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ON THE NUMERICAL RADIUS OF A QUATERNIONIC NORMAL OPERATOR

GOLLA RAMESH

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ABSTRACT. We prove that for a right linear bounded normal operator on a quaternionic Hilbert space (quaternionic bounded normal operator) the norm and the numerical radius are equal. As a consequence of this result we give a new proof of the known fact that a non zero quaternionic compact normal operator has a non zero right eigenvalue. Using this we give a new proof of the spectral theorem for quaternionic compact normal operators. Finally, we show that every quaternionic compact operator is norm attaining and prove the Lindenstrauss theorem on norm attaining operators, namely, the set of all norm attaining quaternionic operators is norm dense in the space of all bounded quaternionic operators defined between two quaternionic Hilbert spaces.

1. INTRODUCTION

It is well known that for a bounded normal operator on a complex Hilbert space, the norm and the numerical radius are the same. In this note, we prove this result for right linear normal operators on a quaternionic Hilbert space. As a consequence of this result, we show that every compact normal operator on a quaternionic Hilbert space has a non zero right eigenvalue. This is a crucial point in proving the spectral representation theorem for such operators.

The spectral theorem for compact normal operators on a quaternionic Hilbert space is appeared in a recent article by Ghiloni et al [4]. The authors mainly used the left multiplication on the space of all bounded right linear operators

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on a quaternionic Hilbert space. The spectral theorem for matrices with quaternionic entries was studied in [1]. In this article, we give a new proof of the spectral theorem for compact operators on general quaternionic Hilbert spaces (see [4, Theorem 1.2]). First, we prove that the norm and the numerical radius of a quaternionic normal operator are the same. To prove this, we associate a unique complex normal operator with the given quaternionic normal operator which preserve the norm and the numerical radius. Using this technique and the classical result, we obtain the result. As a consequence, we prove that a quaternionic compact normal operator has a non zero right eigenvalue. Finally, with this idea, we give a new proof of the spectral theorem for quaternionic compact normal operators. Later, we extend the Lindenstrauss theorem on norm attaining operators to the quaternionic case. A simple proof in the classical case can be found in [6].

Organization of the article: In the second section we give necessary details of quaternionic Hilbert spaces and right linear operators on such spaces. In the third section we prove that for a normal quaternionic operator the norm and the numerical radius are equal. Using this we prove the spectral theorem for quaternionic compact operators. In the final section, we consider the norm attaining operators and prove the well known Lindenstrauss theorem on norm attaining operators in the case of quaternionic operators.

2. Preliminaries

We denote the division ring of real quaternions by $\mathbb{H}$. If $q \in \mathbb{H}$, then $q = q_0 + q_1 i + q_2 j + q_3 k$, where $q_n \in \mathbb{R}$ for $n = 0, 1, 2, 3$ and i, j, k satisfy the following conditions:

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ \text{and } ki = -ik = j.$$

The conjugate of q is $\overline{q} = q_0 - q_1 i - q_2 j - q_3 k$ and $|q| := \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$. The imaginary part of $\mathbb{H}$ is defined by $Im(\mathbb{H}) = \{q \in \mathbb{H} : q = -\overline{q}\}$. The set of all unit imaginary quaternions is denoted by $\mathbb{S}$, that is $\mathbb{S} := \{q \in Im(\mathbb{H}) : |q| = 1\}$ and the unit sphere of $\mathbb{H}$ by $S_{\mathbb{H}}$.

Here we list out some of the properties of quaternions, which we need later.

