Adv. Oper. Theory 2 (2017), no. 1, 59–77
http://doi.org/10.22034/aot.1611-1052
ISSN: 2538-225X (electronic)
http://aot-math.org

VARIOUS NOTIONS OF BEST APPROXIMATION PROPERTY IN SPACES OF BOCHNER INTEGRABLE FUNCTIONS

TANMOY PAUL

Communicated by A. Kamińska

Abstract. We show that a separable proximinal subspace of X, say Y is strongly proximinal (strongly ball proximinal) if and only if $L_p(I,Y)$ is strongly proximinal (strongly ball proximinal) in $L_p(I,X)$, for $1\leq p<\infty$. The $p=\infty$ case requires a stronger assumption, that of 'uniform proximinality'. Further, we show that a separable subspace Y is ball proximinal in X if and only if $L_p(I,Y)$ is ball proximinal in $L_p(I,X)$ for $1\leq p\leq\infty$. We develop the notion of 'uniform proximinality' of a closed convex set in a Banach space, rectifying one that was defined in a recent paper by P.-K Lin et al. [J. Approx. Theory 183 (2014), 72–81]. We also provide several examples having this property; viz. any U-subspace of a Banach space has this property. Recall the notion of 3.2.$I.P.$ by Joram Lindenstrauss, a Banach space X is said to have 3.2.$I.P.$ if any three closed balls which are pairwise intersecting actually intersect in X. It is proved the closed unit ball B_X of a space with 3.2.$I.P$ and closed unit ball of any M-ideal of a space with 3.2.$I.P.$ are uniformly proximinal. A new class of examples are given having this property.

1. Introduction and preliminaries

Let X be a Banach space and C be a closed convex subset of X. For $x\in X$, let $d(x,C)=\inf_{z\in C}\|x-z\|$ and $P_C(x)=\{z\in C:\|x-z\|=d(x,C)\}$. The set valued mapping $P_C:X\to 2^C$ is called the metric projection of C and the points in $P_C(x)$ are called the best approximation from x in C. We call the subset C *proximinal* (or it has best approximation property) if for every point $x\in X\setminus C$, $P_C(x)\neq\emptyset$.

Date: Received: Nov. 6, 2016; Accepted: Jan. 21, 2017.
2010 *Mathematics Subject Classification.* Primary 41A50; Secondary 46B20, 46E40, 46E15.
Key words and phrases. $L_p(I,X)$, proximinality, strong proximinality, ball proximinality.

Let $(\Omega, \mathcal{M}, \mu)$ be a finite measure space. For a Banach space X consider the Banach space of Bochner p-integrable (essentially bounded for $p = \infty$) functions on Ω with values in X, endowed with the usual p-norm viz. $L_p(\Omega, X)$. Let us recall any such function is essentially a strongly measurable function, separably valued and if (s_n) is a sequence of simple functions such that $s_n(t) \to f(t)$ a.e. then $\lim_n \int_I \|s_n(t)\|^p dm(t) = \int_I \|f(t)\|^p dm(t)$. In [8, 9, 16, 17] the authors discussed for a finite measure space how often the property of best approximation of Y in X is stable under the spaces of functions $L_p(\Omega, Y)$ in $L_p(\Omega, X)$. Let us recall the following Theorem in this context.

Theorem 1.1. *Let Y be a subspace of X and $f \in L_p(\Omega, X)$ then,*

(a) [12, Theorem 5]$d(f, L_p(\Omega, Y)) = \|d(f(.), Y)\|_p$ *for* $1 \leq p \leq \infty$.

(b) [17, Theorem 3.4]*For a separable subspace Y of X, $L_p(\Omega, Y)$ is proximinal in $L_p(\Omega, X)$ if and only if Y is proximinal in X, for $1 \leq p \leq \infty$.*

(c) [12, Corollary 2]$f \in P_{L_p(\Omega,Y)}(g)$ *if and only if* $f(t) \in P_Y(g(t))$ *a.e. for* $1 \leq p < \infty$.

(d) [17, Proposition 2.5]$L_\infty(\Omega, Y)$ *is proximinal in* $L_\infty(\Omega, X)$ *if and only if for $f \in L_\infty(\Omega, X)$ there exists $g \in L_\infty(\Omega, Y)$ such that $f(t) \in P_Y(g(t))$ a.e.*

Suppose $I = [0, 1]$, and $(I, \mathcal{B}, m)$ stands for the complete Lebesgue measure space over the Borel σ-field $\mathcal{B}$. One can define $L_p(I, B_X)$, similar to the space $L_p(I, X)$, which represents the set of measurable functions from I to B_X which are p-integrable. After Saidi's paper, [21], people find it is worth investigating about the proximinality of closed unit ball of a proximinal subspace. The authors in [1] investigate the proximinality of $L_p(I, B_Y)$ in $L_p(I, X)$ if B_Y is proximinal in X. Recall the following results from [1, Pg 12].

Theorem 1.2. *Let Y be a separable ball proximinal subspace of X. Then*

(a) $L_\infty(I, Y)$ *is ball proximinal in* $L_\infty(I, X)$.

(b) $L_p(I, B_Y)$ *is proximinal in* $L_p(I, X)$.

A latest article in this context is [16]. It is also relevant to mention here that for a proximinal subspace Y, $L_1(I, Y)$ is not necessarily proximinal in $L_1(I, X)$ if Y is not separable [17]. Light and Cheney also discussed about this best approximation property in the function spaces of type $L_p(\Omega, X)$ in [13, Chapter 2]. Discussion in [13, Chapter 10] is also relevant to the content of this paper. Our aim in this paper is to study various strengthenings of best approximation property, defined in Definition 1.3, of $L_p(I, Y)$ in $L_p(I, X)$. A concise presentation of this work is available in Section 2.

We now state few known Definitions from the literature which are relevant and also have impacts to the main theme of this paper. First recall from [1, 5] the following stronger versions of proximinality.

Definition 1.3. (a) A closed convex subset C of X is said to be *Strongly proximinal* if it is proximinal and for a given $x \in X \setminus C$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that $P_C(x, \delta) \subseteq P_C(x) + \varepsilon B_X$, where $P_C(x, \delta) = \{z \in C : \|x - z\| \leq d(x, C) + \delta\}$.

(b) A subspace Y is said to be *Ball proximinal* if B_Y is proximinal in X.
(c) A subspace Y is said to be *Strongly ball proximinal* if B_Y is strongly proximinal.

Readers can come across the articles [1, 3, 5] for various examples of subspaces having these proximity properties.

Recall the following notions for a set valued map. Here $CB(X)$ stands for the set of all closed and bounded subsets of a Banach space X.

Definition 1.4. [19] Let T be a topological space and $\Gamma : T \to CB(X)$ be a set valued map. Γ is said to be

(a) upper semi-continuous, abbreviated usc (resp. lower semi-continuous, abbreviated lsc) if for any closed (open) subset A of X, the set $\Gamma^{-1}(A) := \{t \in T : \Gamma(t) \cap A \neq \emptyset\}$ is closed (open).
(b) upper Hausdorff semi-continuous, abbreviated uHsc. (resp. lower Hausdorff semi-continuous, abbreviated lHsc) if for every $t \in T$ and every $\varepsilon > 0$, there is a neighborhood N of t, such that $\Gamma(t) \subseteq \Gamma(t_0) + \varepsilon B_X$ (resp. $\Gamma(t_0) \subseteq \Gamma(t) + \varepsilon B_X$) for each $t \in N$.
(c) Γ is continuous if it is both usc and lsc and Hausdorff continuous, abbreviated H-continuous, if it is both uHsc and lHsc.

From the definition of strong proximinality, it is clear that if Y is a strongly proximinal subspace then P_Y is uHsc. In general we have usc $\Rightarrow$ uHsc and lHsc $\Rightarrow$ lsc and if the above Γ is compact valued then usc $\Leftrightarrow$ uHsc and lHsc $\Leftrightarrow$ lsc.

The following notion was introduced by Yost in [23]. The author established some connections between the properties of best approximation and the following for a subspace of a Banach space.

Definition 1.5. [23] A subspace Y of a Banach space X is said to have the $1\frac{1}{2}$ -ball property if, whenever $\|x - y\| < r + s$ where $y \in Y$ and $x \in X$ with $B[x, r] \cap Y \neq \emptyset$ then $B[x, r] \cap B[y, s] \cap Y \neq \emptyset$.

