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NON-ISOMORPHIC C^*-ALGEBRAS WITH ISOMORPHIC UNITARY GROUPS

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Abstract. Dye, [Ann. of Math. (2) 61 (1955), 73–89] proved that the discrete unitary group in a factor determines the algebraic type of the factor. Afterwards, for a large class of simple unital C^*-algebras, Al-Rawashdeh, Booth and Giordano [J. Funct. Anal. 262 (2012), 4711–4730] proved that the algebras are $*$-isomorphic if and only if their unitary groups are isomorphic as abstract groups. In this paper, we give a counterexample in the non-simple case. Indeed, we give two C^*-algebras with isomorphic unitary groups but the algebras themselves are not $*$-isomorphic.

1. Introduction

In [4], H. Dye proved that two von Neumann factors not of type I_{2n} are isomorphic (via a linear or a conjugate linear $*$-isomorphism) if and only if their unitary groups are isomorphic as abstract groups. Indeed, he proved the following main theorem:

Theorem 1.1 ([4], Theorem 2). *Let M and N be factors not of type I_{2n}, and let φ be a group isomorphism between their unitary groups $\mathcal{U}(M)$ and $\mathcal{U}(N)$. Then there exists a linear (or conjugate linear) $*$-isomorphism ψ of M onto N which implements φ in the following sense: for some (possible discontinuous) character λ of $\mathcal{U}(M)$ and all $u \in \mathcal{U}(M)$, $\varphi(u) = \lambda(u)\psi(u)$.*

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In [[3], Theorem 1], M. Broise shows that the unitary group of a factor not of type I_n has no non-trivial characters. Therefore Dye's result can be rewritten as follows:

Theorem 1.2. *If N and M are two von Neumann factors not of type $I_n (n < \infty)$, then any isomorphism between their unitary groups is implemented by a linear or a conjugate linear $*$-isomorphism between the factors.*

Then extending the above result to some cases of simple, unital C^*-algebras, the author in [1] proved that if φ is a continuous automorphism of the unitary group of a UHF-algebra, then φ is implemented by linear or conjugate linear $*$-isomorphism.

In [2], Al-Rawashdeh, Booth and Giordano generalized Dye's approach for a large class of simple, unital C^*-algebras. An isomorphism of the unitary groups, induces an isomorphism of their K-theory. In particular, if A and B are both simple unital AF-algebras, both irrational rotation algebras, or both Cuntz algebras and their unitary groups are isomorphic (as abstract groups), then A and B are isomorphic as C^*-algebras. In general, they proved the following main theorems:

Theorem 1.3 ([2], Theorem 4.10)**.** *Let A and B be two simple, unital AH-algebras of slow dimension growth and of real rank zero. Then A and B are isomorphic if and only if their unitary groups are topologically isomorphic.*

Theorem 1.4 ([2], Corollary 5.7)**.** *Let A and B be two unital Kirchberg algebras belonging to the UCT-class $\mathcal{N}$. Then A and B are isomorphic if and only if their unitary groups are isomorphic (as abstract groups).*

In this paper, we give an example of two C^*-algebras whose unitary groups are isomorphic, however the algebras themselves are not $*$-isomorphic. The counterexample is given in the non-simple C^*-algebra $C(X)$, where X is a compact set. Recall that the unitary group of $C(X)$ is the group of all continuous functions from X to the unit circle $\mathbb{T}$, which is denoted by $C(X, \mathbb{T})$.

2. The Counterexample

Let us recall Milutin's theorem which is stated as follows:

Theorem 2.1 (Milutin)**.** *[[7], p.494] If X and Y are two compact, metrizable spaces which are non-countable, then $C(X, \mathbb{R}) \simeq C(Y, \mathbb{R})$ as Banach spaces.*

Let us recall the following results of V. Pestov in [6]. Let ζ denote the group homomorphism from $C(X, \mathbb{T})$ to the cohomotopy group $\pi^1(X)$ assigning to every mapping its homotopy class. Denote by $C^0(X, \mathbb{T})$ the kernel of ζ. Let X be a topological space and θ be the map of the linear space $C(X, \mathbb{R})$ to the group $C(X, \mathbb{T})$, given by $\theta(f) = \exp(2\pi i f)$. The image of $C(X, \mathbb{R})$ under θ is contained in $C^0(X, \mathbb{T})$ and θ is an additive group homomorphism.

If $x_0 \in X$, then let

$$\begin{array}{rcl} C(X,x_0,\mathbb{R}) & = & \{f\in C(X,\mathbb{R}); f(x_0)=0\}, \\ C(X,x_0,\mathbb{T}) & = & \{f\in C(X,\mathbb{T}); f(x_0)=1\}, \\ C^0(X,x_0,\mathbb{T}) & = & \{f\in C^0(X,\mathbb{T}); f(x_0)=1\}. \end{array}$$

Obviously, θ maps $C(X,x_0,\mathbb{R})$ to $C^0(X,x_0,\mathbb{T})$. Denote by θ_0 the restriction of θ to $C(X,x_0,\mathbb{R})$.