(1) For $p, q \in \mathbb{H}$, we have $\overline{pq} = \overline{q}\,\overline{p}$, $|pq| = |p||q|$ and $|\overline{p}| = |\overline{q}|$.
(2) We define an equivalence relation on $\mathbb{H}$ as, $p \sim q$ if and only if $p = s^{-1}qs$ for some $s \neq 0 \in \mathbb{H}$. The equivalence class of p is $[p] := \{s^{-1}qs : 0 \neq s \in H\}$.
(3) For each $m \in \mathbb{S}, \mathbb{C}_m := \{\alpha + m\beta : \alpha, \beta \in \mathbb{R}\}$ is a real subalgebra of $\mathbb{H}$ and is called as the slice complex plane generated by 1 and m.
(4) We have $\mathbb{C}_m \cap \mathbb{C}_n = \mathbb{R}$ if $m \neq \pm n$, and $\mathbb{H} = \cup_{m \in \mathbb{S}} \mathbb{C}_m$.

A right $\mathbb{H}$-module H is called a quaternionic pre-Hilbert space if there exists a Hermitian quaternionic scalar product; namely a function $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{H}$ satisfying the following:

(1) $\langle u, vp + wq \rangle = \langle u, v \rangle p + \langle u, w \rangle q$ for all $u, v, w \in H$ and $p, q \in \mathbb{H}$
(2) $\langle u, v \rangle = \overline{\langle v, u \rangle}$ for all $u, v \in H$
(3) $\langle u, u \rangle \geq 0$ for all $u \in H$ and $\langle u, u \rangle = 0$ iff $u = 0$.

Let H be a quaternionic pre-Hilbert space with Hermitian quaternionic scalar product $\langle \cdot, \cdot \rangle$ on H. Such an inner product $\langle \cdot, \cdot \rangle$ satisfies the Cauchy-Schwarz inequality:

$$|\langle u, v\rangle|^2 \leq \langle u, u\rangle \langle v, v\rangle \text{ for all } u, v \in H.$$

Define $\|u\| = \langle u, u\rangle^{\frac{1}{2}}$, for every $u \in H$. Then $\|\cdot\|$ is a norm in the usual real sense. If the normed space $(H, \|\cdot\|)$ is complete, then H is called a quaternionic Hilbert space.

The norm induced by this inner product satisfy the parallelogram law:

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2) \text{ for all } u, v \in H.$$

We denote the unit sphere of the Hilbert space H by S_H.

An operator $T : H \to H$ is said to be right linear if

(1) $T(x+y) = Tx + Ty$ for all $x, y \in H$
(2) $T(xq) = (Tx)q$ for all $x \in H,\ q \in \mathbb{H}$.

A right linear operator $T : H \to H$ is said to be bounded if there exists a $M > 0$ such that $\|Tx\| \leq M\|x\|$ for all $x \in H$. For such an operator the norm is defined by

$$\|T\| = \sup\{\|Tu\| : u \in S_H\}.$$

We denote the space of all bounded right linear operators on H by $\mathcal{B}(H)$. For $T \in \mathcal{B}(H)$, the null space is defined by $N(T) = \{x \in H : Tx = 0\}$ and the range space is defined by $R(T) = \{Tx : x \in H\}$.

Let $T \in \mathcal{B}(H)$. Then there exists a unique operator $T^* \in \mathcal{B}(H)$ such that $\langle u, Tv\rangle = \langle T^*u, v\rangle$ for all $u, v \in H$. This operator T^* is called the adjoint of T.

Let $T \in \mathcal{B}(H)$. Then T is said to be self-adjoint if $T = T^*$, anti self-adjoint if $T^* = -T$, normal if $TT^* = T^*T$ and unitary if $TT^* = T^*T = I$. If T is self-adjoint and $\langle x, Tx\rangle \geq 0$ for all $x \in H$, then T is said to be positive.

Let $T \in \mathcal{B}(H)$ be positive. Then there exists a unique positive operator $S \in \mathcal{B}(H)$ such that $S^2 = T$. Such a S is called the square root of T and is denoted by $S = T^{\frac{1}{2}}$.