It is well known that a subspace Y having $1\frac{1}{2}$ ball property is strongly proximinal. There are many function spaces and function algebras in the class of continuous functions having this property.

Recall the notion of $3.2.I.P.$ in this connection, defined in the abstract. Lindenstrauss monograph [15] was the first where the above property was appeared for the first time, although the article [14] by Lima encounters a systematic study of intersection properties of balls in Banach spaces.

2. Main results

The following problems are the origin of this investigation.

Problem 2.1. *Let Y be a subspace of X which is strongly proximinal (ball proximinal). Is $L_p(\Omega, Y)$ strongly proximinal (ball proximinal) in $L_p(\Omega, X)$ for $1 \leq p \leq \infty$?*

The above problem on ball proximinality is asked in [1, Pg 12].

Problem 2.2. *Let $f \in L_p(\Omega, X)$ and Y be a subspace of X. What is the numerical value of $d(f, B_{L_p(\Omega,Y)})$?*

Problem 2.3. *Let Y be a subspace of X having $1\frac{1}{2}$ ball property and $(\Omega, \mathcal{M}, \mu)$ be a finite measure space. Does $L_p(\Omega, Y)$ has $1\frac{1}{2}$ ball property in $L_p(\Omega, X)$ for $p = 1, \infty$?*

Remark 3.10 states if $L_\infty(\Omega, Y)$ is strongly proximinal in $L_\infty(I, X)$ then P_Y must be lHsc, on the other Y would be strongly proximinal in X for the same. Hence P_Y is Hausdorff continuous if $L_\infty(\Omega, Y)$ is strongly proximinal in $L_\infty(I, X)$. Hence it raises the following question.

Problem 2.4. *Let $P_Y : X \to 2^Y$ be Hausdorff continuous. Then what is the appropriate condition on Y in X which makes $L_\infty(\Omega, Y)$ strongly proximinal in $L_\infty(\Omega, X)$ and vice versa ?*

We considered these problems for the measure space $(I, \mathcal{B}, m)$. The results in Section 5 only require that the measure space has to be positive with total variation 1, the other results can be derived for any finite measure space. The main results in this article are the following:

Theorem 2.5 (Theorem 3.6,5.7)**.** *For a separable proximinal subspace Y of X, Y is strongly proximinal (strongly ball proximinal) in X if and only if $L_p(I, Y)$ is strongly proximinal (strongly ball proximinal) in $L_p(I, X)$, for $1 \leq p < \infty$.*

Theorem 2.6 (Theorem 5.4)**.** *For a separable proximinal subspace Y of X, Y is ball proximinal in X if and only if $L_p(I, Y)$ is ball proximinal in $L_p(I, X)$, for $1 \leq p \leq \infty$.*

And also,

Theorem 2.7 (Theorem 4.9)**.** *Let Y be a separable proximinal subspace of X, then consider the following statements.*

(a) $Y(B_Y)$ is uniformly proximinal in X.

(b) $L_\infty(I, Y)(B_{L_\infty(I,Y)})$ is uniformly proximinal in $L_\infty(I, X)$.

(c) $L_\infty(I, Y)(B_{L_\infty(I,Y)})$ is strongly proximinal in $L_\infty(I, X)$.

Then (a) $\Longleftrightarrow$ (b) and (b) $\Longrightarrow$ (c).

We couldn't answer the Problem 2.4, the above Theorem is a partial answer of Problem 2.4. A section-wise illustration of this work is outlined in the next few paragraphs.

In Section 3 we discuss some distance formulas which enable us to conclude the strong proximinality of $L_p(I, Y)$ in $L_p(I, X)$. These distance formulas are proved with the help of pathologies of measurable set valued functions and their measurable selections. Problem 2.3 is answered in Theorem 3.12.

The non-availability of conclusion in Theorem 2.5 for $p = \infty$ invites a uniform version of strong proximinality of Y in X, as discussed in Section 4. To begin with, the content of Section 4 we would like to thank the authors in [16] for drawing our attention towards the notion of 'uniform proximinality' in Banach space. However, a similar notion dates back to the paper by Pai and Nowroji

([19]) in the context of Property-(R_2); nevertheless, the way used in [16, Pg 79] to define 'uniform proximinality' is wrong. A simple geometry in the Euclidean space $\mathbb{R}^2$ clarifies the flaw (Example 4.1).

We adopt the idea introduced in [19] in terms of Property-(R_2) and define 'uniform proximinality' of a closed convex set. Section 4 is devoted to discussing this property. Strong proximinality can now be viewed as a local version of this 'uniform proximinality'. Several examples are given which satisfy this property; the list includes closed convex subsets of uniformly convex space, subspace with $1\frac{1}{2}$-ball property and any U-proximinal subspace (see [11]). An elegant observation in this context is that closed unit ball of a Banach space is not necessarily uniformly proximinal (using Example in [10]), we derive that it is true if X has $3.2.I.P$ (see [14]). Finally, we prove the strong proximinality of $L_\infty(I, Y)$ in $L_\infty(I, X)$ as a necessary condition for uniform proximinality of Y in X (Theorem 2.7). A weaker version of [20, Theorem 15] is also proved here.

Section 5 is devoted to ball proximinality and strong ball proximinality of $L_p(I, Y)$ in $L_p(I, X)$. It is proved for $f \in L_p(I, X)$, $d(f, L_p(I, B_Y)) = d(f, B_{L_p(I,Y)})$ for $1 \leq p \leq \infty$ which answers Problem 2.2. This result together with Theorem 5.6 leads to some interesting observations. The main results in this Section are stated in Theorem 2.6. Our results answer the question raised in [1] after Theorem 4.10.

Since in a Banach space X, B_X is not necessarily strongly proximinal in X we found it is meaningful to identify some cases when the answer is affirmative. From [4] it follows that $B_{L_p(\mu)}$ is strongly proximinal in $L_p(\mu)$ (spaces having reflexivity and Kadec-Klee property) for any positive measure μ when $1 < p < \infty$. From our result it follows that the conclusion is still true for $L_p(\mu)$ where $p = 1, \infty$ (for real scalar); in fact the result holds true for $B_{L_p(I,X)}, 1 \leq p \leq \infty$ when and only when X has the similar property.

A new class of examples is given in Section 6 which are uniformly proximinal.

For a Banach space X, B_X, S_X and $B[x, r]$ denote the closed unit ball, the closed unit sphere and closed ball with centre at x and radius r respectively. All Banach spaces are assumed to be complex unless otherwise stated. Those spaces that have any intersection properties of balls like $3.2.I.P.$, $4.2.I.P.$ are assumed to be real. X will always denote a Banach space and by a subspace we always mean a closed subspace.

3. Strong proximinality of $L_p(I, Y)$ in $L_p(I, X)$

Similar to the Theorem 1.1 we now approach towards a distance formula which is actually stated in Theorem 3.4. To this end we need the following pathologies related to the set valued functions which help us to derive Theorem 3.4.

Lemma 3.1. (a) *Let X be a Banach space and Y be a proximinal subspace of X such that the metric projection P_Y is uHsc. Then the mapping $G : X \times X \to \mathbb{R}$ defined by $G((x, z)) = d(x, P_Y(z))$ is upper semi-continuous in first variable and lower semi-continuous in second variable.*

(b) *Let Y be a subspace as defined in (a) and is also separable, then for any two measurable functions $f : I \to Y$ and $g : I \to X$ the mapping $\varphi : I \to \mathbb{R}$ defined by $\varphi(t) = d(f(t), P_Y(g(t)))$ is measurable.*

Proof. (a). Upper semi continuity of G at it's first variable follows from the fact that, for a closed set A if $h(x) = d(x, A)$ then h defines a continuous (and hence upper semi-continuous) mapping from X to $\mathbb{R}$.

On the other hand let $\varepsilon > 0$. Since P_Y is uHsc, there exists a $\delta > 0$ such that $P_Y(z) \subseteq P_Y(z_0) + \varepsilon B_Y$ whenever $\|z - z_0\| < \delta$. If (z_n) converges to z, there exists an $N \in \mathbb{N}$ such that $\|z_n - z\| < \delta$ for all $n \geq N$. Hence for $n \geq N$ we get, $d(x, P_Y(z_n)) \geq d(x, P_Y(z) + \varepsilon B_Y) \geq d(x, P_Y(z)) - \varepsilon$.

