Adv. Oper. Theory 1 (2016), no. 2, 219–235
http://doi.org/10.22034/aot.1610.1038
ISSN: 2538-225X (electronic)
http://aot-math.org

TSALLIS RELATIVE OPERATOR ENTROPY WITH NEGATIVE PARAMETERS

JUN ICHI FUJII[1] and YUKI SEO[2*]

Communicated by H. Osaka

Abstract. Tsallis relative operator entropy was firstly formulated by Fujii and Kamei as an operator version of Uhlmann's relative entropy. Afterwards, Yanagi, Kuriyama and Furuichi reformulated Tsallis relative operator entropy as an operator version of Tsallis relative entropy. In this paper, we define Tsallis relative operator entropy with negative parameters of (non-invertible) positive operators on a Hilbert space and show some properties.

1. Introduction

Fujii and Kamei [3] introduced the relative operator entropy which is a relative version of the operator entropy defined by Nakamura–Umegaki [12]: For positive invertible operators A and B on a Hilbert space, the relative operator entropy is defined by

$$S(A|B) = A^{\frac{1}{2}} \left(\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}}.$$

In addition, for non-invertible A and B, since $S(A|B+\varepsilon)$ has the right term monotone decreasing property as $\varepsilon \downarrow 0$, the relative operator entropy is defined by

$$S(A|B) = \operatorname*{s-lim}_{\varepsilon \to 0} S(A|B+\varepsilon) \tag{1.1}$$

if the strong operator limit exists as a bounded operator.

Date: Received: Oct. 26, 2016; Accepted: Dec. 18, 2016.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 47A63; Secondary 47A64.
Key words and phrases. Tsallis relative operator entropy, positive operator, operator geometric mean.

As a parametric extension of the relative operator entropy, Yanagi, Kuriyama and Furuichi [14] defined Tsallis relative operator entropy which is an operator version of Tsallis type relative entropy in quantum system due to Abe [1], also see [13, 6]: For two positive invertible operators A and B on a Hilbert space and any real number $t \in (0, 1]$, Tsallis relative operator entropy is defined by

$$T_t(A|B) = \frac{A \sharp_t B - A}{t},$$

where the t-weighted geometric operator mean is defined by

$$A \sharp_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2} \qquad \text{for } t \in [0, 1].$$

We use the notation $\natural_t$ for the binary operation

$$A \natural_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2} \qquad \text{for } t \notin [0, 1], \tag{1.2}$$

whose formula is the same as $\sharp_t$. Though $A \natural_t B$ for $t \notin [0, 1]$ are not operator mean in the sense of Kubo–Ando theory [11], $A \natural_t B$ have operator mean like properties for any positive invertible operators A and B. Thus we call (1.2) the quasi t-geometric mean for $t \notin [0, 1]$. Moreover, Furuichi, Yanagi and Kuriyama [7] considered Tsallis relative operator entropy for the parameter $t < 0$:

$$T_t(A|B) = \frac{A \natural_t B - A}{t} \qquad \text{for } t < 0. \tag{1.3}$$

Also, from viewpoint of Uhlmann's interpolational method, Fujii and Kamei [4] formulated (1.3) for $t \in [-1, 0)$ and showed many operator mean like properties. For example, for positive invertible operators A and B

$$A - AB^{-1}A \le T_t(A|B) \le B - A \qquad \text{for all } t \in [-1, 1] \tag{1.4}$$

and

$$T_t(A|B) \searrow S(A|B) \text{ for } t \searrow 0 \quad \text{and} \quad T_t(A|B) \nearrow S(A|B) \quad \text{for } t \nearrow 0. \tag{1.5}$$

It is meaningful to study properties of Tsallis relative operator entropy for the development of non-commutative statistical physics and quantum information theory. However, Tsallis relative operator entropy with negative parameters requires the invertibility of positive operators in general. So it is necessary for us to formulate (1.3) for general non-invertible positive operators.

The aim of this paper is to study Tsallis relative operator entropy with negative parameters $t \in [-1, 0)$ of (non-invertible) positive operators on a Hilbert space. For this, we investigate the properties of the quasi t-geometric mean $\natural_t$ for $t \in [-1, 0)$ in non-invertible case. By limiting the range of t to $[-1, 0)$, we consider the properties of Tsallis relative operator entropy with negative parameters for non-invertible case.

2. Quasi t-geometric mean $\natural_t$ for $-1 \le t < 0$

In this section, we study the properties of the quasi t-geometric mean $\natural_t$ for $t \in [-1, 0)$ in non-invertible case.

Let A and B be positive operators on a Hilbert space and $t \in [-1,0)$. Then it follows that $A \natural_t (B+\varepsilon)$ is monotone increasing on $\varepsilon \downarrow 0$. In fact, for $\delta > 0$ and $\varepsilon > \varepsilon' > 0$

$$(A+\delta) \natural_t (B+\varepsilon) \le (A+\delta) \natural_t (B+\varepsilon')$$

by Löwner–Heinz theorem. Since $B+\varepsilon, B+\varepsilon'$ are invertible and

$$(A+\delta) \natural_t (B+\varepsilon) = (A+\delta)^{1/2}((A+\delta)^{1/2}(B+\varepsilon)^{-1}(A+\delta)^{1/2})^{-t}(A+\delta)^{1/2},$$

it follows from $\delta \to 0$ that

$$A \natural_t (B+\varepsilon) \le A \natural_t (B+\varepsilon').$$

For non-invertible B, the quasi t-geometric mean $A \natural_t B$ for $t \in [-1,0)$ is defined as the following strong-operator limit if it exists:

$$A \natural_t B = \operatorname*{s\text{-}lim}_{\varepsilon \downarrow 0} A \natural_t (B+\varepsilon). \tag{2.1}$$

By the definition of (2.1), $A \natural_t B$ for $t \in [-1,0)$ exists if a set $\{A \natural_t (B+\varepsilon) : \varepsilon > 0\}$ is bounded above.

For non-invertible case, we have the following properties of quasi t-geometric means $A \natural_t B$ for $t \in [-1,0)$:

Lemma 2.1. *Let A, B, C and D be positive operators. If $A \natural_t B$ and $C \natural_t D$ exist for some $t \in [-1,0)$, then the following properties like operator means hold:*

(1) *right reverse monotonicity: $B \le C$ implies $A \natural_t B \ge A \natural_t C$.*
(2) *super-additivity: $A \natural_t B + C \natural_t D \ge (A+C) \natural_t (B+D)$.*
(3) *homogeneity: $(\alpha A) \natural_t (\alpha B) = \alpha(A \natural_t B)$ for all $\alpha > 0$.*
(4) *jointly convexity: For $\alpha \in [0,1]$*

$$((1-\alpha)A + \alpha C) \natural_t ((1-\alpha)B + \alpha D) \le (1-\alpha)A \natural_t B + \alpha\, C \natural_t D.$$

Proof. (1): Since $B \le C$, we have $B+\varepsilon \le C+\varepsilon$ for all $\varepsilon > 0$ and so

$$(A+\delta) \natural_t (B+\varepsilon) \ge (A+\delta) \natural_t (C+\varepsilon)$$

for all $\delta > 0$ and as $\delta \to 0$ we have

$$A \natural_t (B+\varepsilon) \ge A \natural_t (C+\varepsilon).$$

Since $A \natural_t (B+\varepsilon)$ and $A \natural_t (C+\varepsilon)$ are monotone increasing on $\varepsilon \downarrow 0$ and $A \natural_t B$ exists, it follows that $A \natural_t C$ exists and we have (1) as $\varepsilon \to 0$.

