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TRIGONOMETRIC POLYNOMIALS OVER HOMOGENEOUS SPACES OF COMPACT GROUPS

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Abstract. This paper presents a systematic study for trigonometric polynomials over homogeneous spaces of compact groups. Let H be a closed subgroup of a compact group G. Using the abstract notion of dual space $\widehat{G/H}$, we introduce the space of trigonometric polynomials $\mathrm{Trig}(G/H)$ over the compact homogeneous space G/H. As an application for harmonic analysis of trigonometric polynomials, we prove that the abstract dual space of any homogeneous space of compact groups separates points of the homogeneous space in some sense.

1. Introduction

The abstract aspects of harmonic analysis over homogeneous spaces of compact non-Abelian groups or precisely left coset (resp. right coset) spaces of non-normal subgroups of compact non-Abelian groups is placed as building blocks for coherent states analysis [7, 10], theoretical and particle physics [1]. Over the last decades, abstract and computational aspects of Plancherel formulas over symmetric spaces have achieved significant popularity in geometric analysis, mathematical physics and scientific computing (computational engineering), see [2, 3, 4, 8, 9, 11, 14] and references therein.

Let G be a compact group, H be a closed subgroup of G. Then, the left coset space G/H is considered as a compact homogeneous space, which G acts on it via the left action. The current paper is organized as follows. Section 2 is devoted

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to fixing notations and a brief summary on analytic and abstract aspects of trigonometric polynomials over compact groups and classical results on harmonic analysis of compact homogeneous spaces. Next we present abstract harmonic analysis of function spaces over homogeneous spaces of compact groups. We then introduce the abstract notion of dual space $\widehat{G/H}$ and the space of trigonometric polynomials $\mathrm{Trig}(G/H)$ over the compact homogeneous space G/H. Using the abstract notion of dual space $\widehat{G/H}$, we present a systematic study for the space of trigonometric polynomials $\mathrm{Trig}(G/H)$ over the compact homogeneous space G/H. Then, as a consequence of analytic aspects of trigonometric polynomials over homogeneous spaces, we prove that dual space of any homogeneous space of compact groups separates points of the homogeneous space G/H in some sense.

2. Preliminaries and Notations

Let G be a compact group with the probability Haar measure dx. Each irreducible representation of G is finite dimensional and every unitary representation of G is a direct sum of irreducible representations, see [1, 5]. The set of all unitary equivalence classes of irreducible unitary representations of G is denoted by $\widehat{G}$. This definition of $\widehat{G}$ is in essential agreement with the classical definition, when G is Abelian, since each character of an Abelian group is a one dimensional representation of G. If π is any unitary representation of G, for $u, v \in \mathcal{H}_\pi$ the functions $\pi_{u,v}(x) = \langle \pi(x)u, v\rangle$ are called matrix elements of π. If $\{e_j\}$ is an orthonormal basis for $\mathcal{H}_\pi$, then π_{ij} means π_{e_i,e_j}. The notation $\mathrm{Trig}_\pi(G)$ is used for the linear span of the matrix elements of π and the notation $\mathrm{Trig}(G)$ is used for the linear span of $\bigcup_{[\pi]\in\widehat{G}} \mathrm{Trig}_\pi(G)$. The Peter-Weyl Theorem (see [1, 5, 6]) guarantees that, $\mathrm{Trig}(G)$ is uniformly dense in $\mathcal{C}(G)$, and $\|.\|_{L^p(G)}$-dense in $L^p(G)$.