Hence we have $\liminf_n d(x, P_Y(z_n)) \geq d(x, P_Y(z))$.

(b). Let $D \subseteq Y$ be a countable dense subset of Y. It is clear that the mapping $A : I \to Y \times X$ defined by $A(t) = (f(t), g(t))$ is measurable. We now show that $G : Y \times X \to \mathbb{R}$ defined by $G((y, x)) = d(y, P_Y(x))$ is measurable. Hence $\varphi(t) = G(A(t))$ will be measurable.

To this end we show that $G^{-1}([\alpha, \infty))$ is measurable for all real α's.

Now, $G((y, x)) \geq \alpha \Longleftrightarrow$

$(\forall n \in \mathbb{N})(\exists z_n \in D) \left[\|y - z_n\| < \frac{1}{n} \ \&\ G((z_n, x)) > \alpha - \frac{1}{n}\right] \Longleftrightarrow$

$(y, x) \in \bigcap_n \bigcup_{z \in D} \left[\{y \in Y : \|y - z\| < \frac{1}{n}\} \times \{x \in X : G((z, x)) > \alpha - \frac{1}{n}\}\right]$.

Clearly if $(y, x) \in$ RHS, then there exists a sequence $(z_n) \subseteq D$ such that $G((z_n, x)) > \alpha + \frac{1}{n}$ and $z_n \to y$ and hence $G((y, x)) \geq \limsup_n G((z_n, x)) \geq \alpha$. On the other hand if $G((y, x)) \geq \alpha$, then the sets $\{v \in Y : G((v, x)) < G((y, x)) + \frac{1}{n}\}$ and $\{z \in X : G((y, z)) > \alpha - \frac{1}{n}\}$ are open for all n and contain y, x respectively. This completes the proof. □

Now we need the following technical Theorem which helps us to find a measurable selection of a closed set valued measurable function. We call a set valued map $F : X \to 2^Y$ is measurable if the graph of F, $Gr(F) = \{(x, F(x)) : x \in X\} = \bigcup\{(x, y) : x \in X, y \in F(x)\} \in \mathcal{B}_X \bigotimes \mathcal{B}_Y$. The last set represents the smallest σ-field containing the measurable rectangles $M \times N$, where $M \in \mathcal{B}_X, N \in \mathcal{B}_Y$, where $\mathcal{B}_X, \mathcal{B}_Y$ represent the Borel σ-fields over X, Y respectively.

Theorem 3.2. [22, Corollary 5.5.8.] *Let $(\Omega, \mathfrak{M}, \mu)$ be a complete probability space, Y a polish space and $B \in \mathfrak{M} \bigotimes \mathcal{B}_Y$. Then $\pi_\Omega(B) \in \mathfrak{M}$ and B admits a $\mathfrak{M}$ measurable section.*

The above Theorem is a consequence of Von Naumann's selection Theorem ([22, Theorem 5.5.2]); we may need to apply some other variant of this Theorem, but Theorem 3.2 is crucially used in various places.

Lemma 3.3. *Let Y be a separable proximinal subspace of X for which the map $P_Y : X \to 2^Y$ is uHsc. Let $f : I \to Y, g : I \to X$ are measurable, then for $\delta > 0$ consider the set valued function $\Phi_\delta : I \to 2^Y$ defined by $\Phi_\delta(t) = P_{P_Y(g(t))}(f(t), \delta)$. Then Φ_δ is measurable and it has a measurable selection.*

Proof. Clearly we have $\Phi_\delta(t) = P_Y(g(t)) \cap B[f(t), \varphi(t) + \delta]$, where φ is defined in Lemma 3.1. Since all functions in Φ_δ is measurable, we have the graph $Gr(\Phi_\delta) = \{(t, \Phi_\delta(t)) : t \in I\}$ is measurable. In fact we have the following representation for Φ_δ.

Define $F_1, F_2 : I \to 2^Y$ by $F_1(t) = B[f(t), \varphi(t) + \delta]$ and $F_2(t) = P_Y(g(t))$. Since f and φ both the functions are measurable, $Gr(F_1)$ is measurable. Also

$\{(t,y) : t \in I, y \in F_2(t)\} = \{(t,y) : \|y - f(t)\| = d(f(t),Y)\} = \bigcap_n \{(t,y) : \|y - f(t)\| \leq \|y_n - f(t)\|\}$ where (y_n) is a dense subset of Y. Hence the graph of F_2 is also measurable. Now $Gr(\Phi_\delta) = Gr(F_1) \cap Gr(F_2)$. Hence $Gr(\Phi_\delta)$ is again measurable. From Theorem 3.2 it follows that the last set has a measurable selection. □

We now establish a distance formula between a given point in $L_p(I,Y)$ and the set of best approximation from a given point in $L_p(I,X)$ to $L_p(I,Y)$. Similar to Theorem 1.1 the distance function is an integral of the point wise distance function.

Theorem 3.4. *Let Y be a separable proximinal subspace of X such that P_Y is uHsc. Then for $1 \leq p < \infty$ and $f \in L_p(I,Y), g \in L_p(I,X)$,*
$d(f, P_{L_p(I,Y)}(g)) = \|d(f(.), P_Y(g(.)))\|_p$.

Proof. From Lemma 3.1 it follows that the map $t \mapsto d(f(t), P_Y(g(t)))$ is measurable and hence the above integral is justified. Now for the given range of p,

$$\begin{aligned} d(f, P_{L_p(I,Y)}(g)) &= \inf_{h \in P_{L_p(I,Y)}(g)} \|f - h\|_p \\ &\geq \|d(f(.), P_Y(g(.)))\|_p, \text{ from Theorem 1.1(b).} \end{aligned}$$

Now for each n define $\Phi_n : I \to 2^Y$ by $\Phi_n(t) = P_{P_Y(g(t))}(f(t), \frac{1}{n})$. From Lemma 3.3 it follows that the graph of Φ_n is measurable and hence by Theorem 3.2 it has a measurable selection. Let h_n be such a selection. Clearly for all $t, h_n(t) \in P_Y(g(t))$ hence $h_n \in P_{L_p(I,Y)}(g)$, which leads to the following identity.

$$d(f, P_{L_p(I,Y)}(g)) \leq \liminf_n \|f - h_n\|_p = \|d(f(.), P_Y(g(.)))\|_p.$$

The last equality follows from the Dominated convergence theorem for $p < \infty$ and this establishes the other inequality. □

The following Remark states about the possible relation between $d(f, P_{L_\infty(I,Y)}(g))$ and ∞-norm of the pointwise distance function $t \mapsto d(f(t), P_Y(g(t)))$.

Remark 3.5. Let us define $Z = \{h \in L_\infty(I,Y) : h(t) \in P_Y(g(t)) \text{ a.e.}\}$. It is clear that, $P_{L_\infty(I,Y)}(g) \supseteq Z$. Hence $d(f, P_{L_\infty(I,Y)}(g)) \leq \|d(f(.), P_Y(g(.)))\|_\infty$: In fact,

$$\begin{aligned} d(f, P_{L_\infty(I,Y)}(g)) &\leq d(f, Z) \\ &= \inf_{h \in Z} \mathit{ess}\sup_{t \in I} \|f(t) - h(t)\| \\ &= \operatorname{ess\,sup}_{t \in I} d(f(t), P_Y(g(t))) \\ &= \|d(f(.), P_Y(g(.)))\|_\infty. \end{aligned}$$

Our main results of this section are the following.

Theorem 3.6. *Let Y be a separable proximinal subspace of X. Then Y is strongly proximinal in X if and only if $L_p(I,Y)$ is strongly proximinal in $L_p(I,X)$ for $1 \leq p < \infty$.*

Proof. Let Y be strongly proximinal in X and let for some $p \in [1,\infty)$, $L_p(I,Y)$ be not strongly proximinal in $L_p(I,X)$. Hence there exists $f \in L_p(I,X), \varepsilon > 0$ and $(g_n) \subseteq L_p(I,Y)$ such that $\|f - g_n\|_p \to d(f, L_p(I,Y))$ but $d(g_n, P_{L_p(I,Y)}(f)) \geq \varepsilon$.

