\mathbb{R}$ and $\mathbb{R}^2$, respectively. As $X$ and $Y$ are not homeomorphic topological spaces, the $C^*$-algebras $C(X)$ and $C(Y)$ are not $*$-isomorphic.

**Claim:** $C(X, \mathbb{T}) \simeq C(Y, \mathbb{T})$ as abstract groups.

**Proof.** As $X$ and $Y$ are both contractible subsets of $\mathbb{R}$ and $\mathbb{R}^2$, their cohomology groups $H^q(X) = H^q(Y) = 0$, for all $q > 0$. The cohomotopy groups $\pi^1(X)$ and $\pi^1(Y)$ are trivial. As $X$ and $Y$ are both compact metrizable non-countable spaces, there exists a Banach space-isomorphism $\Phi$ from $C(X, \mathbb{R})$ to $C(Y, \mathbb{R})$, by Milutin’s theorem. We may assume that $\Phi$ maps constant functions onto themselves. Now define

$$
\psi : C(X, x_0, \mathbb{R}) \to C(Y, y_0, \mathbb{R})
$$

$$
f \mapsto \Phi(f) - \Phi(f)(y_0).
$$

It is clear that $\psi$ is a linear. If $g \in C(Y, y_0, \mathbb{R})$, then $h = \Phi^{-1}(g) - \Phi^{-1}(g)(x_0) \in C(X, x_0, \mathbb{R})$ and $\psi(h) = g$, hence $\psi$ is a surjective. If $\psi(f) = 0$, then for all $y \in Y$, $\Phi(f)(y) = \Phi(f)(y_0)$, therefore $\Phi(f)$ is a constant function of $Y$ and then $f = 0$. Hence $\psi$ is an isomorphism. By Proposition (2.2), we have that $C^0(Y, \mathbb{T}) \simeq C^0(Y, \mathbb{T})$, hence $C(X, \mathbb{T}) \simeq C(Y, \mathbb{T})$ and the example is completed. \qed

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**References**


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