Let H be a closed subgroup of the compact group G with the probability Haar measure dh. The left coset space G/H is considered as a compact homogeneous space that G acts on it from the left and $q : G \to G/H$ given by $x \mapsto q(x) := xH$ is the surjective canonical mapping. The function space $\mathcal{C}(G/H)$ consists of all functions $T_H(f)$, where $f \in \mathcal{C}(G)$ and

$$T_H(f)(xH) = \int_H f(xh)dh. \tag{2.1}$$

This equivalently means that the linear map $T_H : \mathcal{C}(G) \to \mathcal{C}(G/H)$ is a surjective bounded linear operator. Let μ be a Radon measure on G/H and $x \in G$. The translation μ_x of μ is defined by $\mu_x(E) = \mu(xE)$, for Borel subsets E of G/H. The measure μ is called G-invariant if $\mu_x = \mu$, for $x \in G$. If G is compact, the homogeneous space G/H has a normalized G-invariant measure μ, satisfying the Weil's formula

$$\int_{G/H} T_H(f)(xH)d\mu(xH) = \int_G f(x)dx, \qquad f \in L^1(G). \tag{2.2}$$

It is well-known that $\|T_H(f)\|_{L^1(G/H,\mu)} \leq \|f\|_{L^1(G)}$, see [1, 5, 13].

3. Abstract Harmonic Analysis over Homogeneous Spaces of Compact Groups

Throughout this paper we assume that G is a compact group with the probability Haar measure dx, H is a closed subgroup of G with the probability Haar measure dh, and μ is the normalized G-invariant measure on the compact homogeneous space G/H satisfying (2.2).

The following proposition shows that the linear map $T_H : \mathcal{C}(G) \to \mathcal{C}(G/H)$ is uniformly continuous.

Proposition 3.1. *Let H be a closed subgroup of a compact group G. The linear map $T_H : \mathcal{C}(G) \to \mathcal{C}(G/H)$ is uniformly continuous.*

Proof. Let $f \in \mathcal{C}(G)$ and $x \in G$. Then we have

$$|T_H(f)(xH)| = \left| \int_H f(xh)dh \right| \leq \int_H |f(xh)|dh \leq \|f\|_{\sup} \left(\int_H dh \right) = \|f\|_{\sup},$$

which implies

$$\|T_H(f)\|_{\sup} = \sup_{xH \in G/H} |T_H(f)(xH)| \leq \|f\|_{\sup}.$$

□

For a function $\psi : G/H \to \mathbb{C}$, define the function $\psi_q : G \to \mathbb{C}$ by

$$\psi_q(x) := \psi \circ q(x) = \psi(xH),$$

for all $xH \in G/H$.

Then we can present the following results.

Corollary 3.2. *Let H be a closed subgroup of a compact group G and $\psi \in \mathcal{C}(G/H)$. Then*

1. *$\psi_q \in \mathcal{C}(G)$ and $T_H(\psi_q) = \psi$.*
2. *$\|\psi\|_{\sup} = \|\psi_q\|_{\sup}$.*

Proof. Let $\psi \in \mathcal{C}(G/H)$. (1) It is easy to see that $\psi_q \in \mathcal{C}(G)$. Let $x \in G$. Then we can write

$$T_H(\psi_q)(xH) = \int_H \psi_q(xh)dh = \int_H \psi(xhH)dh = \int_H \psi(xH)dh = \psi(xH).$$

(2) Also, we can write

$$\|\psi\|_{\sup} = \sup_{xH \in G/H} |\psi(xH)| = \sup_{x \in G} |\psi(xH)| = \sup_{x \in G} |\psi_q(x)| = \|\psi_q\|_{\sup}.$$

□

Next theorem proves that the linear map T_H is norm-decreasing in other L^p-spaces, when $p > 1$.