Now $\|f - g_n\|_p \to d(f, L_p(I,Y))$
$\implies \int_I \|f(t) - g_n(t)\|^p dm(t) \to \int_I d(f(t), Y)^p dm(t)$.
$\implies \int_I |\|f(t) - g_n(t)\|^p - d(f(t), Y)^p|\, dm(t) \to 0$.

A well known property of L_p convergence ensures that there exists a subsequence (g_{n_k}) satisfying $\|f(t) - g_{n_k}(t)\|^p - d(f(t), Y)^p \to 0$ a.e.

Since $\|f(t) - g_{n_k}(t)\| \to d(f(t), Y)$ a.e. we have $d(g_{n_k}(t), P_Y(f(t))) \to 0$ a.e. Since $d(g_{n_k}(t), P_Y(f(t)))^p \leq 2\|f(t)\|^p$, a L_1 function. Hence by Dominated Converge Theorem, $\lim_{k\to 0} \int_I d(g_{n_k}(t), P_Y(f(t)))^p dm(t) = 0$, contradicting our assumption on (g_n). Hence the result follows. □

Since all g_n's in the above proof are separably valued the above proof can be fitted with all such strongly proximinal Y of which all its separable subspaces are also strongly proximinal.

Corollary 3.7. *Let Y be a stronly proximinal subspace of X. If every separable subspace of Y is strongly proximinal in X then $L_p(I,Y)$ is strongly proximinal in $L_p(I,X)$.*

Proof. For such type of (g_n) defined above get a separable subspace $Z \subseteq Y$ such that $d(f, L_p(I,Y)) = d(f, L_p(I,Z))$, $1 \leq p \leq \infty$. From our assumption and Theorem 3.6 it follows $d(g_n, P_{L_p(I,Z)}(f)) \to 0$ and hence $d(g_n, P_{L_p(I,Y)}(f)) \to 0$. □

Remark 3.8. In general the conclusion of the Theorem 3.6 is not true for $p = \infty$, Example 3.9. In next Section we show that a stronger version of strong proximinality of $L_p(I,Y)$ in $L_p(I,X)$ can be achieved from the similar assumption of Y in X and also vice versa.

We now show that strong proximinality of $L_\infty(I,Y)$ in $L_\infty(I,X)$ demands a stronger assumption on Y in X.

From Michael's selection theorem (see [18, Theorem 3.1′]) it is clear that if Y is a finite dimensional subspace of a normed linear space X and the metric projection P_Y is lsc then it has a continuous selection. Now in [2, Example 2.5] the author has shown that there exists a 1 dimensional subspace Y in the 3 dimensional space $\mathbb{R}^3$ with a suitable norm where the metric projection P_Y has no continuous selection. Hence it can not be lsc, and being a compact valued map P_Y is not also lHsc. We now use these observations in the following example for the subspace Y and the corresponding metric projection P_Y to derive the non stability behavior of $L_\infty(I,Y)$ in $L_\infty(I,X)$ in the context of strong proximinality.

Example 3.9. *If Y is strongly proximinal in X then $L_\infty(I,Y)$ not necessarily strongly proximinal in $L_\infty(I,X)$: Let X and Y be the spaces defined in* [2, Example 2.5]. *Then there exists a sequence $(x_n) \subseteq X, x \in X$ such that $x_n \to x$ but $P_Y(x) \nsubseteq P_Y(x_n) + \varepsilon B_Y$ for some $\varepsilon > 0$. Define $z_n = \frac{x_n}{d(x_n,Y)}, z_0 = \frac{x}{d(x,Y)}$. Then $z_n \to z_0$ and $d(z_n, Y) = 1 = d(z_0, Y)$. Also we have,*

$$d(x,Y)P_Y(z_0) \nsubseteq d(x_n,Y)P_Y(z_n) + \varepsilon B_Y, \text{ for all } n \in \mathbb{N}.$$

That is there exists $y_n \in P_Y(z_0)$ *such that* $d\left((d(x,Y), d(x_n,Y)P_Y(z_n)\right) \geq \varepsilon$ *and hence* $d(y_n, \alpha_n P_Y(z_n)) \geq \eta$ *where* $\alpha_n \to 1$ *and some* $\eta > 0$.

It is clear that $|\|y_n - z_n\| - d(z_n, Y)| \to 0$. *Let* (I_n) *be a sequence of pairwise disjoint intervals with* $\cup_n I_n = I$.

Define $f \in L_\infty(I,X), g_k \in L_\infty(I,Y)$ *with* $f|_{I_n} = z_n, g_k|_{I_n} = y_n$ *if* $k = n$ *otherwise* $g_k|_{I_n} \subseteq P_Y(z_k)$. *Clearly we have* $\|f - g_k\|_\infty \to d(f, L_\infty(I,Y))$ *but* $d(g_k, P_{L_\infty(I,Y)}(f)) \geq \eta$, *for all but finitely many* k*'s. The last inequality follows from the fact that,*

$$P_{L_\infty(I,Y)}(f) = \{h \in L_\infty(I,Y) : h|_{I_n} \subseteq P_Y(z_n), \text{ for all } n\}.$$

Remark 3.10. From above example it is clear if $L_\infty(I,Y)$ is strongly proximinal in $L_\infty(I,X)$ then P_Y must be Hausdorff continuous.

We conclude this Section by an application of Theorem 1.1. The scalar field for the Banach spaces considered in rest of this Section is $\mathbb{R}$.

The following result, Theorem 3.12, concludes about strong proximinality of $L_\infty(I,Y)$ in $L_\infty(I,X)$. It is also a strengthening of [20, Theorem 15] which was proved for strong $1\frac{1}{2}$ ball property. Before we go for Theorem 3.12 here is a useful characterization of $1\frac{1}{2}$ ball property.

Theorem 3.11. [6] *For a subspace* Y *of* X, *the following are equivalent.*

- (*a*) Y *has* $1\frac{1}{2}$ *ball property.*
- (*b*) $\|x - y\| = d(x,Y) + d(y, P_Y(x))$, *for* x *in* X *and* $y \in Y$.
- (*c*) $\|x\| = d(x,Y) + d(0, P_Y(x))$, *for* $x \in X$.

Theorem 3.12. *A separable subspace* Y *of* X *has* $1\frac{1}{2}$ *ball property if and only if* $L_1(I,Y)(L_\infty(I,Y))$ *has* $1\frac{1}{2}$ *ball property in* $L_1(I,X)(L_\infty(I,X))$.

Proof. Suppose Y has $1\frac{1}{2}$ ball property in X. We only show that the distance formula in Theorem 3.11(*c*) holds for any $f \in L_1(I,X)$. Now $\|f(t)\| = d(f(t),Y) + d(0, P_Y(f(t)))$ a.e. For $p = 1$, we get the result by integrating both sides and use the distance formulas discussed in Theorem 1.1, 3.4. For $p = \infty$ we take the essential supremum in both sides and use the Remark 3.5 and get $\|f\|_\infty \geq d(f, L_\infty(I,Y)) + d(0, P_{L_\infty(I,Y)}(f))$. The other inequality is obvious.

Conversely, for any $x \in X$ consider the constant function $f(t) = x$ for all $t \in I$. The result now follows from Theorem 3.11 and 3.4. ☐

4. Uniform proximinality of $L_p(I,Y)$ in $L_p(I,X)$

In a recent paper ([16]) the authors has introduced the notion *uniform proximinality* and it is claimed that closed unit ball of any uniformly convex space is uniformly proximinal. We first observe that the property does not holds even for the 2 dimensional Euclidean space.

Example 4.1. *Let* C *be the closed unit ball of* $(\mathbb{R}^2, \|.\|_2)$, $x = (2,0)$. *Then* $P_C((2,0)) = \{(1,0)\}$. *Let* $\alpha = 2$ *and* $\varepsilon = 1/2$. *Then there does not exist* $\delta > 0$ *satisfying the condition in* [16], *pg 79, which makes* C *uniformly proximinal. In fact, if such a* $\delta > 0$ *exists then* $\|(0,0) - (2,0)\| < \alpha + \delta$ *but* $\|(0,0) - (1,0)\| > \varepsilon$.

We now define a stronger version of proximinality, viz. *uniform proximinality* which is in fact stated in [19] in the context of centres of closed bounded sets.