(2): For $\delta > 0$ and $\varepsilon > 0$, put $X_\delta = (A+\delta)^{1/2}(A+C+2\delta)^{-1/2}$ and $Y_\delta = (C+\delta)^{1/2}(A+C+2\delta)^{-1/2}$. It follows that

$$\begin{aligned}&X_\delta^*((A+\delta)^{-1/2}(B+\varepsilon)(A+\delta)^{-1/2})^t X_\delta + Y_\delta^*((C+\delta)^{-1/2}(D+\varepsilon)(C+\delta)^{-1/2})^t Y_\delta \\ &\ge ((A+C+2\delta)^{-1/2}(B+D+2\varepsilon)(A+C+2\delta)^{-1/2})^t,\end{aligned}$$

so that

$$(A+\delta) \natural_t (B+\varepsilon) + (C+\delta) \natural_t (D+\varepsilon) \ge (A+C+\delta) \natural_t (B+D+2\varepsilon).$$

Hence as $\delta \to 0$ it follows from the invertibility of $B+\varepsilon$ and $D+\varepsilon$ that

$$A \natural_t (B+\varepsilon) + C \natural_t (D+\varepsilon) \ge (A+C) \natural_t (B+D+2\varepsilon)$$

and as $\varepsilon \to 0$ $(A+C) \natural_t (B+D)$ exists and we have (2) since $A \natural_t B$ and $C \natural_t D$ exist.

(3) follows from the definition of the quasi t-geometric means for $t \in [-1, 0)$.

(4) follows from (2) and (3). $\square$

For non-invertible case, the quasi t-geometric mean $A \natural_t B$ for $t \in [-1, 0)$ have the following information monotonicity:

Theorem 2.2. *Let A and B be positive operators and Φ a normal positive linear map. If $A \natural_t B$ exists for some $t \in [-1, 0)$, then*

$$information monotonicity: \quad \Phi(A \natural_t B) \geq \Phi(A) \natural_t \Phi(B).$$

In particular,

$$transformer inequality: \quad T^*AT \natural_t T^*BT \leq T^*(A \natural_t B)T \quad \text{for any operators } T$$

and the equality holds for invertible T.

Proof. For $n \in \mathbb{N}$, put $\Phi_n(X) = \Phi(X) + \frac{1}{n}\varphi(X)I$ where φ is a state. Then the linear map Φ_n is strictly positive for all $n \in \mathbb{N}$, i.e., $X > 0$ implies $\Phi_n(X) > 0$ for all $n \in \mathbb{N}$. Moreover, for each $\varepsilon > 0$ put

$$\Psi_n(X) = \Phi_n(B+\varepsilon)^{-1/2}\Phi_n((B+\varepsilon)^{1/2}X(B+\varepsilon)^{1/2})\Phi_n(B+\varepsilon)^{-1/2} \quad \text{for } n \in \mathbb{N}.$$

Then Ψ_n is a unital positive linear map for all $n \in \mathbb{N}$ and the Jensen operator inequality for $1 < 1-t \leq 2$ implies

$$\Psi_n(X^{1-t}) \geq \Psi_n(X)^{1-t} \qquad \text{for } X > 0,$$

also see [9, p22,Theorem 1.20]. Hence we have

$$\begin{aligned}
\Phi_n(A \natural_t (B+\varepsilon)) &= \Phi_n(B+\varepsilon)^{1/2}\Psi_n(((B+\varepsilon)^{-1/2}A(B+\varepsilon)^{-1/2})^{1-t})\Phi_n(B+\varepsilon)^{1/2} \\
&\geq \Phi_n(B+\varepsilon)^{1/2}\Psi_n((B+\varepsilon)^{-1/2}A(B+\varepsilon)^{-1/2})^{1-t}\Phi_n(B+\varepsilon)^{1/2} \\
&= \Phi_n(B+\varepsilon) \natural_{1-t} \Phi_n(A) \\
&= \Phi_n(A) \natural_t \Phi_n(B+\varepsilon).
\end{aligned}$$

By the right reverse monotonicity of $\natural_t$ in Lemma 2.1, we have

$$\Phi_n(A \natural_t (B+\varepsilon)) \geq \Phi_n(A) \natural_t \Phi_n(B+\varepsilon) \geq \Phi_n(A) \natural_t (\Phi_n(B+\varepsilon)+\delta)$$

for all $\delta > 0$. Since $\Phi(B+\varepsilon)+\delta$ is invertible, as $n \to \infty$ we have

$$\Phi(A \natural_t (B+\varepsilon)) \geq \Phi(A) \natural_t (\Phi(B+\varepsilon)+\delta).$$

Since $A \natural_t B = \text{s-lim}_{\varepsilon \downarrow 0} A \natural_t (B+\varepsilon)$ and $\Phi(B)+\delta$ is invertible, as $\varepsilon \to 0$ it follows from the normality of Φ that

$$\Phi(A \natural_t B) \geq \Phi(A) \natural_t (\Phi(B)+\delta).$$

Since $\{\Phi(A) \natural_t (\Phi(B)+\delta) : \delta > 0\}$ is bounded above for all $\delta > 0$, $\Phi(A) \natural_t \Phi(B)$ exists and $\Phi(A \natural_t B) \geq \Phi(A) \natural_t \Phi(B)$. $\square$

For $t \in [-1, 0)$, since $1 \natural_t \varepsilon$ is not bounded above for $\varepsilon > 0$, $1 \natural_t 0$ does not make sense. Thus we consider an existence condition such that $A \natural_t B$ exists as a bounded operator, which is expressed by the boundedness of tangent lines: For $\alpha > 0$ and $t \in [-1, 0)$, put

$$L_{\alpha,t}(A, B) = (1-t)\alpha^{-t}A + t\alpha^{1-t}B. \tag{2.2}$$

Then we have $L_{\alpha,t}(A, B) \le A \natural_t B$ for all $\alpha > 0$ and positive invertible A, B.

Lemma 2.3. *Let A and B be positive operators and $t \in [-1, 0)$. Then $A \natural_t B$ exists as a bounded operator if and only if*

$$\sup_{\alpha>0} L_{\alpha,t}(A, B) = \sup_{\alpha>0} \left[(1-t)\alpha^{-t}A + t\alpha^{1-t}B\right] < +\infty. \tag{2.3}$$

The convention (2.3) means that there is a scalar constant c with $\varphi(L_{\alpha,t}(A, B)) \le c$ for all states φ and $\alpha > 0$. As we will see in the proof, we have $A \natural_t B \le c$.

Proof. Suppose that $A \natural_t B$ exists as a bounded operator for some $t \in [-1, 0)$. Then for each $\varepsilon > 0$

$$\begin{aligned} A \natural_t B \ge A \natural_t (B+\varepsilon) &= (B+\varepsilon)^{1/2}\left[(B+\varepsilon)^{-1/2}A(B+\varepsilon)^{-1/2}\right]^{1-t}(B+\varepsilon)^{1/2} \\ &\ge (B+\varepsilon)^{1/2}\left[(1-t)\alpha^{-t}(B+\varepsilon)^{-1/2}A(B+\varepsilon)^{-1/2} + t\alpha^{1-t}\right](B+\varepsilon)^{1/2} \\ &= (1-t)\alpha^{-t}A + t\alpha^{1-t}(B+\varepsilon) = L_{\alpha,t}(A, B+\varepsilon) \end{aligned}$$

and as $\varepsilon \to 0$ we have

$$A \natural_t B \ge (1-t)\alpha^{-t}A + t\alpha^{1-t}B = L_{\alpha,t}(A, B)$$

for all $\alpha > 0$.