If $S \in \mathcal{B}(H)$, then the operator $|S| := (S^*S)^{\frac{1}{2}}$ is called as the modulus of S. In fact, there exists a partial isometry V ($\|Vx\| = \|x\|$ for all $x \in N(V)^{\perp}$) such that $T = V|T|$ and $N(V) = N(T)$. This decomposition is unique and is known as the *polar decomposition* of T (We refer to [2, Theorem 2.20] for more details).

Let $T \in \mathcal{B}(H)$ and $q \in \mathbb{H}$. Define

$$\Delta_q(T) := T^2 - T(q + \overline{q}) + I|q|^2,$$

where, if $r \in \mathbb{R}$, the operator $Tr \in \mathcal{B}(H)$ is defined by setting $(Tr)x := (Tx)r$ for all $x \in H$. The spherical spectrum of T is defined as

$$\sigma_S(T) := \{q \in \mathbb{H} : \Delta_q(T) \text{ is not invertible in } \mathcal{B}(H)\}.$$

The spherical point spectrum is defined as

$$\sigma_{pS}(T) := \{q \in \mathbb{H} : \Delta_q(T) \text{ is not one-to-one }\}.$$

All the above mentioned material can be found in [2].

The numerical range and the numerical radius of T are defined by

$$W(T) = \{\langle x, Tx\rangle : x \in S_H\}$$

and

$$w(T) = \sup\{|\langle x, Tx\rangle| : x \in S_H\},$$

respectively.

For a normal operator on a complex Hilbert space the numerical range is convex, whereas, this is not the case for normal operators on quaternionic Hilbert spaces (see [3] for details).

Let $T \in \mathcal{B}(H)$. Then T is said to be compact if $T(B)$ is pre-compact for every bounded subset B of H. Equivalently, $(T(x_n))$ has a convergent subsequence for every bounded sequence (x_n) of H.

3. Numerical radius of a normal operator

Suppose that H is a non zero quaternionic Hilbert space with Hermitian quaternionic scalar product $\langle\cdot,\cdot\rangle$. Let $m \in \mathbb{S}$ and $J \in \mathcal{B}(H)$ be an anti self-adjoint, unitary operator. Define $H_{\pm}^{Jm} := \{u \in H : Ju = \pm um\}$. Then $H_{\pm}^{Jm}$ is a non-zero closed subset of H. The restriction of the inner product on H to $H_{\pm}^{Jm}$ is a $\mathbb{C}_m$-valued inner product and with respect to this inner product $H_{\pm}^{Jm}$ is a Hilbert space. In fact, if we consider H as a $\mathbb{C}_m$ linear space, H has the decomposition: $H = H_{+}^{Jm} \oplus H_{-}^{Jm}$ (see [2, pages 21-22] for details). We need the following results to prove our main theorem.

Proposition 3.1. [2, Proposition 3.11] *If $T\colon H_{+}^{Jm} \to H_{+}^{Jm}$ is a bounded $\mathbb{C}_m-$ linear operator, then there exists unique bounded, right $\mathbb{H}-$ linear operator $\widetilde{T}\colon H \to H$ such that $\widetilde{T}(u) = T(u)$, for every $u \in H_{+}^{Jm}$.*

Furthermore,

1. $\|\widetilde{T}\| = \|T\|$
2. $J\widetilde{T} = \widetilde{T}J$
3. $(\widetilde{T})^* = \widetilde{T^*}$
4. *If $S\colon H_{+}^{Jm} \to H_{+}^{Jm}$ is a bounded $\mathbb{C}_m-$ linear operator, then $\widetilde{ST} = \widetilde{S}\widetilde{T}$*
5. *If S is the inverse of T, then $\widetilde{S}$ is the inverse of $\widetilde{T}$.*

On the other hand, if $V \in \mathcal{B}(H)$, then there exists a unique $U \in \mathcal{B}(H_{+}^{Jm})$ such that $\widetilde{U} = V$ if and only if $JV = VJ$.