Theorem 3.3. *Let H be a closed subgroup of a compact group G, μ be the normalized G-invariant measure on G/H associated to the Weil's formula, and $p \geq 1$. The linear map $T_H : \mathcal{C}(G) \to \mathcal{C}(G/H)$ has a unique extension to a bounded linear map from $L^p(G)$ onto $L^p(G/H, \mu)$.*

Proof. We shall show that, each $f \in \mathcal{C}(G)$ satisfies

$$\|T_H(f)\|_{L^p(G/H,\mu)} \leq \|f\|_{L^p(G)}. \tag{3.1}$$

Using compactness of H, and the Weil's formula, we can write

$$\begin{aligned}
\|T_H(f)\|^p_{L^p(G/H,\mu)} &= \int_{G/H} |T_H(f)(xH)|^p d\mu(xH) \\
&= \int_{G/H} \left| \int_H f(xh) dh \right|^p d\mu(xH) \\
&\leq \int_{G/H} \left(\int_H |f(xh)| dh \right)^p d\mu(xH) \\
&\leq \int_{G/H} \int_H |f(xh)|^p dh d\mu(xH) \\
&= \int_{G/H} \int_H |f|^p(xh) dh d\mu(xH) \\
&= \int_{G/H} T_H(|f|^p)(xH) d\mu(xH) = \int_G |f(x)|^p dx = \|f\|^p_{L^p(G)},
\end{aligned}$$

which implies (3.1). Thus, we can extend T_H to a bounded linear operator from $L^p(G)$ onto $L^p(G/H,\mu)$, which still will be denoted by T_H. □

As an immediate consequence of Theorem 3.3 we deduce the following corollary.

Corollary 3.4. *Let H be a closed subgroup of a compact group G, μ be the normalized G-invariant measure on G/H associated to the Weil's formula, and $p \geq 1$. Let $\varphi \in L^p(G/H,\mu)$ and $\varphi_q := \varphi \circ q$. Then $\varphi_q \in L^p(G)$ with*

$$\|\varphi_q\|_{L^p(G)} = \|\varphi\|_{L^p(G/H,\mu)}. \tag{3.2}$$

Proof. Indeed, using the Weil's formula, we can write

$$\begin{aligned}
\|\varphi_q\|^p_{L^p(G)} &= \int_G |\varphi_q(x)|^p dx \\
&= \int_{G/H} T_H\left(|\varphi_q|^p\right)(xH) d\mu(xH) = \int_{G/H} \left(\int_H |\varphi_q(xh)|^p dh \right) d\mu(xH),
\end{aligned}$$

and since H is compact and dh is a probability measure, we get

$$\begin{aligned}
\int_{G/H} \left(\int_H |\varphi_q(xh)|^p dh \right) d\mu(xH) &= \int_{G/H} \left(\int_H |\varphi(xhH)|^p dh \right) d\mu(xH) \\
&= \int_{G/H} \left(\int_H |\varphi(xH)|^p dh \right) d\mu(xH) \\
&= \int_{G/H} |\varphi(xH)|^p \left(\int_H dh \right) d\mu(xH) \\
&= \int_{G/H} |\varphi(xH)|^p d\mu(xH) = \|\varphi\|^p_{L^p(G/H,\mu)},
\end{aligned}$$

which implies (3.2). □

Let $\mathcal{J}^2(G,H) := \{f \in L^2(G) : T_H(f) = 0\}$ and $\mathcal{J}^2(G,H)^\perp$ be the orthogonal complement of the closed subspace $\mathcal{J}^2(G,H)$ in $L^2(G)$.

Proposition 3.5. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H associated to the Weil's formula. Then $T_H : L^2(G) \to L^2(G/H,\mu)$ is a partial isometric linear map.*

Proof. Let $\varphi \in L^2(G/H,\mu)$. Then, we claim that $T_H^*(\varphi) = \varphi_q$, and hence $T_H T_H^*(\varphi) = \varphi$. Indeed, using the Weil's formula, we can write

$$\begin{aligned}\langle T_H^*(\varphi), f\rangle_{L^2(G)} &= \langle \varphi, T_H(f)\rangle_{L^2(G/H,\mu)} \\ &= \int_{G/H} \varphi(xH)\overline{T_H(f)(xH)}d\mu(xH) \\ &= \int_{G/H} \varphi(xH)T_H(\overline{f})(xH)d\mu(xH) \\ &= \int_{G/H} T_H(\varphi_q.\overline{f})(xH)d\mu(xH) = \int_G \varphi_q(x)\overline{f(x)}dx = \langle \varphi_q, f\rangle_{L^2(G)},\end{aligned}$$

for all $f \in L^2(G)$, which implies that $T_H^*(\varphi) = \varphi_q$. Now a straightforward calculation shows that $T_H = T_H T_H^* T_H$. Then, by Theorem 2.3.3 of [12], T_H is a partial isometric operator. $\square$

The following corollaries are straightforward consequences of Proposition 3.5.