Conversely, suppose that $\sup_{\alpha>0} L_{\alpha,t}(A, B) < +\infty$; there is the scalar upper bound c. Then we have

$$c \ge L_{\alpha,t}(A, B) \ge L_{\alpha,t}(A, B+\varepsilon)$$

for all $\varepsilon > 0$ since $t < 0$ and this implies

$$c(B+\varepsilon)^{-1} \ge L_{\alpha,t}((B+\varepsilon)^{-1/2}A(B+\varepsilon)^{-1/2}, I)$$

and hence

$$c(B+\varepsilon)^{-1} \ge \sup_{\alpha>0} L_{\alpha,t}((B+\varepsilon)^{-1/2}A(B+\varepsilon)^{-1/2}, I) = \left[(B+\varepsilon)^{-1/2}A(B+\varepsilon)^{-1/2}\right]^{1-t}.$$

Therefore we have $c \ge A \natural_t (B+\varepsilon)$ for all $\varepsilon > 0$. Since $\{A \natural_t (B+\varepsilon) : \varepsilon > 0\}$ is bounded above and monotone increasing for $\varepsilon \to 0$, there exists the strong-operator limit of $\{A \natural_t (B+\varepsilon) : \varepsilon > 0\}$ and so $A \natural_t B$ exists as a bounded operator. □

In order to show one of sufficient conditions that $A \natural_t B$ for some $t \in [-1, 0)$ exists, we need some preliminaries. The following lemma says that the quasi t-geometric mean for $t \in [-1, 0)$ has normalization:

Lemma 2.4. *Let A be a positive operator and $t \in [-1, 0)$. Then $A \natural_t A = A$.*

Proof. Put $F(\alpha) = (1-t)\alpha^{-t} + t\alpha^{1-t}$. Since $L_{\alpha,t}(A, A) = F(\alpha)A \le max_{\alpha>0}F(\alpha)A = F(1)A = A$, we have

$$A \ge (1-t)\alpha^{-t}A + t\alpha^{1-t}A \ge (1-t)\alpha^{-t}A + t\alpha^{1-t}(A+\varepsilon)$$

for all $\varepsilon > 0$ and we have $A \ge A \natural_t (A+\varepsilon)$ and so $A \natural_t A$ exists and $A \ge A \natural_t A$.

Conversely, by the supper-additivity of $\natural_t$, we have

$$\begin{aligned} A+\varepsilon &= (A+\varepsilon) \natural_t (A+\varepsilon) \le A \natural_t A + \varepsilon \natural_t \varepsilon \\ &= A \natural_t A + \varepsilon \end{aligned}$$

and so $A \le A \natural_t A$. Therefore, we have $A \natural_t A = A$. □

The following lemma shows that a kind of arithmetic-geometric mean inequality holds, also see [8, p129, Theorem 2]:

Lemma 2.5. *Let A and B be positive operators and $t \in [-1, 0)$. If $A \natural_t B$ exists, then*

$$A \natural_t B \ge (1-t)A + tB$$

Proof. Since $A \natural_t (B+\varepsilon) \ge (1-t)\alpha^{-t}A + t\alpha^{1-t}(B+\varepsilon)$ for all $\alpha > 0$, if we put $\alpha = 1$, then $A \natural_t (B+\varepsilon) \ge (1-t)A + t(B+\varepsilon)$ and as $\varepsilon \to 0$ we have the desired inequality. □

If A is majorized by B in the sense of Douglas, i.e., $A \le cB$ for some $c > 0$, then $A \natural_t B$ exists for all $t \in [-1, 0)$:

Theorem 2.6. *Let A and B be positive operators. If there is a scalar $c > 0$ such that $A \le cB$, then $A \natural_t B$ exists for all $t \in [-1, 0)$, and*

$$(1-t)A + tB \le A \natural_t B \le c^{-t}A. \tag{2.4}$$

Proof. For $t \in [-1, 0)$, put $F(\alpha) = (1-t)\alpha^{-t}c + t\alpha^{1-t}$ for $\alpha > 0$. Then

$$F'(\alpha) = t(1-t)\alpha^{-t-1}(\alpha - c)$$

and $F(\alpha)$ is maximum at $\alpha = c$. Hence

$$\begin{aligned} L_{\alpha,t}(A, B) &= (1-t)\alpha^{-t}A + t\alpha^{1-t}B \le ((1-t)\alpha^{-t}c + t\alpha^{1-t})B \\ &\le \max_{\alpha>0} F(\alpha)B = F(c)B = c^{1-t}B. \end{aligned}$$

Therefore, since $L_{\alpha,t}(A, B)$ is bounded above for all $\alpha > 0$, it follows from Lemma 2.3 that $A \natural_t B$ exists. Then we have the LHS of (2.4) by Lemma 2.5. Since $\frac{1}{c}A \le B$, it follows from (i) of Lemma 2.1 and Lemma 2.4 that $A \natural_t B \le A \natural_t (\frac{1}{c}A) = c^{-t}A \natural_t A = c^{-t}A$ and so we have the RHS of (2.4). □

We have the following relations around existence conditions:

Theorem 2.7. *The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) hold for any positive operators A, B and $t \in [-1, 0)$, and each converse does not always hold.*

(1) *majorization or range inclusion: $A \le cB$ for some $c > 0$, i.e.,* $\mathrm{ran}A^{\frac{1}{2}} \subset \mathrm{ran}B^{\frac{1}{2}}$.

(2) *existence condition:* $A \natural_t B$ *exists as a bounded operators, i.e.,*

$$\sup_{\alpha>0}\left[(1-t)\alpha^{-t}A + t\alpha^{1-t}B\right] < +\infty.$$

(3) *kernel inclusion:* $\ker A \supset \ker B$.

Proof. (1)$\Longrightarrow$(2) follows from Theorem 2.6.

(2)$\Longrightarrow$(3): For every $x \in \ker B$, we have

$$\langle((1-t)\alpha^{-t}A + t\alpha^{1-t}B)x, x\rangle = (1-t)\alpha^{-t}\langle Ax, x\rangle.$$

If $\langle Ax, x\rangle > 0$, then the LHS above diverges as $\alpha \to \infty$ because $-1 \le t < 0$ and it contradicts (2). Hence we have $Ax = 0$ and $x \in \ker A$.

The majorization (1) is stronger than existence condition (2): If A is a positive operator with $\sigma(A) = [0,1]$, then A is not majorized by A^2, while we see that $A \natural_t A^2 = A^{1+t}$. In fact, for each $\varepsilon > 0$, since $A + \varepsilon$ is invertible, we have

$$(A+\varepsilon) \natural_t (A+\varepsilon)^2 = (A+\varepsilon)^{1-t}(A+\varepsilon)^{2t} = (A+\varepsilon)^{1+t} \to A^{1+t}$$

as $\varepsilon \to 0$ and so $A \natural_t A^2$ exists and $A \natural_t A^2 = A^{1+t}$.

The existence condition (2) is stronger than kernel inclusion (3): If B is a positive operator with $\sigma(B) = [0,1]$ and 0 is not an eigenvalue, then I and B have trivial kernel. On the other hand, since $t \in [-1,0)$, for each $\varepsilon > 0$, $I \natural_t (B+\varepsilon) = (B+\varepsilon)^t \nearrow B^t$ diverges as $\varepsilon \to 0$ and so $I \natural_t B$ does not exist as a bounded operator. $\square$

Remark 2.8. If both ranges of A and B are closed, in particular, for the case of matrices, the above conditions in Theorem 2.7 are all equivalent since the relation $\mathrm{ran} A^{\frac{1}{2}} = \overline{\mathrm{ran}} A = (\ker A)^\perp$ holds for all positive operators A.