Definition 4.2. Let C be a closed convex subset of X. We call C is uniformly proximinal if given $\varepsilon > 0$ and $R > 0$ there exists $\delta(\varepsilon, R) > 0$ such that for any $x \in X, d(x, C) \leq R$ and $y \in C$ with $\|x - y\| < R + \delta$, there exists $y' \in C$ with $\|y - y'\| < \varepsilon$ and $\|x - y'\| \leq R$.

Here are some examples of uniformly proximinal sets.

Example 4.3. (a) *It is clear that a Banach space X having 3.2.I.P., $B_X(B_{L_\infty(I,X)})$ is uniformly proximinal in $X(L_\infty(I,X))$.*

(b) [19, Proposition 3.5] *Any w^*-closed convex subset of ℓ_1 is uniformly proximinal.*

(c) [19, Proposition 3.7] *Any closed convex proximinal subset of a LUR space is uniformly proximinal.*

(d) *Any subspace Y of X having $1\frac{1}{2}$ ball property is uniformly proximinal: Let $R, \varepsilon > 0$ such that $d(x, Y) \leq R$ and $\|x - y\| < R + \varepsilon$ for some $y \in Y$, from the Definition 1.5 we have $B[x, R] \cap B[y, \varepsilon] \cap Y \neq \emptyset$. Any point from this intersection solve our purpose.*

(e) [11] *Any subspace Y of X which is U-proximinal is also uniformly proximinal: Let $\eta, R > 0$, suppose $\varepsilon : \mathbb{R} \to \mathbb{R}$ be the continuous function corresponding to the subspace Y in [11]. Get $\theta > 0$ satisfying $\varepsilon(\theta) < \eta/R$, let $\delta = R\theta$. Let $x \in X$ such that $d(x, Y) \leq R$ and $y \in Y$ be such that $\|x - y\| < R + \delta$.*

Claim: *There exists $y' \in Y$ such that $\|y - y'\| < \eta$ and $\|x - y'\| \leq R$.*

Now $d(\frac{x}{R}, Y) \leq 1$ and $\|\frac{x}{R} - \frac{y}{R}\| < 1 + \theta$, in other words $\frac{x}{R} \in Y + B_X$ and $\frac{x}{R} - \frac{y}{R} \in (1 + \theta)B_X$ and hence $\frac{x}{R} - \frac{y}{R} \in Y + B_X$. And finally there exists $y_1 \in \varepsilon(\theta)B_Y$ such that $\|\frac{x}{R} - \frac{y}{R} - y_1\| \leq 1$. Define $y' = y + Ry_1$, this y' satisfies the desired requirements.

We refer [19] to the reader for many other interesting uniformly proximinal subsets of Banach spaces.

Remark 4.4. (a) In the Definition 4.2 if we demand to have $\delta = \varepsilon$ for all $R > 0$ we get back $1\frac{1}{2}$ ball property.

(b) From the Definition 4.2 it is clear that uniform proximinality of C forces the set to be strongly proximinal.

(c) From the example by Godefroy in [10, Pg. 89] it is clear that the closed unit ball of a Banach space not necessarily have uniformly proximinal property.

We now claim that converse of Remark 4.4(b) is not true. First observe the following.

Proposition 4.5. *If a closed convex set C in X is uniformly proximinal then the metric projection $P_C : X \to 2^C$ is continuous in the Hausdorff metric.*

Proof. Let $x_n \to x$ in X, without loss of generality we may assume $d(x, C) = 1, d(x_n, C) = 1$ for all n. Let $\delta(1, \varepsilon) > 0$ be the number corresponding to uniform proximinality of C. If possible let $P_C(x) \nsubseteq P_C(x_n)+\varepsilon B_Y$ for all but finitely many n's, for some $\varepsilon > 0$. Hence there exists $y_n \in P_C(x)$ such that $d(y_n, P_C(x_n)) \geq \varepsilon$. Get a N such that $|\|x_n - y_n\| - d(x_n, C)| < \delta$ for all $n > N$. Now using the property of uniform proximinality of C there exists $y'_n \in P_C(x_n)$ such that $\|y_n - y'_n\| < \varepsilon$, contradicting our hypothesis $d(y_n, P_C(x_n)) \geq \varepsilon$. This proves P_C is lHsc.

The uHsc of P_C follows from strong proximinality of C. □

From Proposition 4.5 and the arguments used before Example 3.9, it now follows that the subspace Y in [2, Example 2.5] can not be uniformly proximinal, while on the other hand being a finite dimensional subspace it is always strongly proximinal.

We now show that similar to proximinality and strong proximinality, the closed unit ball of a subspace by virtue of being uniformly proximinal forces the subspace to be uniformly proximinal.

Proposition 4.6. *For a subspace Y of X, if B_Y is uniformly proximinal then Y is also uniformly proximinal.*

Proof. We use the technique used in [1, Lemma 2.3]. If possible let B_Y is uniformly proximinal and Y is not. From the definition there exist $R > 0, \varepsilon > 0, x \in X$ where $d(x, Y) \leq R$ and also there exists $(y_n) \subseteq Y$ such that $\|x - y_n\| < R + \frac{1}{n}$ but for all $y \in B(y_n, \varepsilon), \|x - y\| > R$.

Choose $\lambda > \|x\| + R + 2\varepsilon$, then $d(x, \lambda B_Y) = d(x, Y)$. From our assumption on y_n it follows that $\|y_n\| < \|x\| + R + \frac{1}{n}$ and hence $y_n \in \lambda B_Y$.

Uniform proximinality of λB_Y (and hence B_Y) would be contradicted if we can show that $B_Y(y_n, \varepsilon) \subseteq \lambda B_Y$, for all n. And It follows from the following observation.

$\|y_n\| + \varepsilon < \|x\| + R + \varepsilon + \frac{1}{n} \leq \|x\| + R + 2\varepsilon < \lambda$, for large n.

This completes the proof. □

We now propose the following problem which is relevant to the subsequent matter.

Problem 4.7. *Let Y be a subspace of X which is uniformly proximinal. Is it necessary that B_Y is also uniformly proximinal in X ?*

Remark 4.8.
- (a) It is clear from the Definition 4.2 that uniform proximinality of C is a uniform version of strong proximinality for the points which are of finite distance away from C. Hence due to the Example by Godefroy in [10, Pg. 89] it is clear that closed unit ball of a Banach space not necessarily uniformly proximinal.
- (b) We do not know whether the converse of Example 4.3(e) is true or not.
- (c) From Theorem 3.12 we have if Y is separable and also has $1\frac{1}{2}$ ball property in X then $L_p(I, Y)$ has $1\frac{1}{2}$ ball property (hence uniformly proximinal) in $L_p(I, X)$ for $p = 1, \infty$.

From the Definition 4.2 we now have the following.

Theorem 4.9. *Let Y be a separable proximinal subspace of X, Consider the following statements.*

(a) $Y(B_Y)$ is uniformly proximinal in X.
(b) $L_\infty(I,Y)(B_{L_\infty(I,Y)})$ is uniformly proximinal in $L_\infty(I,X)$.
(c) $L_\infty(I,Y)(B_{L_\infty(I,Y)})$ is strongly proximinal in $L_\infty(I,X)$.
Then (a) $\Longleftrightarrow$ (b) and (b) $\Longrightarrow$ (c).

Proof. It is clear that $(b) \Longrightarrow (a)$ and $(b) \Longrightarrow (c)$. We only show that $(a) \Longrightarrow (b)$. We prove the result for the subspace Y, case for B_Y follows from that with obvious modifications.

Let us choose $R > 0$ and $\varepsilon > 0$. Choose $\delta(R, \varepsilon) > 0$ for the subspace Y. We claim that this δ will also work for $L_\infty(I,Y)$. Let $f \in L_\infty(I,X)$ with $d(f, L_\infty(I,Y)) \leq R$. Let $g \in L_\infty(I,Y)$ be such that $\|f-g\|_\infty < R+\delta$. Then from the property of uniform proximinality it follows that $B[f(t),R] \cap B[g(t),\varepsilon] \cap Y \neq \emptyset$ a.e. Consider the set valued map $\varphi : t \mapsto B[f(t),R] \cap B[g(t),\varepsilon] \cap Y$ from $[0,1]$ to 2^Y. It is clear that the graph of this map $\{(t, \phi(t) : t \in I)\}$ is measurable and whence by Theorem 3.2 it follows it has a measurable selection, let us call it h. We have $h \in L_\infty(I,Y)$ and satisfies the requirements. $\square$

Theorem 4.9 leads to the following problem.