Corollary 3.6. *Let H be a closed subgroup of a compact group G. Let $P_{\mathcal{J}^2(G,H)}$ and $P_{\mathcal{J}^2(G,H)^\perp}$ be the orthogonal projections onto the closed subspaces $\mathcal{J}^2(G,H)$ and $\mathcal{J}^2(G,H)^\perp$ respectively. Then, for each $f \in L^2(G)$ and a.e. $x \in G$, we have*

(1) $P_{\mathcal{J}^2(G,H)^\perp}(f)(x) = T_H(f)(xH)$.
(2) $P_{\mathcal{J}^2(G,H)}(f)(x) = f(x) - T_H(f)(xH)$.

Corollary 3.7. *Let H be a compact subgroup of a locally compact group G and μ be the normalized G-invariant measure on G/H associated to the Weil's formula. Then*

(1) $\mathcal{J}^2(G,H)^\perp = \{\psi_q = \psi \circ q : \psi \in L^2(G/H,\mu)\}$.
(2) *For $f \in \mathcal{J}^2(G,H)^\perp$ and $h \in H$, we have $R_h f = f$.*
(3) *For $f, g \in \mathcal{J}^2(G,H)^\perp$, we have $\langle T_H(f), T_H(g)\rangle_{L^2(G/H,\mu)} = \langle f, g\rangle_{L^2(G)}$.*

Remark 3.8. Invoking Corollary 3.7, one can regard the Hilbert space $L^2(G/H,\mu)$ as a closed subspace of $L^2(G)$, that is the closed subspace consists of all $f \in L^2(G)$ which satisfies $R_h f = f$, for all $h \in H$. Then Theorem 3.3 and Proposition 3.5 guarantee that the linear map

$$T_H : L^2(G) \to L^2(G/H,\mu) \subset L^2(G)$$

is an orthogonal projection onto $L^2(G/H,\mu)$.

4. Trigonometric Polynomials over Homogeneous Spaces of Compact Groups

In this section we shall introduce the abstract notion of trigonometric polynomials over homogeneous spaces of compact groups. We then study basic properties of trigonometric polynomials.

For a closed subgroup H of G, define

$$H^{\perp} := \left\{[\pi] \in \widehat{G} : \pi(h) = I \quad \text{for all } h \in H\right\}, \tag{4.1}$$

If G is Abelian, each closed subgroup H of G is normal and the locally compact group G/H is Abelian, then the character group $\widehat{G/H}$ is the set of all characters (one dimensional irreducible representations) of G which are constant on H, that is precisely $H^{\perp}$. If G is a non-Abelian group and H is a closed normal subgroup of G, then the dual space $\widehat{G/H}$ which is the set of all unitary equivalence classes of unitary representations of G/H, has meaning and it is well-defined. Indeed, G/H is a non-Abelian group. In this case, the map $\Phi : \widehat{G/H} \to H^{\perp}$ defined by $\sigma \mapsto \Phi(\sigma) := \sigma \circ q$ is a Borel isomorphism and $\widehat{G/H} = H^{\perp}$, see [1, 5]. Thus, if H is normal, $H^{\perp}$ coincides with the classic definitions of the dual space either when G is Abelian or non-Abelian.