In [5], we show a kernel property $\ker(A \sharp_t B) = \ker A \vee \ker B$ for the geometric mean $A \sharp_t B$ for $t \in (0,1)$. Thus, we observe a kernel property for the quasi t-geometric mean $A \natural_t B$ for $t \in [-1,0)$. To show it, we need the following lemma:

Lemma 2.9. *Let A and B be positive operators. If $A \natural_t B = 0$ for some $t \in [-1,0)$, then $A = 0$.*

Proof. By the information monotonicity in Theorem 2.2, we have

$$0 = \varphi(A \natural_t B) \ge \varphi(A) \natural_t \varphi(B) \ge 0$$

for all state φ and so $\varphi(A) \natural_t \varphi(B) = 0$. Since $0 \le \varphi(A)^{1-t}(\varphi(B)+\varepsilon)^t \nearrow \varphi(A) \natural_t \varphi(B) = 0$ as $\varepsilon \to 0$, we have $\varphi(A) = 0$ for all state φ and thus $A = 0$. $\square$

Theorem 2.10. *Let A and B be positive operators. If $A \natural_t B$ exists for some $t \in [-1,0)$, then*

$$\ker(A \natural_t B) \vee \ker B \subset \ker A.$$

Proof. By the transformer inequality in Theorem 2.2,

$$P(A \natural_t B)P \ge (PAP) \natural_t (PBP) \ge 0 \qquad \text{for all projections } P.$$

If P is the projection on $\ker(A \natural_t B)$, then $P(A \natural_t B)P = 0$ and so $(PAP) \natural_t (PBP) = 0$. By Lemma 2.9, we have $PAP = 0$. If $x \in \ker(A \natural_t B)$, then

$$0 = \langle PAPx, x\rangle = \langle Ax, x\rangle = \left\|A^{1/2}x\right\|^2$$

and so $A^{1/2}x = 0$. Thus $x \in \ker A$ and so

$$\ker(A \natural_t B) \subset \ker A.$$

By Theorem 2.7, we have $\ker B \subset \ker A$ and thus

$$\ker(A \natural_t B) \vee \ker B \subset \ker A.$$

□

By readers' convenience, we recall the following well-known 'monotone convergence lemma' for monotone double sequences:

Lemma 2.11. *Let $\{a_{\delta_1,\delta_2}\}$ be a bounded double sequence of real numbers for $\delta_1, \delta_2 \in (0, 1]$. If $\{a_{\delta_1,\delta_2}\}$ is monotone decreasing for $\delta_1, \delta_2 \downarrow 0$, then there exists the limit with*

$$\lim_{\delta_1,\delta_2\downarrow 0} a_{\delta_1,\delta_2} = \lim_{\delta_1\downarrow 0}\lim_{\delta_2\downarrow 0} a_{\delta_1,\delta_2} = \lim_{\delta_2\downarrow 0}\lim_{\delta_1\downarrow 0} a_{\delta_1,\delta_2}.$$

We have a right lower semi-continuity of the quasi t-geometric mean:

Theorem 2.12. *Let A, B, B_n be positive operators for $n = 1, 2, \dots$ and $A \natural_t B$ exists for some $t \in [-1, 0)$. If $B_n \searrow B$ as $n \to \infty$, then $A \natural_t B_n \nearrow A \natural_t B$ as $n \to \infty$.*

Proof. Since $B_n \searrow B$, it follows from Lemma 2.1 that $A \natural_t B_n$ exist for all n and $A \natural_t B_n \le A \natural_t B$. For $n \le n'$ and $m \le m'$, we have

$$A \natural_t B_n \le A \natural_t B_{n'} \quad \text{and} \quad A \natural_t (B_n + \frac{1}{m}) \le A \natural_t (B_n + \frac{1}{m'})$$

and $A \natural_t (B_n+\frac{1}{m}) \le A \natural_t (B+\frac{1}{m}) \le A \natural_t B$. Put $s(n,m) = \langle (A \natural_t (B_n+\frac{1}{m})x, x\rangle$ for $x \in H$ and then $s(n,m)$ is monotone increasing double sequence of real numbers and bounded above. By Lemma 2.11, we have

$$\lim_{m\to\infty}\lim_{n\to\infty} s(n,m) = \lim_{n\to\infty}\lim_{m\to\infty} s(n,m) = \lim_{n,m\to\infty} s(n,m)$$

and so s-$\lim_{n\to\infty} A \natural_t B_n = A \natural_t B$. □

Here, we recall Izumino's construction of operator means [10, 2]: Let A and B be positive operators. Put $R = (A + B)^{1/2}$. Since $A, B \le A + B$, it follows from Douglas majorization theorem that there exist the derivatives D and E such that $A^{1/2} = DR$ and $B^{1/2} = ER$ with $\ker R \subset \ker D \cap \ker E$ and $\ker D^* = \ker A$ and $\ker E^* = \ker B$. For the range projection P on $\operatorname{ran} R$, we have

$$R(D^*D + E^*E)R = A + B = R^2 = RPR$$

and $\ker R \subset \ker D \cap \ker E = \ker D^*D \cap \ker E^*E$, and hence we have $D^*D + E^*E = P$. Moreover, since P commutes both D^*D and E^*E, it follows that D^*D commutes with E^*E and so we may assume that $D^*D + E^*E = I_R$ on $\overline{\operatorname{ran} R}$.

We have the following transformer equality of the quasi-t geometric mean for some $t \in [-1, 0)$:

Theorem 2.13. *Let A and B be positive operators. Under the situation above, if $A \natural_t B$ exists for some $t \in [-1, 0)$, then*

$$A \natural_t B = R(D^*D \natural_t E^*E)R = \operatorname*{s-lim}_{\varepsilon \to 0} R[D^*D \natural_t (E^*E + \varepsilon)]R.$$

Proof. For all $\varepsilon > 0$ and $\alpha > 0$, it follows from Lemma 2.3 that

$$A \natural_t B \geq A \natural_t (B + \varepsilon) \geq (1-t)\alpha^{-t}A + t\alpha^{1-t}(B + \varepsilon)$$

and so

$$\begin{aligned} A \natural_t B &\geq (1-t)\alpha^{-t}A + t\alpha^{1-t}B \\ &= R\left((1-t)\alpha^{-t}D^*D + t\alpha^{1-t}E^*E\right)R \\ &\geq R\left((1-t)\alpha^{-t}D^*D + t\alpha^{1-t}(E^*E + \varepsilon)\right)R \\ &= R(E^*E+\varepsilon)^{1/2}L_{\alpha,t}((E^*E+\varepsilon)^{-1/2}D^*D(E^*E+\varepsilon)^{-1/2}, I)(E^*E+\varepsilon)^{1/2}R, \end{aligned}$$

where $L_{\alpha,t}$ is defined as (2.2). Hence it follows that

$$\begin{aligned} A \natural_t B &\geq R(E^*E+\varepsilon)^{1/2}\left((E^*E+\varepsilon)^{-1/2}D^*D(E^*E+\varepsilon)^{-1/2}\right)^{1-t}(E^*E+\varepsilon)^{1/2}R \\ &= R(D^*D \natural_t(E^*E+\varepsilon))R \end{aligned}$$

for all $\varepsilon > 0$. Since $R(D^*D \natural_t(E^*E+\varepsilon))R$ are monotone increasing as $\varepsilon \to 0$ and bounded above for all $\varepsilon > 0$, there exists the strong-operator limit

$$G = \operatorname*{s-lim}_{\varepsilon \to 0} R(D^*D \natural_t(E^*E+\varepsilon))R$$

and we may write $G = R(D^*D \natural_t E^*E)R$ and so

$$A \natural_t B \geq R(D^*D \natural_t E^*E)R.$$

On the other hand, it follows from transformer inequality in Theorem 2.2 that

$$\begin{aligned} A \natural_t B &\geq R[D^*D \natural_t (E^*E+\varepsilon)]R \\ &\geq (RD^*DR) \natural_t (RE^*ER + \varepsilon R^2) \\ &\geq (RD^*DR) \natural_t (RE^*ER + \varepsilon\|R^2\|) \to A \natural_t B \end{aligned}$$

as $\varepsilon \to 0$. Hence we have

$$A \natural_t B = R(D^*D \natural_t E^*E)R.$$

□

3. Tsallis relative operator entropy

In this section, we study properties of Tsallis relative operator entropy with negaitive parameters for non-invertible case.

Let A and B be positive operators and $t \in [-1, 0)$. Since $A\natural_t(B+\varepsilon)$ is monotone incraesing on $\varepsilon \searrow 0$, it follows that Tsallis relative operator entropy

$$T_t(A|B+\varepsilon) = \frac{A\natural_t(B+\varepsilon) - A}{t}$$

is monotone decreasing on $\varepsilon \searrow 0$. Then we define Tsallis relative operator entropy with negative parameters as

$$T_t(A|B) = \operatorname*{s\text{-}lim}_{\varepsilon\downarrow 0} T_t(A|B+\varepsilon)$$

if the strong-operator limit exists.