Problem 4.10. *Let $L_\infty(I,Y)$ is strongly proximinal in $L_\infty(I,X)$. Is it true that Y is uniformly proximinal in X ?*

5. Ball Proximinality of $L_p(I,Y)$ in $L_p(I,X)$

We first prove the distance formula analogous to Theorem 3.4 for the closed unit ball of $L_p(I,Y)$, for $1 \leq p \leq \infty$.

Theorem 5.1. *Let $f \in L_p(I,X)$ be a strongly measurable function then*
$d(f, B_{L_p(I,Y)}) = \|d(f(.), B_Y)\|_p$, for $1 \leq p \leq \infty$.

Proof. Case for $p = \infty$ is already observed in [1], it remains to prove when $p < \infty$.

Step 1: Let $f(t) = x$ for all $t \in I$ and for some $x \in X$. Clearly $d(f, B_{L_p(I,Y)}) \leq d(f, L_p(I, B_Y)) = d(x, B_Y)$.

Let $g \in B_{L_p(I,Y)}$ and $\varepsilon > 0$, then there is a sequence of simple functions $(s_n) \subseteq B_{L_p(I,Y)}$ such that $s_n \to g$ in $L_p(I,Y)$. Without loss of generality we may assume each s_n has a following representation. $s_n = \sum_{i=1}^{k_n} y_{i,n}\chi_{E_{i,n}}$, where $y_{i,n} \in Y, \cup_i E_{i,n} = I$ and $E_{i,n} \cap E_{j,n} = \emptyset$ for $i \neq j$.

Define $z_n = \sum_i m(E_{i,n}) y_{i,n}$, then $\|z_n\|^p \leq \sum_i m(E_{i,n})\|y_{i,n}\|^p = \|s_n\|_p \leq 1$, first inequality follows from $x \mapsto \|x\|^p$ is a convex function. Hence $z_n \in B_Y$.

Now $d(x, B_Y)^p \leq \|x - z_n\|^p = \int_I \|f(t) - s_n(t)\|^p dm(t) = \|f - s_n\|_p^p \leq \|f - g\|_p^p + \varepsilon$ for all but finitely many n's. Taking infimum over $g \in B_{L_p(I,Y)}$ we get the result.

Step 2: Let $f = \sum_{i=1}^n x_i \chi_{E_i}$, where $x_i \in X$, $\cup_i E_i = I$ and $E_i \cap E_j = \emptyset$ for $i \neq j$.

Now

$$\begin{aligned}
d(f, B_{L_p(I,Y)})^p &\leq \int_I d(f(t), B_Y)^p dm(t) \\
&= \sum_1^n d(x_i, B_Y)^p m(E_i) \\
&= \sum_1^n d(x_i, B_{L_p(I,Y)})^p m(E_i) \text{ follows from Step 1} \\
&= \inf_{g \in B_{L_p(I,Y)}} \sum_1^n \int_{E_i} \|x_i - g(t)\|^p dm(t) \\
&= \inf_{g \in B_{L_p(I,Y)}} \int_I \|f(t) - g(t)\|^p dm(t) = d(f, B_{L_p(I,Y)})^p
\end{aligned}$$

Step 3: Let $f \in L_p(I, X)$ and $\varepsilon > 0$. Get a sequence of simple functions $(s_n) \subseteq L_p(I, X)$ such that $s_n \to f$ in $L_p(I, X)$. Without loss of generality assume s_n converges to f pointwise and $\|s_n(t)\| \leq \|f(t)\|$ a.e.
Now

$$\begin{aligned}
d(f, B_{L_p(I,Y)}) &= \inf_{g \in B_{L_p(I,Y)}} \|f - g\|_p \\
&\geq \inf_{g \in B_{L_p(I,Y)}} \|s_n - g\|_p - \|s_n - f\|_p \\
&= d(s_n, B_{L_p(I,Y)}) - \|s_n - f\|_p \\
&= \left(\int_I d(s_n(t), B_Y)^p dm(t)\right)^{1/p} - \|s_n - f\|_p; \text{ from Step 2} \\
&\geq \left(\int_I d(s_n(t), B_Y)^p dm(t)\right)^{1/p} - \varepsilon; \text{ for large } n \\
&\geq \left(\int_I d(f(t), B_Y)^p dm(t)\right)^{1/p} - 2\varepsilon; \text{ for large } n
\end{aligned}$$

The last inequality follows from the following observation.

$$\begin{aligned}
\|d(f(.), B_Y)\|_p &\leq \|d(f(.), B_Y) - d(s_n(.), B_Y)\|_p + \|d(s_n(.), B_Y)\|_p \\
&= \left(\int_I |d(f(t), B_Y) - d(s_n(t), B_Y)|^p \, dm(t)\right)^{1/p} + \\
&\qquad \|d(s_n(.), B_Y)\|_p \\
&\leq \left(\int_I \|f(t) - s_n(t)\|^p dm(t)\right)^{1/p} + \|d(s_n(.), B_Y)\|_p
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the result follows. □

Remark 5.2. (*a*) In [1] it is observed that for $f \in L_p(I, X)$, $d(f, L_p(I, B_Y)) = \|d(f(.), B_Y)\|_p$, hence from Theorem 5.1 it follows $P_{L_p(I,B_Y)}(f) \subseteq P_{B_{L_p(I,Y)}}(f)$ for $1 \leq p \leq \infty$.

(b) For a $g \in L_p(I, B_Y)$ we have, $g \in P_{B_{L_p(I,Y)}}(f) \iff$
$g(t) \in P_{B_Y}(f(t))$a.e. $\iff g \in P_{L_p(I,B_Y)}(f)$ for $1 \le p < \infty$.

Remark 5.2(a) leads to the following question.

Problem 5.3. *For a subspace Y of X what are the functions $f \in L_p(I, X)$ for $1 \le p < \infty$ for which $P_{B_{L_p(I,Y)}}(f) = P_{L_p(I,B_Y)}(f)$?*

We now prove the main result of this Section.

Theorem 5.4. *Let Y be a separable proximinal subspace of X. Then the following are equivalent.*

(a) Y is ball proximinal in X.
(b) $L_p(I, B_Y)$ is proximinal in $L_p(I, X)$, for $1 \le p \le \infty$.
(c) $L_p(I, Y)$ is ball proximinal in $L_p(I, X)$, for $1 \le p \le \infty$.

Proof. From [1] and Remark 5.2 it is now clear that $(a) \implies (b)$ and $(b) \implies (c)$. We now show that $(c) \implies (a)$. Now the Case for $p = \infty$ is already observed in [1], it remains to prove the result for $p < \infty$. Hence it is enough to prove that Y is ball proximinal in X if $L_p(I, Y)$ is same in $L_p(I, X)$ for some $p \in [1, \infty)$.

Let $x \in X$ and define $f(t) = x$ for all $t \in I$. Then $f \in L_p(I, X)$ and $d(f, B_{L_p(I,Y)}) = d(x, B_Y)$. Choose $g \in B_{L_p(I,Y)}$ satisfying $\|f - g\|_p = d(x, B_Y)$. Now choose a sequence of simple functions (s_n) such that $\|s_n - g\|_p \to 0$ where $\|s_n\|_p \le \|g\|_p$. Let $s_n = \sum_{i=1}^{k_n} x_i^n \chi_{E_i^n}$ where $x_i^n \in Y$ and $\cup_i E_i^n = I$. Let $y_n = \sum_{i=1}^{k_n} x_i^n m(E_i^n)$. Since $\sum_{i=1}^{k_n} \|x_i^n\|^p m(E_i^n) \le 1$ and $t \mapsto t^p$ is a convex function on $\mathbb{R}$ we have $y_n \in B_Y$. Now we have,

$$\begin{aligned}
d(x, B_Y)^p &\le \|x - y_n\|^p \\
&= \|x - \sum_{i=1}^{k_n} x_i^n m(E_i^n)\|^p \\
&= \|\sum_{i=1}^{k_n} (x - x_i^n) m(E_i^n)\|^p \\
&\le \sum_{i=1}^{k_n} \|x - x_i^n\|^p m(E_i^n) \\
&= \|x - s_n\|_p^p \\
&\to d(x, B_Y)^p
\end{aligned}$$

,which ensures that (y_n) is a minimizing sequence in B_Y for x. Clearly (y_n) is cauchy; in fact $\lim_n y_n = \int_I g(t) dm(t)$, and hence there exists $y_0 \in B_Y$ such that $\|x - y_0\| = d(x, B_Y)$. $\square$

The arguments involved in the proof of Corollary 3.7 lead to the following conclusion.