If T is normal (but not self-adjoint), there exists an anti self-adjoint, unitary $J \in \mathcal{B}(H)$ such that $TJ = JT$ (see [2, Theorem 5.9] for details). Hence Proposition 3.1 holds with $V = T$. If T is self-adjoint, then the existence of an anti self-adjoint, unitary $J \in \mathcal{B}(H)$ such that $TJ = JT$ is guaranteed by [2, Theorem 5.7(b)].

Remark 3.2. If $S, T \in \mathcal{B}(H_{+}^{Jm})$, then it can be easily shown that $\widetilde{S+T} = \tilde{S} + \tilde{T}$ by following the same steps as in [2, Proposition 3.11].

Theorem 3.3. *Let $T \in \mathcal{B}(H)$ be normal. Then $w(T) = \|T\|$.*

Proof. First note that $S_{H_+^{Jm}} \subseteq S_H$. Let $T_+ \in \mathcal{B}(H_+^{Jm})$ be such that $\widetilde{T_+} = T$ as in Proposition 3.1. Then

$$\begin{aligned} w(T) = \sup\{|\langle Tx, x\rangle| : x \in S_H\} &\geq \sup\{|\langle Tx, x\rangle| : x \in S_{H_+^{Jm}}\} \\ &= \sup\{|\langle T_+x, x\rangle| : x \in S_{H_+^{Jm}}\} \\ &= w(T_+). \end{aligned}$$

Since T_+ is normal, we have $\|T_+\| = w(T_+)$. But $\|T_+\| = \|T\|$. This shows that $w(T) \geq \|T\|$. But the other inequality is clear. Thus $w(T) = \|T\|$. □

As a consequence we obtain a new proof of the following known result.

Theorem 3.4. [4, Theorem 1.1] *If $T \in \mathcal{B}(H)$ is compact and normal, then there exists a $q \in \sigma_{pS}(T)$ such that $|q| = \|T\|$.*

Proof. If $T = 0$, then it suffices to set $q = 0$. Suppose $T \neq 0$. By Theorem 3.3, there exists a sequence (x_n) in S_H such that $|\langle x_n, Tx_n\rangle| \to \|T\|$ as $n \to \infty$. If necessary, choose a subsequence of (x_n), (we again denote it by (x_n)) such that $\langle x_n, Tx_n\rangle \to q$ for some $q \in \mathbb{H} \setminus \{0\}$ with $|q| = \|T\|$. Since T is compact, there exists a subsequence (x_{n_k}) of (x_n) such that (Tx_{n_k}) is convergent. Let $y := \lim\limits_{k\to\infty} Tx_{n_k}$. Observe that $\|Tx_{n_k}\| \leq \|T\|\|x_{n_k}\| \leq \|T\|$ for every k. It follows that $\|y\| \leq \|T\| = |q|$. Then

$$\begin{aligned} \|Tx_{n_k} - x_{n_k}q\|^2 &= \langle Tx_{n_k} - x_{n_k}q, Tx_{n_k} - x_{n_k}q\rangle \\ &= \|Tx_{n_k}\|^2 - \overline{\langle x_{n_k}, Tx_{n_k}\rangle} q - \bar{q}\langle x_{n_k}, Tx_{n_k}\rangle + |q|^2 \\ &\to \|y\|^2 - |q|^2 \leq 0. \end{aligned}$$

Hence $\|y\| = |q|$; in particular, $y \neq 0$. So $T(x_{n_k}) - x_{n_k}q \to 0$ as $n \to \infty$. Since (Tx_{n_k}) converges to y, it follows that $x_{n_k}q \to y$. Thus $Ty = \lim\limits_{k\to\infty} T(x_{n_k}q) = \lim\limits_{k\to\infty} T(x_{n_k})q = yq$. Thanks to Proposition 4.5 of [4], we have that $q \in \sigma_{pS}(T)$. □

Using Theorem 3.4 as in the case of complex compact operators, we can give a new proof of the spectral theorem for quaternionic compact normal operators (see [4, Theorem 1.2]).