Definition 4.1. Let H be a closed subgroup of a compact group G and $\Omega \subseteq \widehat{G}$. The homogeneous space G/H satisfies the Ω-separation property, if $\pi(x) = I_{\mathcal{H}_\pi}$ for all $[\pi] \in \Omega$ imply $x \in H$. In this case, we say that Ω separates points of G/H, at times. It is evident to check that, Ω separates points of G/H if and only if for any $xH \neq yH$ one can find $[\pi] \in \Omega$ such that $\pi(x) \neq \pi(y)$.

The following theorem shows that the homogeneous space G/H does not satisfy the $H^{\perp}$-separation property, if H is not normal in G.

Theorem 4.2. *Let H be a closed subgroup of a compact group G. Then, $H^{\perp}$ separates points of G/H if and only if H is normal.*

Proof. Let H be a normal subgroup of G. Then, G/H is a compact group. Thus, Gelfand–Raikove Theorem guarantees that the dual space of G/H, which is precisely $H^{\perp}$, separates points of G/H. Conversely, assume that $H^{\perp}$ separates points of G/H. Let $x \in G$ and $h \in H$. Then we have

$$\pi(x^{-1}hx) = \pi(x^{-1})\pi(h)\pi(x) = \pi(x)^*\pi(h)\pi(x) = \pi(x)^*\pi(x) = I_{\mathcal{H}_\pi},$$

for all $[\pi] \in H^{\perp}$. Then, the $H^{\perp}$-separation property of G/H implies $x^{-1}hx \in H$. Since $x \in G$ and $h \in H$ are arbitrary, we obtain that H is normal in G. □

Remark 4.3. Theorem 4.2 guarantees that $H^{\perp}$ is not the appropriate candidate to be considered as the dual space of the homogeneous space G/H, when H is not a normal subgroup of G. In fact, $H^{\perp}$ is not large enough with respect to the left coset space G/H, if H is a given closed subgroup of G unless H be normal in G.

For a closed subgroup H of G and a continuous unitary representation $(\pi, \mathcal{H}_\pi)$ of G, define

$$T_H^\pi := \int_H \pi(h)dh, \tag{4.2}$$

where the operator valued integral (4.2) is considered in the weak sense.

In other words,

$$\langle T_H^\pi \zeta, \xi\rangle = \int_H \langle \pi(h)\zeta, \xi\rangle dh, \quad \text{for } \zeta, \xi \in \mathcal{H}_\pi. \tag{4.3}$$

The function $h \mapsto \langle \pi(h)\zeta, \xi\rangle$ is bounded and continuous on H. Since H is compact, the right integral is the ordinary integral of a function in $L^1(H)$. Hence, T_H^π is a bounded linear operator on $\mathcal{H}_\pi$ with $\|T_H^\pi\| \leq 1$.

Remark 4.4. Let $(\pi, \mathcal{H}_\pi)$ be a continuous unitary representation of G with $T_H^\pi \neq 0$. Let $(\sigma, \mathcal{H}_\sigma)$ be a continuous unitary representation of G such that $[\pi] = [\sigma]$. Let $S : \mathcal{H}_\pi \to \mathcal{H}_\sigma$ be the unitary operator which satisfies $\sigma(x)S = S\pi(x)$ for all $x \in G$. Then we have

$$\begin{aligned} ST_H^\pi &= S\left(\int_H \pi(h)dh\right) = \int_H S\pi(h)dh \\ &= \int_H \sigma(h)Sdh = \left(\int_H \sigma(h)dh\right) S = T_H^\sigma S, \end{aligned}$$

which implies that $T_H^\sigma \neq 0$ as well. Thus we deduce that the non-zero property of T_H^π depends only on $[\pi]$, that is the unitary equivalence class of $(\pi, \mathcal{H}_\pi)$.