By Lemma 2.1 and Theorem 2.2, we have the following properties of Tsallis relative operator entropy with negative parameters for non-invertible case:

Theorem 3.1. *Let A, B, C, D be positive operators. If $T_t(A|B)$ and $T_t(C|D)$ exist for some $t \in [-1, 0)$, then the following properties of Tsallis relative operator entropy with negative parameters hold:*

(1) *right monotonicity:* If $B \le C$, then $T_t(A|B) \le T_t(A|C)$.

(2) *transformer inequality:* $X^*T_t(A|B)X \le T_t(X^*AX|X^*BX)$ *for all* X (*the equality holds for invertible* X).

(2′) *information monotonicity:* $\Phi(T_t(A|B)) \le T_t(\Phi(A)|\Phi(B))$ *for all normal positive linear maps* Φ.

(3) *sub-additivity:* $T_t(A|B) + T_t(C|D) \le T_t(A + C|B + D)$.

(3′) *jointly concavity:* *For all* $s \in [0, 1]$
$$(1-s)T_t(A|B) + sT_t(C|D) \le T_t((1-s)A + sC|(1-s)B + sD).$$

(4) *homogeneity:* $T_t(\alpha A|\alpha B) = \alpha T_t(A|B)$ *for all* $\alpha > 0$.

(5) *affine parametrization:* $T_t(A|A \natural_s B) = sT_{ts}(A|B)$ *for* $t, s \in \mathbb{R}$ *with* $s, t \ne 0$.

(6) *orthogonality:* $$T_t\left(\bigoplus_k A_k|\bigoplus_k B_k\right) = \bigoplus_k T_t(A_k|B_k).$$

Proof. (1) follows from (1) of Lemma 2.1. (2) and (2') follows from Theorem 2.2. (3),(3') and (4) follows from Lemma 2.1. (5): By the definition of the quasi t-geomtric means, we have $A \natural_t(A\natural_s B) = A\natural_{st}B$ for $s, t \in \mathbb{R}$ with $s, t \ne 0$ and so

$$T_t(A|A \natural_s B) = \frac{A \natural_t(A\natural_s B) - A}{t} = \frac{A\natural_{st}B - A}{t} = s\frac{A\natural_{st}B - A}{st} = sT_{st}(A|B).$$

(6): Since the quasi t-geometric means $\natural_t$ for $t \in [-1, 0)$ satisfy the orthogonality in the invertible case, we have (6) under the existence of $T_t(A_k|B_k)$ for each k. □

By Lemma 2.3, we have the following existence condition such that $T_t(A|B)$ for $t \in [-1, 0)$ exists as a bounded operator:

Lemma 3.2. *Let A and B be positive operators and $t \in [-1, 0)$. Then $T_t(A|B)$ exists if and only if*

$$\inf_{\alpha>0}\left\{\frac{(1-t)\alpha^{-t}A + t\alpha^{1-t}B - A}{t}\right\} > -\infty.$$

Next, by Theorem 2.7, we have sufficient conditions that $T_t(A|B)$ exists for non-invertible case by virtue of the quasi t-geometric means.

Theorem 3.3. *The following implications* (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) *hold for any positive operators A, B and $t \in [-1, 0)$, and each converse does not always hold.*

(1) *majorization or range inclusion:* $A \le cB$ *for some* $c > 0$, *i.e.,*
$$\mathrm{ran}A^{\frac{1}{2}} \subset \mathrm{ran}B^{\frac{1}{2}}.$$
(2) *existence condition:* $T_t(A|B)$ *exists as a bounded operators, i.e.,*
$$\inf_{\alpha>0}\left\{\frac{(1-t)\alpha^{-t}A + t\alpha^{1-t}B - A}{t}\right\} > -\infty$$
(3) *kernel condition:* $\ker A \supset \ker B$.

Furuichi, Yanagi and Kuriyama [7] showed bounds of Tsallis relative operator entropy with $t \in (0,1]$ for invertible case. Thus, we study on bounds of Tsallis relative operator entropy with negative parameters for non-invertible case.

Let A be a positive operator and $t \in [-1,0)$. Put $H_t(A) = T_t(A|I) = \frac{A^{1-t}-A}{t}$ and it is called Tsallis operator entropy. Then $H_t(A)$ converges to the operator entropy $H(A) = H_0(A) = -A\log A$ as $t \nearrow 0$. For $x \ge 0$ and $t \in [-1,0)$, we denote the generalized logarithmic function by $\ln_t(x) = \frac{x^t-1}{t}$.

Theorem 3.4. *Let* A *and* B *be positive operators.*

(1) *If* $T_t(A|B)$ *exists for some* $t \in [-1,0)$, *then* $T_t(A|B) \le H_t(A) + A^{1-t}\ln_t\|B\|$.
(2) $A \le cB$ *for some* $c > 0$ *implies* $T_t(A|B) \ge (1-c)A$ *for all* $t \in [-1,0)$.

Proof. Since $T_t(A|B)$ exists for some $t \in [-1,0)$ and $B \le \|B\|$, it follows from the right monotonicity in Theorem 3.1 that
$$T_t(A|B) \le T_t(A|\|B\|) = \frac{A^{1-t}\|B\|^t - A}{t} = H_t(A) + A^{1-t}\ln_t\|B\|.$$
Next, suppose that $A \le cB$ for some $c > 0$. It follows from Theorem 3.3 that $T_t(A|B)$ exists for all $t \in [-1,0)$. Since $1 - \frac{1}{x} \le \frac{x^t-1}{t}$ for $x > 0$ and $t \in [-1,0)$, we have $A - A(B+\varepsilon)^{-1}A \le T_t(A|B+\varepsilon)$ for all $\varepsilon > 0$ and
$$A - A(B+\varepsilon)^{-1}A \ge A - A(\frac{A}{c}+\varepsilon)^{-1}A \to (1-c)A$$
as $\varepsilon \to 0$. Hence we have $T_t(A|B) \ge (1-c)A$. □

We have an upper semi-continuity of Tsallis relative operator entropy with negative parameters:

Lemma 3.5. *Let* A *and* B *be positive operators. If* $T_t(A|B)$ *exists for some* $t \in [-1,0)$, *then*
$$T_t(A+\varepsilon|B+\varepsilon) \searrow T_t(A|B)$$
as $\varepsilon \searrow 0$.

Proof. For $\varepsilon > \delta > 0$, it follows from the super-additivity of $\natural_t$ that
$$\begin{aligned} T_t(A+\varepsilon|B+\varepsilon) &= \frac{1}{t}\left[(A+\delta+\varepsilon-\delta)\ \natural_t\ (B+\delta+\varepsilon-\delta) - (A+\varepsilon)\right] \\ &\ge \frac{1}{t}\left[(A+\delta)\ \natural_t\ (B+\delta) + (\varepsilon-\delta)\ \natural_t\ (\varepsilon-\delta) - (A+\varepsilon)\right] \\ &= \frac{(A+\delta)\ \natural_t\ (B+\delta) - (A+\delta)}{t} = T_t(A+\delta|B+\delta). \end{aligned}$$