Definition 3.5. Let H_0 be a quaternionic closed subspace of a quaternionic Hilbert space H. Then H_0 is said to be invariant under $T \in \mathcal{B}(H)$ if $T(H_0) \subseteq H_0$. If H_0 and $H_0^{\perp}$ are both invariant under T, then H_0 is said to be a reducing subspace for T.

Example 3.6. Let $T \in \mathcal{B}(H)$ and $T\phi = \phi q$, where $|q| = \|T\|$ and $\phi \in S_H$. Then $H_0 := \text{span}_{\mathbb{H}}\{\phi\}$ is a non trivial reducing subspace for T. As H_0 is right linear we can see that H_0 is an invariant subspace for T. To show that H_0 reduces T, it is enough to prove H_0 to be invariant under T^*. Note that

$$|q|^2 = \|T\phi\|^2 = \langle T^*T\phi, \phi\rangle = \bar{q}\langle T^*\phi, \phi\rangle.$$

Thus $\langle T^*\phi, \phi\rangle = q$. As $|\langle T^*\phi, \phi\rangle| = |q|$, we have $T^*\phi = \phi p$ for some $p \in \mathbb{H}$. Then $q = \langle T^*\phi, \phi\rangle = \langle \phi p, \phi\rangle = \bar{p}$. Thus $p = \bar{q}$. As T^* is right linear, we can conclude that $T^*(H_0) \subseteq H_0$.

Lemma 3.7. *Let $T \in \mathcal{B}(H)$ be normal and H_0, a reducing subspace for T. Let $T_0 := T|_{H_0}$. Then*

(1) $T_0^* = T^*|_{H_0}$
(2) T_0 *is normal.*

Proof. Let $x, y \in H_0$. Then

$$\langle T_0^* x, y\rangle = \langle x, T_0 y\rangle = \langle x, Ty\rangle = \langle T^* x, y\rangle.$$

We can conclude that $T_0^* x - T^* x \in H_0^{\perp}$. Also, since H_0 reduces both T and T^*, it follows that $T_0^* x - T^* x \in H_0$. That is $T_0^* x = T^* x$ for each $x \in H_0$. This completes the proof of (1).

To prove (2), let $x \in H_0$. Then we have

$$\begin{aligned}\langle T_0^* T_0 x, x\rangle &= \langle Tx, Tx\rangle \\ &= \langle x, T^*Tx\rangle \\ &= \langle x, TT^*x\rangle \\ &= \langle TT^*x, x\rangle \\ &= \langle T_0 T^*|_{H_0} x, x\rangle \\ &= \langle T_0 T_0^* x, x\rangle.\end{aligned}$$

Now the conclusion follows from the polarization identity. □

Theorem 3.8. *Let $T \in \mathcal{B}(H)$ be compact and normal. Then there exists a system of eigenvectors (ϕ_n) and corresponding right quaternion eigenvalues (q_n) such that*

$$Tu = \sum_{n=1}^{\infty} \phi_n q_n \langle \phi_n | u\rangle \text{, for all } u \in H. \tag{3.1}$$

Moreover, if (q_n) is infinite, then $q_n \to 0$ as $n \to \infty$.

The series on the right hand side of Equation (3.1) converges in the operator norm of $\mathcal{B}(H)$.