Let

$$\mathcal{K}_\pi^H := \{\zeta \in \mathcal{H}_\pi : \pi(h)\zeta = \zeta \ \forall h \in H\}. \tag{4.4}$$

Then, $\mathcal{K}_\pi^H$ is a closed subspace of $\mathcal{H}_\pi$ and $\mathcal{R}(T_H^\pi) = \mathcal{K}_\pi^H$, where

$$\mathcal{R}(T_H^\pi) = \{T_H^\pi \zeta : \zeta \in \mathcal{H}_\pi\}.$$

It is easy to see that $[\pi] \in H^\perp$ if and only if $\mathcal{K}_\pi^H = \mathcal{H}_\pi$.

Proposition 4.5. *The linear operator T_H^π is an orthogonal projection.*

Proof. Using compactness of H, it can be easily checked that $(T_H^\pi)^* = T_H^\pi$. As well as, we can write

$$\begin{aligned} T_H^\pi T_H^\pi &= \left(\int_H \pi(h)dh\right)\left(\int_H \pi(t)dt\right) \\ &= \int_H \pi(h)\left(\int_H \pi(t)dt\right)dh \\ &= \int_H \left(\int_H \pi(h)\pi(t)dt\right)dh \\ &= \int_H \left(\int_H \pi(ht)dt\right)dh = \int_H T_H^\pi dt = T_H^\pi. \end{aligned}$$

□

Invoking Remark 4.3, we can suggest the following dual space for G/H.

Definition 4.6. Let H be a closed subgroup of a compact group G. Then, we define the dual space of G/H, as the subset of $\widehat{G}$ which is given by

$$\widehat{G/H} := \left\{[\pi] \in \widehat{G} : T_H^\pi \neq 0\right\} = \left\{[\pi] \in \widehat{G} : \int_H \pi(h)dh \neq 0\right\}. \tag{4.5}$$

Evidently, any closed subgroup H of G satisfies

$$H^\perp \subseteq \widehat{G/H}. \tag{4.6}$$

Let $(\pi, \mathcal{H}_\pi)$ be a continuous unitary representation of G such that $T_H^\pi \neq 0$. Then, the functions $\pi^H_{\zeta,\xi} : G/H \to \mathbb{C}$ defined by

$$\pi^H_{\zeta,\xi}(xH) := \langle \pi(x)T_H^\pi \zeta, \xi\rangle \quad \text{for } xH \in G/H, \tag{4.7}$$

for $\xi, \zeta \in \mathcal{H}_\pi$ are called H-matrix elements of $(\pi, \mathcal{H}_\pi)$.

For $xH \in G/H$ and $\zeta, \xi \in \mathcal{H}_\pi$, we have

$$\begin{aligned}|\pi^H_{\zeta,\xi}(xH)| &= |\langle \pi(x)T_H^\pi \zeta, \xi\rangle| \\ &\leq \|\pi(x)T_H^\pi \zeta\| \cdot \|\xi\| \\ &\leq \|T_H^\pi \zeta\| \cdot \|\xi\| \leq \|\zeta\| \cdot \|\xi\|.\end{aligned}$$

Also, we can write

$$\pi^H_{\zeta,\xi}(xH) = \langle \pi(x)T_H^\pi \zeta, \xi\rangle = \pi_{T_H^\pi \zeta,\xi}(x). \tag{4.8}$$

Invoking definition of the linear map T_H and T_H^π, we have

$$\begin{aligned}T_H(\pi_{\zeta,\xi})(xH) &= \int_H \pi_{\zeta,\xi}(xh)dh \\ &= \int_H \langle \pi(xh)\zeta, \xi\rangle dh \\ &= \int_H \langle \pi(x)\pi(h)\zeta, \xi\rangle dh \\ &= \langle \pi(x)\left(\int_H \pi(h)dh\right)\zeta, \xi\rangle = \langle \pi(x)T_H^\pi \zeta, \xi\rangle,\end{aligned}$$

which implies

$$T_H(\pi_{\zeta,\xi}) = \pi^H_{\zeta,\xi}. \tag{4.9}$$

The linear span of the H-matrix elements of a continuous unitary representation $(\pi, \mathcal{H}_\pi)$ satisfying $T_H^\pi \neq 0$, is denoted by $\mathrm{Trig}_\pi(G/H)$ which is a subspace of $\mathcal{C}(G/H)$.