Put $a_{\delta_1,\delta_2} = \langle T_t(A+\delta_1|B+\delta_2+\delta_1)x, x\rangle$ and then a_{δ_1,δ_2} are monotone decreasing for $\delta_1, \delta_2 \downarrow 0$, i.e., $(\delta_1', \delta_2') \le (\delta_1, \delta_2)$ implies $a_{\delta_1',\delta_2'} \le a_{\delta_1,\delta_2}$. Since $B+\delta_2$ is invertible, we have $a_{\delta_1,\delta_2} \searrow a_{0,\delta_2}$ as $\delta_1 \to 0$. Moreover, since $T_t(A|B)$ exists, we have $a_{0,\delta_2} \searrow a_{0,0} = \langle T_t(A|B)x, x\rangle$ as $\delta_2 \to 0$. Also, we have $a_{\delta_1,\delta_2} \searrow a_{\delta_1,0}$ as $\delta_2 \searrow 0$. By Lemma 2.11, there exists the limit and we have

$$\lim_{\delta_1,\delta_2\downarrow 0} a_{\delta_1,\delta_2} = \lim_{\delta_1\downarrow 0} \big(\lim_{\delta_2\downarrow 0} a_{\delta_1,\delta_2} \big) = \lim_{\delta_2\downarrow 0} \big(\lim_{\delta_1\downarrow 0} a_{\delta_1,\delta_2} \big)$$

and we have the desired result. $\square$

Lemma 3.6. *Let A, B, B_n be positive operators for $n = 1, 2, \ldots$ and $T_t(A|B)$ exists for some $t \in [-1, 0)$. If $B_n \searrow B$ as $n \to \infty$, then $T_t(A|B_n) \searrow T_t(A|B)$ as $n \to \infty$.*

Proof. Since $B_n \searrow B$ as $n \to \infty$ and $T_t(A|B)$ exists, it follows from (1) of Theorem 3.1 that $T_t(A|B_n)$ exists for all n and $T_t(A|B) \le T_t(A|B_n)$. By Theorem 2.12, we have

$$T_t(A|B_n) = \frac{A \natural_t B_n - A}{t} \searrow \frac{A \natural_t B - A}{t} = T_t(A|B)$$

as $n \to \infty$. $\square$

For positive invertible operators A, B and $t \in [-1, 0)$, we have (1.4) in Introduction and so the positivity (resp. negativity) of $T_t(A|B)$ is equivalent to $B \ge A$ (resp. $B \le A$) and hence $T_t(A|B) = 0$ if and only if $A = B$. By virtue of Izumino's construction, we have the following same results for non-invertible case:

Theorem 3.7. *Let A and B be positive operators. Suppose that $T_t(A|B)$ exists for some $t \in [-1, 0)$. Then $T_t(A|B) \ge 0$ (resp. $T_t(A|B) \le 0$) if and only if $B \ge A$ (resp. $B \le A$).*

Proof. If $T_t(A|B) \ge 0$ for some $t \in [-1, 0)$, then $A \natural_t B - A \le 0$ and so

$$0 \ge A \natural_t B - A \ge (1-t)A + tB - A = t(B - A).$$

Hence we have $A \le B$. Conversely, if $B \ge A$, then we have

$$T_t(A+\varepsilon|B+\varepsilon) \ge T_t(A+\varepsilon|A+\varepsilon) = 0$$

for all $\varepsilon > 0$ and so it follows from Lemma 3.5 that $T_t(A|B) \ge 0$ as $\varepsilon \to 0$.

Next, suppose that $A \ge B$. Then we have

$$T_t(A+\varepsilon|B+\varepsilon) \le T_t(A+\varepsilon|A+\varepsilon) = 0$$

and so it follows from Lemma 3.5 that $0 \ge T_t(A+\varepsilon|B+\varepsilon) \searrow T_t(A|B)$ as $\varepsilon \to 0$. Hence we have $T_t(A|B) \le 0$.

Suppose that $T_t(A|B) \le 0$. Put $R = (A+B)^{1/2}$. By Douglas majorization theorem, there exist the derivatives D and E such that $A^{1/2} = DR$ and $B^{1/2} = ER$ with $\ker R \subset \ker D \cap \ker E$ and $\ker D^* = \ker A$ and $\ker E^* = \ker B$. In this case, we have $D^*D + E^*E = I_R$ and D^*D commutes with E^*E. For each $\varepsilon > 0$, we define an operator $(E^*E)_\varepsilon : \overline{\mathrm{ran} R} \mapsto \overline{\mathrm{ran} R}$ by

$$(E^*E)_\varepsilon x = \begin{cases} E^*Ex & \text{for } x \in \mathrm{ran} Q_{[0,1-\varepsilon)} \\ \varepsilon x & \text{for } x \in (\mathrm{ran} Q_{[0,1-\varepsilon)})^\perp \end{cases} \tag{3.1}$$

where $Q_{[0,1-\varepsilon)}$ is the spectral projection of D^*D corresponding to $[0,1-\varepsilon)$. Then $(E^*E)_\varepsilon$ is invertible for all $\varepsilon > 0$ and $(E^*E)_\varepsilon \searrow E^*E$ as $\varepsilon \to 0$. Moreover, we have

$$R(D^*D \natural_t (E^*E)_\varepsilon)R \nearrow R(D^*D \natural_t E^*E)R \quad \text{as } \varepsilon \to 0.$$

In fact, since $E^*E \le (E^*E)_\varepsilon \le E^*E + \varepsilon$, it follows that

$$\begin{aligned} RD^*DR \natural_t R(E^*E+\varepsilon)R &\le R[D^*D \natural_t (E^*E+\varepsilon)]R \\ &\le R[D^*D \natural_t (E^*E)_\varepsilon]R \\ &\le R[D^*D \natural_t E^*E]R \end{aligned}$$

for all $\varepsilon > 0$ and by Theorem 2.13, we have

$$\operatorname*{s\text{-}lim}_{\varepsilon\to 0} R[D^*D \natural_t (E^*E)_\varepsilon]R = A \natural_t B = R(D^*D \natural_t E^*E)R.$$

Now, suppose that there exists an interval $[a,b]$ such that $b < 1/2$ and $[a,b] \subset \sigma(D^*D)$. For $R\xi \in \mathrm{ran}Q_{[a,b]}$, where $Q_{[a,b]}$ is the spectral projection of D^*D corresponding to $[a,b]$ we have

$$a\langle R\xi, R\xi\rangle \le \langle A\xi, \xi\rangle = \langle D^*DR\xi, R\xi\rangle \le b\langle R\xi, R\xi\rangle$$

By definition of $(E^*E)_\varepsilon$, there exists a constant $c \in \mathbb{R}$ such that $\langle R[D^*D \natural_t (E^*E)_\varepsilon]R\xi, \xi\rangle = c$ for sufficient small $\varepsilon > 0$. Hence we have

$$\begin{aligned} &\langle R[D^*D \natural_t (E^*E)_\varepsilon]R\xi, \xi\rangle = c = \langle R[D^*D \natural_t E^*E]R\xi, \xi\rangle \\ &= \langle A \natural_t B\xi, \xi\rangle \ge \langle A\xi, \xi\rangle = \langle D^*DR\xi, R\xi\rangle. \end{aligned}$$

Let $C^*(D^*D)$ be a commutative C*-algebra generated by D^*D and I_R, and then by spectral theorem there is an isometric isomorphism Ψ of $C^*(D^*D)$ onto $C[0,1]$ a set of real valued continuous function on $[0,1]$ such that $\Psi(D^*D) = f$, $\Psi(E^*E) = g$ and $\Psi((E^*E)_\varepsilon) = g_\varepsilon$, where $f(x) = x$, $g(x) = 1-x$ and

$$g_\varepsilon(x) = \begin{cases} 1-x & \text{for } x \in [0, 1-\varepsilon) \\ \varepsilon & \text{for } x \in [1-\varepsilon, 1]. \end{cases}$$

The inequality

$$\langle R[D^*D \natural_t (E^*E)_\varepsilon]R\xi, \xi\rangle \ge \langle D^*DR\xi, R\xi\rangle$$

corresponds to $x^{1-t}g_\varepsilon(x)^t \ge x$ and so $x \ge 1/2$. Hence we have $\langle D^*DR\xi, R\xi\rangle \ge 1/2\langle R\xi, R\xi\rangle$ and this fact contradicts $\langle D^*DR\xi, R\xi\rangle \le b\langle R\xi, R\xi\rangle < \frac{1}{2}\langle R\xi, R\xi\rangle$. Hence we have $\sigma(D^*D) \subset [1/2, 1]$. Thus $D^*D \ge \frac{1}{2}I_R$ and $E^*E \le \frac{1}{2}I_R$ and so $A = RD^*DR \ge RE^*ER = B$. □

Finally, to show a lower semi-continuity of Tsallis relative operator entropy with negative parameters for non-invertible case, we need the following Lemma. Let A and B be positive operators. Put $R = (A+B)^{1/2}$ and D, E be derivatives such that $A^{1/2} = DR$ and $B^{1/2} = ER$. Then we have $D^*D + E^*E = I_R$.