Proof. If $T = 0$, then there is nothing to prove. Hence assume that $T \neq 0$. Set $T_1 = T$ and $H_1 = H$. Since T_1 is compact and normal by Theorem 3.4, there exists a $\phi_1 \in H_1 \setminus \{0\}$ and $q_1 \in \mathbb{H} \setminus \{0\}$ such that $T\phi_1 = \phi_1 q_1$. Also, note that $\|T_1\| = |q_1|$. By Example 3.6, the space $H_1 := \text{span}_{\mathbb{H}}\{\phi_1\}^{\perp}$ is a reducing subspace for T. Next, let $T_2 := T_1|_{H_2}$. Then either $T_2 = 0$ or $T_2 \neq 0$. If $T_2 = 0$, there is nothing to proceed further. If $T_2 \neq 0$, then T_2 is normal by (2) of Lemma 3.7. Thus, again by Theorem 3.4 there exists $q_2 \in \mathbb{H} \setminus \{0\}$ with $|q_2| = \|T_2\|$, $\phi_2 \in H_2 \setminus \{0\}$ such that $T\phi_2 = T_2\phi_2 = \phi_2 q_2$. Note that $|q_2| \leq |q_1|$ and ϕ_1 and ϕ_2 are orthogonal by construction.

Let $H_3 := \text{span}_{\mathbb{H}}\{\phi_1, \phi_2\}^{\perp}$. Since T is normal and H_3 is a reducing subspace for $T_3 := T|_{H_3}$, we have that T_3 is normal and compact. Now either $T_3 = 0$ or $T_3 \neq 0$. If $T_3 \neq 0$, then there exists a quaternion $q_3 \in \mathbb{H} \setminus \{0\}$ and $\phi_3 \in H_3 \setminus \{0\}$ such that $T\phi_3 = T_3\phi_3 = \phi_3 q_3$ and $|q_3| \leq |q_2|$. By construction we have that ϕ_3 is orthogonal to both ϕ_1 and ϕ_2.

Proceeding in this way, we end up with either $T_n = 0$ for some $n \in \mathbb{N}$ or there exists a sequence (q_n) of non zero quaternions and a sequence of vectors $(\phi_n) \subset H$ satisfying:

(1) $T\phi_n = \phi_n q_n$ and $|q_n| = \|T_n\|$ for each $n \in \mathbb{N}$,
(2) $|q_{n+1}| \leq |q_n|$ for each $n \in \mathbb{N}$,
(3) ϕ_r is orthogonal to ϕ_s for each $r, s \in \mathbb{N}$ and $r \neq s$.

Next, we claim that if (q_n) is infinite, then $q_n \to 0$ as $n \to \infty$. If this is not the case, there exists $\epsilon > 0$ such that $|q_n| > \epsilon$ for infinitely many $n \in \mathbb{N}$. Let $S = \{r \in \mathbb{N} : |q_r| > \epsilon\}$. Then we have $T\phi_r = \phi_r q_r$ for each $r \in S$. Thus, for $r, s \in S$, we have

$$\|T\phi_r - T\phi_s\|^2 = \|\phi_r q_r - \phi_s q_s\|^2 = |q_r|^2 + |q_s|^2 > 2\epsilon^2.$$

This shows that $(T\phi_r)$ is not Cauchy in H. But, this contradicts the fact that T is compact. Hence our assumption that $q_n \nrightarrow 0$ is wrong.

Next, we obtain the representation of T as in Equation (3.1).

For $x \in H$, define $x_n := x - \sum_{r=1}^{n-1} \phi_r \langle \phi_r, x\rangle$ for each $n \in \mathbb{N}$. Then $\langle x_n, \phi_r\rangle = 0$ for each $r = 1, 2, \ldots, n-1$.

We have the following two cases:

Case 1: $T_n = 0$ for some $n \in \mathbb{N}$

In this case, we have $0 = T_n x_n = T x_n$. Thus, $Tx = \sum_{r=1}^{n-1} \phi_r q_r \langle \phi_r, x\rangle$ for each $x \in H$. That is, T is a finite rank operator with rank $n-1$.