Next proposition lists basic properties of H-matrix elements.

Proposition 4.7. *Let H be a closed subgroup of a compact group G, μ be the normalized G-invariant measure on G/H associated to the Weil's formula, and $(\pi, \mathcal{H}_\pi)$ be a continuous unitary representation of G. Then,*

(1) $T_H^\pi = 0$ *if and only if* $\mathrm{Trig}_\pi(G) \subseteq \mathcal{J}^2(G, H)$.
(2) *If* $T_H^\pi \neq 0$ *then* $T_H(\mathrm{Trig}_\pi(G)) = \mathrm{Trig}_\pi(G/H)$.
(3) $\mathrm{Trig}_\pi(G) \subseteq \mathcal{J}^2(G, H)^\perp$ *if and only if* $\pi(h) = I$ *for all* $h \in H$.

Proof. Let $(\pi, \mathcal{H}_\pi)$ be a continuous unitary representation of G. (1) Let $T_H^\pi = 0$ and $\zeta, \xi \in \mathcal{H}_\pi$. Then, it is straightforward to see that $T_H(\pi_{\zeta,\xi}) = 0$ and hence we deduce that $\mathrm{Trig}_\pi(G) \subseteq \mathcal{J}^2(G,H)$. Conversely, let $\mathrm{Trig}_\pi(G) \subseteq \mathcal{J}^2(G,H)$ and also $\zeta, \xi \in \mathcal{H}_\pi$. Then we have

$$\begin{aligned}\langle T_H^\pi \zeta, \xi\rangle &= \langle \left(\int_H \pi(h)dh\right)\zeta, \xi\rangle \\ &= \int_H \langle \pi(h)\zeta, \xi\rangle dh \\ &= \int_H \pi_{\zeta,\xi}(h)dh = T_H(\pi_{\zeta,\xi})(H) = 0,\end{aligned}$$

which implies that $T_H^\pi = 0$.

(2) Let $T_H^\pi \neq 0$. Then, it is easy to check that $T_H(\mathrm{Trig}_\pi(G)) \subseteq \mathrm{Trig}_\pi(G/H)$. Now let $\zeta, \xi \in \mathcal{H}_\pi$ and $\psi := \pi^H_{\zeta,\xi} = T_H(\pi_{\zeta,\xi})$ be a H-matrix element. Then, we have $\psi_q = \pi_{\zeta',\xi}$, where $\zeta' = T_H^\pi\zeta$. Hence, we get $\psi_q \in \mathrm{Trig}_\pi(G)$. Since $T_H(\psi_q) = \psi$ we deduce that $\mathrm{Trig}_\pi(G/H) \subseteq T_H(\mathrm{Trig}_\pi(G))$. Therefore, we get

$$T_H(\mathrm{Trig}_\pi(G)) = \mathrm{Trig}_\pi(G/H).$$

(3) Let $\pi(h) = I$ for all $h \in H$. Also, let $\zeta, \xi \in \mathcal{H}_\pi$ and $f := \pi_{\zeta,\xi}$ be a matrix element. Then, for $h \in H$, we have

$$\begin{aligned}R_h f(x) &= f(xh) \\ &= \pi_{\zeta,\xi}(xh) \\ &= \langle \pi(x)\pi(h)\zeta, \xi\rangle \\ &= \langle \pi(x)\zeta, \xi\rangle = f(x).\end{aligned}$$