Lemma 3.8. *Let A and B be positive operators. If $T_t(A|B)$ exists for some $t \in [-1, 0)$, then*

$$RT_t(D^*D|(E^*E)_\varepsilon)R \searrow RT_t(D^*D|E^*E)R = T_t(RD^*DR|RE^*ER) = T_t(A|B)$$

as $\varepsilon \to 0$, where $(E^*E)_\varepsilon$ is defined by (3.1) for $\varepsilon > 0$.

Proof. Since $T_t(A|B)$ exists, we have

$$\begin{aligned} T_t(A|B) &= \frac{A \natural_t B - A}{t} \\ &\leq \frac{(1-t)\alpha^{-t}A + t\alpha^{1-t}B - A}{t} \\ &= R\frac{(1-t)\alpha^{-t}D^*D + t\alpha^{1-t}E^*E - D^*D}{t}R \\ &\leq R\frac{(1-t)\alpha^{-t}D^*D + t\alpha^{1-t}(E^*E)_\varepsilon - D^*D}{t}R \end{aligned}$$

for all $\alpha > 0$ and so

$$\begin{aligned} T_t(A|B) &\leq RT_t(D^*D|(E^*E)_\varepsilon)R \\ &\leq T_t(RD^*DR|R(E^*E)_\varepsilon R) \\ &\to T_t(RD^*DR|RE^*ER) = T_t(A|B) \end{aligned}$$

as $\varepsilon \to 0$. Hence we have

$$RT_t(D^*D|(E^*E)_\varepsilon)R \searrow RT_t(D^*D|E^*E)R = T_t(A|B).$$

$\square$

Similarly, we have the relative operator entropy version of Lemma 3.8:

Lemma 3.9. *Let A and B be positive operators. If the relative operator entropy $S(A|B)$ exists, then*

$$RS(D^*D|(E^*E)_\varepsilon)R \searrow RS(D^*D|E^*E)R = S(RD^*DR|RE^*ER) = S(A|B)$$

as $\varepsilon \to 0$, where $S(A|B)$ is defined by (1.1) and $(E^*E)_\varepsilon$ is defined by (3.1) for $\varepsilon > 0$.

For positive invertible case, we have (1.5). Let A and B be positive operators. For $t \in (0,1]$, Tsallis relative operator entropy $T_t(A|B)$ always exists and $T_t(A|B) \searrow S(A|B)$ as $t \searrow 0$. Similarly, we show a lower semi-continuity of Tsallis relative operator entropy with negative parameters for non-invertible case:

Theorem 3.10. *Let A and B be positive operators. If $T_{t_0}(A|B)$ exists for some $t_0 \in [-1,0)$, then*

$$T_t(A|B) \nearrow S(A|B) \qquad \textit{for } t_0 \leq t \nearrow 0.$$

Proof. Firstly, we show the monotonicity of Tsallis relative entropy with negative parameters for non-invertible case. For $-1 \leq t_0 < t < s < 0$ and $x > 0$, we have

$$\frac{x^t - 1}{t} \leq \frac{x^s - 1}{s} \leq \log x$$

and hence

$$\begin{aligned} &\frac{(A+\delta)\natural_t(B+\varepsilon+\delta) - (A+\delta)}{t} \leq \frac{(A+\delta)\natural_s(B+\varepsilon+\delta) - (A+\delta)}{s} \\ &\leq (A+\delta)^{1/2}\left[\log(A+\delta)^{-1/2}(B+\varepsilon+\delta)(A+\delta)^{-1/2}\right](A+\delta)^{1/2} \end{aligned}$$

for all $\delta > 0$ and $\varepsilon > 0$. Since $B+\varepsilon$ is invertible, as $\delta \to 0$ we have

$$\frac{A \natural_t (B+\varepsilon) - A}{t} \le \frac{A \natural_s (B+\varepsilon) - A}{s} \le S(A|B+\varepsilon).$$

Hence for $-1 \le t_0 < t < s < 0$

$$T_{t_0}(A|B+\varepsilon) \le T_t(A|B+\varepsilon) \le T_s(A|B+\varepsilon) \le S(A|B+\varepsilon)$$

and since $T_{t_0}(A|B)$ exists, as $\varepsilon \to 0$, it follows that $T_t(A|B), T_s(A|B)$ and $S(A|B)$ exist and we have the desired monotonicity

$$T_{t_0}(A|B) \le T_t(A|B) \le T_s(A|B) \le S(A|B).$$

Since $T_t(A|B)$ is monotone increasing for $t \nearrow 0$ and has an upper bound $S(A|B)$, there exists a strong operator limit $T_0(A|B) = \text{s-lim}_{t\nearrow 0} T_t(A|B)$ and $T_0(A|B) \le S(A|B)$. By definition, it follows that $T_0(A|B)$ has the following properties:

(1) right monotonicity: $B \le B' \quad \Longrightarrow \quad T_0(A|B) \le T_0(A|B')$;
(2) transformer inequality: $X^*T_0(A|B)X \le T_0(X^*AX|X^*BX)$ for every X;
(3) right upper semi-continuity: $B_n \searrow B \quad \Longrightarrow \quad T_0(A|B_n) \searrow T_0(A|B)$.

The third property follows from the same way in the proof of Lemma 3.6 and Theorem 2.12.

For a given $\varepsilon > 0$, let $Q_{[0,1-\varepsilon)}$ be the spectral projection of D^*D corresponding to $[0, 1-\varepsilon)$. For $R\xi \in \text{ran}Q_{[0,1-\varepsilon)}$ we have

$$\langle T_t(D^*D|(E^*E)_\varepsilon)R\xi, R\xi\rangle = \langle RT_t(D^*D|E^*E)R\xi, \xi\rangle = \langle T_t(A|B)\xi, \xi\rangle.$$

In fact, for all $\varepsilon > \varepsilon' > 0$

$$\langle T_t(D^*D|(E^*E)_{\varepsilon'})R\xi, R\xi\rangle = \langle T_t(D^*D|(E^*E)_\varepsilon)R\xi, R\xi\rangle \ge \langle RT_t(D^*D|E^*E)R\xi, \xi\rangle$$

and by Lemma 3.8, we have $RT_t(D^*D|(E^*E)_\varepsilon)R \searrow RT_t(D^*D|E^*E)R = T_t(A|B)$ and so $\langle T_t(D^*D|(E^*E)_\varepsilon)R\xi, R\xi\rangle = \langle RT_t(D^*D|E^*E)R\xi, \xi\rangle$.

Also, since $T_t(A|B)$ exists, it follows from Theorem 3.3 that $\ker A \supset \ker B$ and so $\ker R = \ker(A+B) = \ker A \cap \ker B = \ker B$. Hence D^*D and E^*E have a trivial kernel on $\overline{\text{ran}R}$.