Case 2: $T_n \neq 0$ for any $n \in \mathbb{N}$

Since $x_n \in H_n^{\perp}$ for each $n \in \mathbb{N}$, it can be easily checked by the Pythagorean property that $\|x_n\| \leq \|x\|$ for each $n \in \mathbb{N}$. Thus,

$$\|Tx - \sum_{r=1}^{n} \phi_r q_r \langle \phi_r, x\rangle\| = \|T_n x_n\| \leq |q_n|\|x_n\| \leq |q_n|\|x\| \to 0 \text{ as } n \to \infty.$$

That is $Tx = \sum_{n=1}^{\infty} \phi_n q_n \langle \phi_n, x\rangle$ for each $x \in H$. □

Remark 3.9. Note that if q is a right eigenvalue for T and $p \in [q]$, then p is also a right eigenvalue for T. Hence by Theorem 3.8, we have that $\sigma_{pS}(T) = \{[q_n] : n \in \mathbb{N}\}$ and $\sigma_S(T) \subseteq \sigma_{pS}(T) \cup \{0\}$.

4. Norm attaining Operators

In this section we extend the Lindenstrauss theorem on norm attaining operators from the classical case to the quaternionic case. Explicitly, we show that the set of all quaternionic norm attaining operators is dense in the space of all bounded quaternionic operators with respect to the operator norm.

Recall that a bounded right linear operator T is said to be norm attaining if there exists a $x_0 \in S_H$ such that $\|Tx_0\| = \|T\|$. As $\|Tx\| = \||T|x\|$ for all $x \in H$

and $\|T\| = \||T|\|$, it follows that T is norm attaining if and only if $|T|$ is norm attaining.

We show that every quaternionic compact operator is norm attaining. In the case of operators on a complex Hilbert space, this can be proved by the help of Banach-Alaouglu's theorem. In our case we prove it by using Theorem 3.4.

We denote the set of all norm attaining operators defined between H_1 and H_2 by $\mathcal{N}(H_1, H_2)$ and $\mathcal{N}(H, H)$ by $\mathcal{N}(H)$.

Proposition 4.1. *Let $T \in \mathcal{B}(H)$ be compact. Then $T \in \mathcal{N}(H)$.*

Proof. Since T is compact, $|T|$ is compact as well (see [5]). The operator $|T|$ is also self-adjoint and hence normal because it is positive by definition (see [4, Proposition 2.17(b)]). Hence by Theorem 3.4, $\|T\|$ is a right eigenvalue. Hence the result follows. □

Lemma 4.2. *Let $T \in \mathcal{B}(H)$ be normal and T_+ be such that $\tilde{T}_+ = T$ as in Proposition 3.1. Then $T_+ \in \mathcal{N}(H_+^{Jm})$ if and only if $T \in \mathcal{N}(H)$.*

Proof. If $T_+ \in \mathcal{N}(H_+^{Jm})$, then there exists $x_0 \in S_{H_+^{Jm}}$ such that $\|T_+ x_0\| = \|T_+\|$. Since $\|T_+\| = \|T\|$, the conclusion follows.

On the other hand, suppose $T \in \mathcal{N}(H)$. Choose $x_0 \in S_H$ such that $\|Tx_0\| = \|T\|$. Let $y_0 := \dfrac{1}{\sqrt{2}}(x_0 - (Jx_0)m)$. Then $y_0 \in H_+^{Jm}$. Also,

$$\begin{aligned}
\|y_0\|^2 &= \frac{1}{2}\|(x_0 - (Jx_0)m)\|^2 \\
&= \frac{1}{2}\langle (x_0 - (Jx_0)m), (x_0 - (Jx_0)m)\rangle \\
&= \frac{1}{2}\Big(\langle x_0, x_0\rangle - \langle (Jx_0)m, x_0\rangle - \langle x_0, (Jx_0)m\rangle + \langle (Jx_0)m, (Jx_0)m\rangle\Big) \\
&= \frac{1}{2}\Big(1 - \langle (Jx_0)m, x_0\rangle - \langle x_0, (Jx_0)m\rangle + 1\Big).
\end{aligned}$$

Note that as J is anti self-adjoint, $\langle x_0, Jx_0\rangle = -\overline{\langle x_0, Jx_0\rangle}$. With this, we have $\langle (Jx_0)m, x_0\rangle = -\langle x_0, Jx_0\rangle m$. Hence $\|y_0\| = 1$.