Corollary 5.5. *(a) Let Y be a ball proximinal subspace of X, if every separable subspace of Y is ball proximinal in X then $L_p(I, Y)$ is ball proximinal in $L_p(I, X)$ for $1 \le p \le \infty$.*

(*b*) *Let Y be a reflexive subspace of X then $L_p(I, B_Y)$ (and hence $B_{L_p(I,Y)}$) is proximinal in $L_p(I, X)$ for $1 \leq p \leq \infty$.*

Proof. We only prove (a), (b) follows from (a). It remains to prove for a given $f \in L_p(I, X)$, $P_{L_p(I,B_Y)}(f) \neq \emptyset$. Choose $(g_n) \subseteq L_p(I, B_Y)$ such that $\|f - g_n\|_p \to d(f, L_p(I, B_Y))$. Get a separable subspace $Z \subseteq Y$ such that $g_n(I) \subseteq Z$ for all n. It is clear that $d(f, L_p(I, B_Y)) = d(f, L_p(I, B_Z))$. Since $P_{L_p(I,B_Z)}(f) \neq \emptyset$ the result follows. □

We now come to the strong proximinality of closed unit ball of $L_p(I, Y)$. A few routine modifications of Theorem 3.4 lead to the following result.

Theorem 5.6. *Let Y be a strongly ball proximinal subspace of X and $f \in L_p(I, X), g \in L_p(I, X)$ then, $d(f, P_{B_{L_p(I,Y)}}(g)) = \|d(f(.), P_{B_Y}(g(.)))\|_p$, for $1 \leq p < \infty$.*

Combining Theorem 5.6 and the routine modifications in Theorem 3.6, one can have the following.

Theorem 5.7. *Let Y be a separable proximinal subspace of X. Then the following are equivalent.*

(*a*) *Y is strongly ball proximinal subspace of X.*
(*b*) *$L_p(I, B_Y)$ is strongly proximinal in $L_p(I, X)$, for $1 \leq p < \infty$.*
(*c*) *$L_p(I, Y)$ is strongly ball proximinal in $L_p(I, X)$, for $1 \leq p < \infty$.*

Proof. It remains to prove $(c) \implies (a)$. Choose $p \in [1, \infty)$ arbitrarily. Let $x \in X$ and $(y_n) \subseteq B_Y$ be such that $\|x - y_n\| \to d(x, B_Y)$. Define $f(t) = x$ and $g_n(t) = y_n$ for all $t \in I$ then $\|f - g_n\|_p \to d(f, B_{L_p(I,Y)}) = d(x, B_Y)$ and hence $d(g_n, P_{B_{L_p(I,Y)}}(f)) \to 0$. Choose $h_n \in P_{B_{L_p(I,Y)}}(f)$ such that $\|g_n - h_n\|_p \to 0$. Hence there exists $(z_n) \subseteq B_Y$ where $z_n = \int_I h_n(t)dm(t)$.

CLAIM: $z_n \in P_{B_Y}(x)$ and $\|y_n - z_n\| \to 0$.

$$
\begin{aligned}
d(x, B_Y)^p \leq \|x - z_n\|^p &= \|x - \int_I h_n(t)dm(t)\|^p \\
&= \|\int_I (h_n(t) - x)dm(t)\|^p \\
&\leq \int_I \|h_n(t) - x\|^p dm(t) \\
&= \int_I d(x, B_Y)^p dm(t), \text{ follows from Theorem 1.1} \\
&= d(x, B_Y)
\end{aligned}
$$

And finally,

$$\begin{aligned}
\|y_n - z_n\|^p &= \|y_n - \int_I h_n(t)dm(t)\|^p \\
&= \|\int_I (y_n - h_n(t))dm(t)\|^p \\
&\leq \int_I \|y_n - h_n(t)\|^p dm(t) \\
&\leq \|g_n - h_n\|_p^p \to 0
\end{aligned}$$

This completes the proof. □

For the case $p = \infty$ the result follows under an additional assumption on B_Y. The Banach spaces considered for rest of this Section are assumed to be real.

Now it is clear from the above observations that,

Corollary 5.8. *Let X be a separable Banach space.*

(a) *For $1 \leq p < \infty$, if B_X is strongly proximinal in X then $B_{L_p(I,X)}$ is stronly proximinal in $L_p(I, X)$.*

(b) *If X has $3.2.I.P.$ then $B_{L_p(I,X)}$ is stronly proximinal in $L_p(I, X)$ for $1 \leq p < \infty$.*

Proof. Since X is separable, Theorem 5.7 is true for $Y = X$ and hence (a) follows. If X has $3.2.I.P.$ then B_X is strongly proximinal in X (Example 6.8(a)). (b) is now follows from (a). □

Remark 5.9. (a) Uniform convexity of $L_p(I, X)$ for $1 < p < \infty$ follows from uniform convexity of X and vice versa. Hence Corollary 5.8 ensures the strong ball proximinality of $L_p(I, X)$ beyond the class of uniformly convex Banach space X.

(b) It is not necessarily true that $B_{L_\infty(I,Y)}$ is strongly proximinal in $L_\infty(I, X)$ if B_Y is same in X (Example 3.9).

6. A new class of uniformly proximinal subsets

Motivated from the property defined in Definition 1.5 we define the following for a closed unit ball of a subspace but more generally it can be defined for a closed convex subset.

Definition 6.1. We call the closed unit ball B_Y of a subspace Y in X has $1\frac{1}{2}$ ball property if for $x \in X, y \in B_Y$ and $r_1, r_2 > 0$ $B[x, r_1] \cap B_Y \neq \emptyset, \|x - y\| < r_1 + r_2$ implies $B[x, r_1] \cap B[y, r_2] \cap B_Y \neq \emptyset$.

Similar to our earlier observation Remark 4.4(a), the ball B_Y having $1\frac{1}{2}$-ball property is uniformly proximinal for $\delta = \varepsilon$. Here are few immediate consequences of the above property.

Theorem 6.2. *Let Y be a subspace of X. Then,*

(a) *If B_Y has $1\frac{1}{2}$ ball property then Y has $1\frac{1}{2}$ ball property.*

(b) *If B_Y has $1\frac{1}{2}$ ball property in X then Y is ball proximinal in X.*

The proofs of the above Theorem follow from the similar arguments used to prove for a subspace for a similar claim. One can revisit the proofs in [1, Proposition 2.4] for (a) and [23, Lemma 1.1] for (b).

Remark 6.3. The converse of Theorem 6.2(a) is not necessarily true. It is clear that a M-ideal has $1\frac{1}{2}$ ball property but not necessarily ball proximinal as is observed in [7].

We now derive a characterization, similar to Theorem 3.11, for $1\frac{1}{2}$ ball property of B_Y in X. An almost similar arguments can be used to prove the following, for the sake of completeness we briefly outline it here.

Notation 6.4. *For a subset C of X, define $C_\varepsilon = \{x \in X : d(x, B) \leq \varepsilon\}$.*

Theorem 6.5. *Let Y be a subspace of X, then the following are equivalent.*
  (a) *B_Y has $1\frac{1}{2}$ ball property.*
  (b) *$P_{B_Y}(x,\delta) = P_{B_Y}(x)_\delta \cap B_Y$. For all $x \in X$ and $\delta > 0$.*
  (c) *$d(y, P_{B_Y}(x)) = \|y - x\| - d(x, B_Y)$. For all $x \in X, y \in B_Y$.*

Proof. $(a) \implies (b)$: Let $d = d(x,Y)$ and $\|x - y\| \leq d + \delta$ for some $y \in B_Y$. By (a), $B[x, d] \cap B[y, \delta'] \cap B_Y \neq \emptyset$ for all $\delta' > \delta$. That is $B[y, \delta'] \cap P_{B_Y}(x) \neq \emptyset$ and hence $d(y, P_{B_Y}(x)) \leq \delta'$, true for all $\delta' > \delta$, thus $d(y, P_{B_Y}(x)) \leq \delta$. The other inclusion follows trivially from the definition of the sets involved in it.