Now, it remains to show that $T_0(A|B) = S(A|B)$. Conversely, suppose that $T_0(A|B) \ne S(A|B)$. Then there exist a constant $\delta_0 > 0$ and $\xi_0 \in H$ such that

$$\langle S(A|B)\xi_0, \xi_0\rangle - \langle T_0(A|B)\xi_0, \xi_0\rangle = \delta_0 > 0. \tag{3.2}$$

On the other hand, by Lemma 3.9

$$S(A|B) = RS(D^*D|E^*E)R = \operatorname*{s-lim}_{\varepsilon\to 0} RS(D^*D|(E^*E)_\varepsilon)R$$

and moreover

$$T_0(A|B) = RT_0(D^*D|E^*E)R = \operatorname*{s-lim}_{\varepsilon\to 0} RT_0(D^*D|(E^*E)_\varepsilon)R.$$

In fact, since $T_t(A|B) \le RT_t(D^*D|E^*E+\varepsilon)R$ for all $t_0 < t < 0$, we have

$$\begin{aligned} T_0(A|B) &\le RT_0(D^*D|E^*E+\varepsilon)R \\ &\le T_0(RD^*DR|RE^*ER+\varepsilon R^2) \\ &\le T_0(A|B+\varepsilon\left\|R^2\right\|) \to T_0(A|B) \quad \text{as } \varepsilon \to 0 \end{aligned}$$

and so $T_0(A|B) = RT_0(D^*D|E^*E)R = \text{s-lim}_{\varepsilon\to 0} RT_0(D^*D|E^*E+\varepsilon)R$. Note that
$$T_0(RD^*DR|R(E^*E)_\varepsilon R) = RT_0(D^*D|(E^*E)_\varepsilon)R.$$
Because by the invertibility of $(E^*E)_\varepsilon$, for $t \in [-1,0)$ we have
$$\begin{aligned}
&T_t(RD^*DR|R(E^*E)_\varepsilon R)\\
&\quad\le \frac{1}{t}\left[(1-t)\alpha^{-t}RD^*DR + t\alpha^{1-t}R(E^*E)_\varepsilon R - RD^*DR\right]\\
&\quad= R\cdot\frac{1}{t}\left[(1-t)\alpha^{-t}D^*D + t\alpha^{1-t}(E^*E)_\varepsilon - D^*D\right]R\\
&\quad= R\cdot\frac{1}{t}\left[(E^*E)_\varepsilon^{1/2}L_{\alpha,t}((E^*E)_\varepsilon^{-1/2}D^*D(E^*E)_\varepsilon^{-1/2}, I)(E^*E)_\varepsilon^{1/2} - D^*D\right]R
\end{aligned}$$
for all $\alpha > 0$ and so
$$T_t(RD^*DR|R(E^*E)_\varepsilon R) \le R\left[\frac{D^*D \natural_t (E^*E)_\varepsilon - D^*D}{t}\right]R = RT_t(D^*D|(E^*E)_\varepsilon)R.$$
By the transformer inequality of $T_t(A|B)$, we have
$$RT_t(D^*D|(E^*E)_\varepsilon)R \le T_t(RD^*DR|R(E^*E)_\varepsilon R)$$
and so $RT_t(D^*D|(E^*E)_\varepsilon)R = T_t(RD^*DR|R(E^*E)_\varepsilon R)$ for all $t \in [-1,0)$. Hence as $\varepsilon \to 0$ we have $T_0(RD^*DR|R(E^*E)_\varepsilon R) = RT_0(D^*D|(E^*E)_\varepsilon)R$.

By the discussion above, we have
$$\begin{aligned}
T_0(A|B) &\le T_0(RD^*DR|R(E^*E)_\varepsilon R) = RT_0(D^*D|(E^*E)_\varepsilon)R\\
&\le T_0(RD^*DR|R(E^*E+\varepsilon)R)\\
&\le T_0(A|B+\varepsilon\left\|R^2\right\|)\\
&\to T_0(A|B) \qquad \text{as } \varepsilon \to 0
\end{aligned}$$
by the right upper semi-continuity of $T_0(A|B)$ and so
$$T_0(A|B) = \operatorname*{s-lim}_{\varepsilon\to 0} RT_0(D^*D|(E^*E)_\varepsilon)R.$$
Since $T_t(D^*D|(E^*E)_\varepsilon) \to S(D^*D|(E^*E)_\varepsilon)$ as $t \nearrow 0$ by the invertibility of $(E^*E)_\varepsilon$, we have
$$\begin{aligned}
T_t(RD^*DR|R(E^*E)_\varepsilon R) &= RT_t(D^*D|(E^*E)_\varepsilon)R\\
&\qquad\to RT_0(D^*D|(E^*E)_\varepsilon)R = RS(D^*D|(E^*E)_\varepsilon)R
\end{aligned}$$
as $t \nearrow 0$. For ξ_0 in (3.2), we can take $\varepsilon > 0$ such that
$$0 < \langle RT_0(D^*D|(E^*E)_\varepsilon)R\xi_0,\xi_0\rangle - \langle T_0(A|B)\xi_0,\xi_0\rangle < \frac{1}{2}\delta_0.$$
However, we have
$$\begin{aligned}
0 &= \langle RS(D^*D|(E^*E)_\varepsilon)R\xi_0,\xi_0\rangle - \langle RT_0(D^*D|(E^*E)_\varepsilon)R\xi_0,\xi_0\rangle\\
&> \langle S(A|B)\xi_0,\xi_0\rangle - \langle T_0(A|B)\xi_0,\xi_0\rangle - \frac{1}{2}\delta_0 = \frac{1}{2}\delta_0
\end{aligned}$$
and this contradicts. Hence we have $T_0(A|B) = S(A|B)$ and so
$$T_t(A|B) \nearrow S(A|B) \qquad \text{for } t_0 \le t \nearrow 0.$$

□

Acknowledgments. The authors would like to express their hearty thanks to the referees for their valuable suggestions and comments for revising the manuscript. This work is partially supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Scientific Research (C), JSPS KAKENHI Grant Number JP 16K05253.

References

1. S. Abe, *Monotonic decrease of the quantum nonadditive divergence by projective measurements*, Phys. Lett. A **312** (2003), no. 5-6, 336–338.
2. J. I. Fujii, *On Izumino's view of operator means*, Math. Japon. **33** (1988), no. 5, 671–675.
3. J. I. Fujii and E. Kamei, *Relative operator entropy in noncommutative information theory*, Math. Japon. **34** (1989), no. 3, 341–348.
4. J. I. Fujii and E. Kamei, *Uhlmann's interpolational method for operator means*, Math. Japon. **34** (1989), no. 4, 541–547.
5. J. I. Fujii and Y. Seo, *The relative operator entropy and the Karcher mean*, Linear Algebra Appl. (to appear), DOI: 10.1016/j.laa.2016.11.018.
6. S. Furuichi, K. Yanagi, and K. Kuriyama, *Fundamental properties of Tsallis relative entropy*, J. Math. Phys. **45** (2004), no. 12, 4868–4877.
7. S. Furuichi, K. Yanagi and K. Kuriyama, *A note on operator inequalities of Tsallis relative operator entropy*, Linear Algebra Appl. **407** (2005), 19–31.
8. T. Furuta, *Invitation to linear operators*, Taylor & Francis, Ltd., London, 2001.
9. T. Furuta, J.Mićić, J.E. Pečarić, and Y. Seo, *Mond-Pečarić Method in Operator Inequalities*, Monographs in Inequalities **1**, Element, Zagreb, 2005.
10. S. Izumino, *Quotients of bounded operators*, Proc. Amer. Math. Soc. **106** (1989),no. 2, 427–435.
11. F. Kubo and T. Ando, *Means of positive linear operators*, Math. Ann. **246**(1980), no. 3, 205–224.
12. M. Nakamura and H. Umegaki, *A note on the entropy for operator algebras*, Proc. Japan Acad. **37** (1961), 149–154.
13. C. Tsallis, *Possible generalization of Boltzmann–Gibbs statistics*, J. Statist. Phys. **52** (1988), no. 1-2, 479–487.
14. K. Yanagi, K. Kuriyama, and S. Furuichi, *Generalized Shannon inequalities based on Tsallis relative operator entropy*, Linear Algebra Appl. **394** (2005), 109–118.

[1]Department of Arts and Sciences (Information Science), Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582, Japan.
E-mail address: fujii@cc.osaka-kyoiku.ac.jp

[2]Department of Mathematics Education, Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582, Japan.
E-mail address: yukis@cc.osaka-kyoiku.ac.jp