Proposition 2.2 ([6], Pro.13)**.** *Let X be a path-connected space and let $x_0\in X$. Then the map $\theta_0 : C(X,x_0,\mathbb{R}) \to C^0(X,x_0,\mathbb{T})$ is an algebraic isomorphism.*

For every element $x_0 \in X$, the groups $C^0(X,\mathbb{T})$ and $C^0(X,x_0,\mathbb{T})\oplus\mathbb{T}$ are isomorphic under the mapping $f \mapsto (f.f(x_0)^{-1}, f(x_0))$. Similarly, the groups $C(X,x_0,\mathbb{R})\oplus\mathbb{R}$ and $C(X,\mathbb{R})$ under the mapping $f\mapsto (f-f(x_0), f(x_0))$, (see [[6], Lemma 7]).

Consider the following short exact sequence:

$$0 \longrightarrow C^0(X,\mathbb{T}) \xrightarrow{\iota} C(X,\mathbb{T}) \xrightarrow{\zeta} \pi^1(X) \longrightarrow 0.$$

If X is compact, then $C(X,\mathbb{T})$ splits, i.e. $C(X,\mathbb{T}) = C^0(X,\mathbb{T})\oplus \pi^1(X)$. Now let us prove the following lemma:

Lemma 2.3. *Let X and Y be two compact spaces. If $C(Y,\mathbb{R})$ and $C(X,\mathbb{R})$ are isomorphic as Banach spaces, then there is an isomorphism between $C(Y,\mathbb{R})$ and $C(X,\mathbb{R})$ which sends 1 (as a constant function) to itself and hence sends all constant functions to constants.*

Proof. Let ψ denote the isomorphism from $C(Y,\mathbb{R})$ onto $C(X,\mathbb{R})$. If $x_0\in X$, and $k\in\mathbb{R}\backslash\{-1\}$, then we define

$$\varphi_k : C(X,\mathbb{R}) \to C(X,\mathbb{R})$$

$$g \mapsto g + kg(x_0).$$

It is clear that φ_k is a linear map and $\varphi_k(1) = 1+k$.
The map φ_k is surjective: If $h\in C(X,\mathbb{R})$, then $h - \frac{k}{k+1}h(x_0) \in C(X,\mathbb{R})$ and

$$\varphi_k(h - \frac{k}{k+1}h(x_0)) = h + kh(x_0) - \frac{k}{k+1}h(x_0)\varphi_k(1) = h.$$

Now to show that φ_k is injective, let $g\in\ker(\varphi_k)$. Then for every $x\in X$, $g(x)+kg(x_0)=0$ and in particular, $(k+1)g(x_0)=0$, therefore $g=0$, hence φ_k is a bijective.

Let $\psi(1) = f$. As f is a non-zero function which belongs to $C(X,\mathbb{R})$, there exists $x_0\in X$ such that $|f(x_0)| = \|f\|_\infty$. Let $k = 2\text{sign}(f(x_0))$. Then for all $x\in X$,

$$\begin{array}{rcl} \varphi_k(f)(x) & = & f(x) + kf(x_0) \\ & = & f(x) + 2\text{sign}(f(x_0)).f(x_0) \\ & = & f(x) + 2|f(x_0)| > 0. \end{array}$$

The map $\psi_1 = \varphi_k \circ \psi$ is an isomorphism from $C(Y, \mathbb{R})$ onto $C(X, \mathbb{R})$ with $\psi_1(1) > 0$. Then define $\Phi : C(Y, \mathbb{R}) \to C(X, \mathbb{R})$ by $g \mapsto \frac{1}{\psi_1(1)}\psi_1(g)$ and hence the lemma is checked. $\square$

Finally, let us introduce the following main counterexample:

Example 2.4. Consider $X = [0, 1]$ and $Y = [0, 1] \times [0, 1]$ as subspaces of the usual topology of $\mathbb{R}$ and $\mathbb{R}^2$, respectively. As X and Y are not homeomorphic topological spaces, the C^*-algebras $C(X)$ and $C(Y)$ are not $*$-isomorphic.

Claim: $C(X, \mathbb{T}) \simeq C(Y, \mathbb{T})$ as abstract groups.

Proof. As X and Y are both contractible subsets of $\mathbb{R}$ and $\mathbb{R}^2$, their cohomology groups $H^q(X) = H^q(Y) = 0$, for all $q > 0$. the cohomotopy groups $\pi^1(X)$ and $\pi^1(Y)$ are trivial. As X and Y are both compact metrizable non-countable spaces, there exists a Banach space-isomorphism Φ from $C(X, \mathbb{R})$ to $C(Y, \mathbb{R})$, by Milutin's theorem. We may assume that Φ maps constant functions onto themselves. Now define

$$\begin{aligned} \psi : C(X, x_0, \mathbb{R}) &\to C(Y, y_0, \mathbb{R}) \\ f &\mapsto \Phi(f) - \Phi(f)(y_0). \end{aligned}$$

It is clear that ψ is a linear. If $g \in C(Y, y_0, \mathbb{R})$, then $h = \Phi^{-1}(g) - \Phi^{-1}(g)(x_0) \in C(X, x_0, \mathbb{R})$ and $\psi(h) = g$, hence ψ is a surjective. If $\psi(f) = 0$, then for all $y \in Y$, $\Phi(f)(y) = \Phi(f)(y_0)$, therefore $\Phi(f)$ is a constant function of Y and then $f = 0$. Hence ψ is an isomorphism. By Proposition (2.2), we have that $C^0(X, \mathbb{T}) \simeq C^0(Y, \mathbb{T})$, hence $C(X, \mathbb{T}) \simeq C(Y, \mathbb{T})$ and the example is completed. $\square$

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