$(b) \implies (c)$: Let $\varepsilon = \|y - x\| - d(x, B_Y)$, for $y \in B_Y$. Then $y \in P_{B_Y}(x,\varepsilon) = P_{B_Y}(x)_\varepsilon \cap B_Y$. Hence $d(y, P_{B_Y}(x)) \leq \varepsilon = \|y-x\| - d(x, B_Y)$. The other inequality is obvious.

$(c) \implies (a)$: Let $B[x, r_1] \cap B_Y \neq \emptyset$ and $\|x-y\| < r_1 + r_2$ for some $y \in B_Y$. Then $r_1 = d + \delta$ for some $\delta \geq 0$, where $d = d(x, B_Y)$. If possible let $B[x, r_1] \cap B[y, r_2] \cap B_Y = \emptyset$, that is $P_{B_Y}(x,\delta) \cap B[y, r_2] = \emptyset$. But then $P_{B_Y}(x)_\delta \cap B[y, r_2] = \emptyset$, that is $d(y, P_{B_Y}(x)) > r_2 + \delta$. By (c) $\|x - y\| - d > r_2 + \delta$ and finally $\|x - y\| > r_1 + r_2$, a contradiction. $\square$

We now show that the converse of Theorem 6.2(a) is not true.

Example 6.6. *Consider the space $X = (\mathbb{R}^2, \|.\|_2)$ and let $Z = X \bigoplus_\infty \mathbb{R}$. Then X is an M-ideal in Z but for $x = ((1,1),0) \in Z$, $\|x\| = \sqrt{2}$. Now for $y = ((\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), 1) \in B_Z$. we have, $1 = \|x - y\| < d(x, B_X) + d(y, P_{B_X}(x)) = \sqrt{2}$ and hence from Theorem 6.5 it follows that B_X can not have $1\frac{1}{2}$ ball property in Z.*

Remark 6.7.
  (a) From the above characterizations it is clear that $1\frac{1}{2}$ ball property of B_Y forces the subspace Y to be strongly ball proximinal.
  (b) From the example by Godefroy in [10] it is clear that the closed unit ball of a Banach space not necessarily have $1\frac{1}{2}$ ball property.

Remark 6.7(b) motivate us to investigate the class of Banach spaces and its subspaces whose closed unit balls are uniformly proximinal. The following examples are class of such spaces.

Example 6.8.
  (a) *If X has 3.2.I.P. then B_X has $1\frac{1}{2}$ ball property in X, hence the closed unit ball of such a space is strongly proximinal. Hence for any real measure μ, $L_1(\mu)$ or its isometric preduals have this property:*

Let $B[x,r] \cap B_X \neq \emptyset$ and $\|x-z\| < r+s$ for some $z \in B_X$. The balls $B[x,r], B[z,s], B_X$ are pairwise intersecting and hence has non empty intersection.

(b) *Let Y be a M-ideal in a 3.2.I.P space X then B_Y has $1\frac{1}{2}$-ball property in X: Let $B[x,r_1] \cap B_Y \neq \emptyset$ and $\|x-y\| < r_1+r_2$ for some $y \in B_Y$. Hence we have 3 balls $B[x,r_1], B[y,r_2], B_X$ in X intersect pairwise. From the property of 3.2.I.P. we have $B[x,r_1] \cap B[y,r_2] \cap B_X \neq \emptyset$. Now from* [7, Theorem 4.7] *it follows Y has strong 3-ball property. Hence considering above 3 balls once again one can have $B[x,r_1] \cap B[y,r_2] \cap B_X \cap Y \neq \emptyset$ which in turn equivalent to $B[x,r_1] \cap B[y,r_2] \cap B_Y \neq \emptyset$.*

From the Definition 6.1, Theorem 3.12 and the distance formulas proved in Theorem 5.1, 5.6, we have,

Corollary 6.9. *Let X be a separable Banach space. Then the following are equivalent.*

(a) *B_X has $1\frac{1}{2}$ ball property in X.*
(b) *$B_{L_1(I,X)}$ has $1\frac{1}{2}$ ball property in $L_1(I,X)$.*
(c) *$B_{L_\infty(I,X)}$ has $1\frac{1}{2}$ ball property in $L_\infty(I,X)$.*

Acknowledgments. The research was supported by DST-SERB, India. Award No. MA/2013-14/003/DST/TPaul/0104.

References

1. P. Bandyopadhyay, B.-L. Lin, T. S. S. R. K. Rao, *Ball proximinality in Banach spaces*, Banach spaces and their applications in analysis, Walter de Gruyter, Berlin, 2007, 251–264.
2. A. L. Brown, *Metric projections in spaces of integrable functions*, J. Approx. Theory **81** (1995), no. 1, 78–103.
3. S. Dutta and D. Narayana, *Strongly proximinal subspaces in Banach spaces*, Contemp. Math. 435, Amer. Math. Soc., Providence, RI, 2007, 143–152.
4. S. Dutta and P. Shunmugaraj, *Strong proximinality of closed convex sets*, J. Approx. Theory **163** (2011), no. 4, 547-553.
5. G. Godefroy and V. Indumathi, *Strong proximinality and polyhedral spaces*, Rev. Mat. Complut. **14** (2001), no. 1, 105-125.
6. G. Godini, *Best Approximation and Intersection of Balls*, Banach space theory and its applications, Lecture Notes in Mathematics, **991** (1983), 44–54.
7. C. R. Jayanarayanan and T. Paul, *Strong proximinality and intersection properties of balls in Banach spaces*, J. Math. Anal. Appl. **426** (2015), no. 2, 1217–1231.
8. R. Khalil, *Best Approximation in $L^p(I,X)$*, Math. Proc. Cambridge Phil. Soc. **94** (1983), 277–289.
9. R. Khalil and W. Deeb, *Best Approximation in $L^p(I,X), II$* J. Approx. Theory **59** (1989), 296–299.
10. S. Lalithambigai, *Ball proximinality of equable space*, Collect Math. **60** (2009), no. 1, 79–88.
11. Ka-Sing Lau, *On a sufficient condition of proximity*, Trans. Amer. Math. Soc. **251** (1979), 343–356.
12. W. A. Light, *Proximinality in $Lp(S,Y)$*, Rocky Mountain J. Math. **19** (1989), 251–259.
13. W. A. Light and E. W. Cheney, *Approximation Theory in Tensor Product Spaces* Lecture Notes in Mathematics **1169** Springer-Verlag.
14. A. Lima, *Intersection properties of balls and subspaces in Banach spaces*, Trans. Amer. Math. Soc. **227** (1977), 1–62.

15. J. Lindenstrauss *Extension of Compact Operators*, Mem Amer. Math. Soc. **48**, 1964, 112 pp.
16. P.-K. Lin, W. Zhang, and B. Zheng, *Stability of ball proximinality*, J. Approx. Theory **183** (2014), 72–81.
17. J. Mendoza *Proximinality in* $L_p(\mu, X)$, J. Approx. Theory **93** (1998), 331–343.
18. E. Michael, Continuous selection. I. Ann. of Math. **63** (1956), no. 2, 361–382.
19. D. V. Pai and P. T. Nowroji, *On restricted centers of sets*, J. Approx. Theory **66** (1991), no. 2, 170–189.
20. T. S. S. R. K. Rao, *The one and half ball property in spaces of vector-valued functions*, J. Convex Anal. **20** (2013), no. 1, 13–23.
21. F. B. Saidi, *On the proximinality of the unit ball and of proximinal subspaces of Banach space: A counterexample*, Proc. Amer. Math. Soc. **133** (2005), no. 9, 2697–2703.
22. S. M. Srivastava, *A Course on Borel Sets*, Grad. Texts in Math. 180, Springer, New York, 1998.
23. D. Yost *Best approximation and intersection of balls in Banach spaces*, Bull. Austral. Math. Soc. **20** (1979), 285–300.

Department of Mathematics, Indian Institute of Technology Hyderabad, Kandi Campus, Sangareddy, Telangana 502285, India.
E-mail address: tanmoy@iith.ac.in