Next using the fact that J commutes with T and T^* and $T(H_+^{Jm}) \subseteq H_+^{Jm}$, we can conclude that

$$\|T_+ y_0\|^2 = \|Tx_0\|^2 = \|T\|^2 = \|T_+\|^2.$$

Thus T_+ attains norm at y_0. □

Proposition 4.3. *Let $B \in \mathcal{B}(H)$ be positive. Then for given $\epsilon > 0$, there exists a rank one positive operator C with $\|C\| \leq \epsilon$ and a unit vector $y \in N(B)^\perp$ such that*

$$(B + C)y = \|B + C\|y.$$

That is $B + C$ attains norm at y.

Proof. Let $\epsilon > 0$. Since $B \geq 0$, there exists an anti-self-adjoint, unitary operator J such that $JB = BJ$. Let B_+ be the unique operator on H_+^{Jm} such that $\tilde{B}_+ = B$. Now, by the classical theorem ([6, Lemma 1]), there exists a rank one positive operator, denote it by C_+ such that $\|C_+\| \leq \epsilon$ and $B_+ + C_+ \in \mathcal{N}(H_+^{Jm})$. In fact, there exists a unit vector $y \in \underbrace{N(B_+)}^{\perp}$ such that $(B_+ + C_+)y = \|B_+ + C_+\|y$. Now, by Remark 3.2, $B + C = \widetilde{B_+ + C_+}$ and $B + C \in \mathcal{N}(H)$ by Lemma 4.2. It is clear that C is a positive, rank one operator on H.

Also, we have that $(B + C)y = (B_+ + C_+)y = \|B_+ + C_+\|y = \|B + C\|y$. As $N(B) = N(B_+)$, we can conclude that $y \in N(B)^{\perp}$. □

Next, we extend Proposition 4.3 to the general case. For this purpose we use the polar decomposition of a quaternionic operator. Here we give the details.

Theorem 4.4. *The set $\mathcal{N}(H)$ is dense in $\mathcal{B}(H)$ with respect to the operator norm of $\mathcal{B}(H)$.*

Proof. Let $T = V|T|$ be the polar decomposition of T. Given $n \in \mathbb{N}$, By Proposition 4.3, there exists a rank one, positive operator C_n with $\|C_n\| \leq \frac{1}{n}$ such that $|T| + C_n \in \mathcal{N}(H)$. In fact, there exists a sequence $(x_n) \subset H$ of unit vectors such that $(|T| + C_n)x_n = \||T| + C_n\|x_n$ for each $n \in \mathbb{N}$. Define $K_n := VC_n$ for each $n \in \mathbb{N}$. Note that $x_n \in N(T)^{\perp} = N(V)^{\perp}$. Since $V|_{N(V)^{\perp}}$ is an isometry, we have that $\|(T + K_n)x_n\| = \|V(|T| + C_n)x_n\| = \|(|T| + C_n)x_n\| = \||T| + C_n\|$. But $\|T + K_n\| \leq \||T| + C_n\| = \|(T + K_n)x_n\|$. This shows that $T + K_n$ attains norm at x_n. □

Remark 4.5. We can also prove that $\mathcal{N}(H_1, H_2)$ is norm dense in $\mathcal{B}(H_1, H_2)$ following the similar steps as in Theorem 4.4. For this purpose one has to obtain the polar decomposition theorem for $T \in \mathcal{B}(H_1, H_2)$. Again this can be done by following the similar steps in [2, Theorem 2.20].

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Department of Mathematics, I. I. T. Hyderabad, Kandi(V), Sangareddy, Telangana 502 285. India.

E-mail address: rameshg@iith.ac.in