Thus, using Corollary 3.7, we deduce that $f = \pi_{\zeta,\xi} \in \mathcal{J}^2(G,H)^\perp$. Since $\mathrm{Trig}_\pi(G)$ is the linear subspace spanned by matrix elements, we get that $\mathrm{Trig}_\pi(G) \subseteq \mathcal{J}^2(G,H)^\perp$. Conversely, let $\mathrm{Trig}_\pi(G) \subseteq \mathcal{J}^2(G,H)^\perp$. Also, let $h \in H$ and $\zeta, \xi \in \mathcal{H}_\pi$. Then, $f := \pi_{\zeta,\xi} \in \mathrm{Trig}_\pi(G)$. Thus, we have $f \in \mathcal{J}^2(G,H)^\perp$. Invoking Corollary 3.7, we get $R_h f = f$. Hence, we have $\langle \pi(h)\zeta, \xi\rangle = \langle \zeta, \xi\rangle$. Since $\zeta, \xi \in \mathcal{H}_\pi$ was arbitrary, we have $\pi(h) = I$. □

We then define

$$\mathrm{Trig}(G/H) := \text{the linear span of } \bigcup_{[\pi]\in\widehat{G/H}} \mathrm{Trig}_\pi(G/H). \tag{4.10}$$

Functions in $\mathrm{Trig}(G/H)$ are called *trigonometric polynomials* over G/H.

The following result shows that T_H maps $\mathrm{Trig}(G)$ onto $\mathrm{Trig}(G/H)$.

Proposition 4.8. *The linear operator T_H maps* $\mathrm{Trig}(G)$ *onto* $\mathrm{Trig}(G/H)$.

Proof. Invoking definitions of $\mathrm{Trig}(G)$ and $\mathrm{Trig}(G/H)$, using Proposition 4.7 we deduce that the linear operator T_H maps $\mathrm{Trig}(G)$ onto $\mathrm{Trig}(G/H)$. □

The following theorem presents basic properties of trigonometric polynomials.

Theorem 4.9. *Let H be a closed subgroup of a compact group G, μ be the normalized G-invariant measure and $(\pi, \mathcal{H}_\pi)$ be a continuous unitary representation of G such that $T_H^\pi \neq 0$. Then*

1. *The subspace $\mathrm{Trig}_\pi(G/H)$ depends on the unitary equivalence class of π.*
2. *The subspace $\mathrm{Trig}_\pi(G/H)$ is a closed left invariant subspace of $L^1(G/H, \mu)$.*

Proof. (1) Let $(\sigma, \mathcal{H}_\sigma)$ be a continuous unitary representation of G such that $[\pi] = [\sigma]$. Let $S : \mathcal{H}_\pi \to \mathcal{H}_\sigma$ be the unitary operator which satisfies $\sigma(x)S = S\pi(x)$ for all $x \in G$. Remark 4.4 guarantees that $ST_H^\pi = T_H^\sigma S$ and also $T_H^\sigma \neq 0$. Thus for $x \in G$ and $\zeta, \xi \in \mathcal{H}_\pi$ we can write

$$\begin{aligned}\pi^H_{\zeta,\xi}(xH) &= \langle \pi(x)T_H^\pi \zeta, \xi\rangle_{\mathcal{H}_\pi}\\ &= \langle S^{-1}\sigma(x)ST_H^\pi \zeta, \xi\rangle_{\mathcal{H}_\pi}\\ &= \langle \sigma(x)ST_H^\pi \zeta, S\xi\rangle_{\mathcal{H}_\sigma}\\ &= \langle \sigma(x)T_H^\sigma S\zeta, S\xi\rangle_{\mathcal{H}_\sigma} = \sigma^H_{S\zeta,S\xi}(xH),\end{aligned}$$

which implies that $\mathrm{Trig}_\pi(G/H) = \mathrm{Trig}_\sigma(G/H)$.
(2) It is straightforward. □

Next theorem presents some analytic aspects of trigonometric polynomials over G/H as a function space.

Theorem 4.10. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H associated to the Weil's formula. Then,*

1. *$\mathrm{Trig}(G/H)$ is $\|.\|_{L^p(G/H,\mu)}$-dense in $L^p(G/H, \mu)$.*
2. *$\mathrm{Trig}(G/H)$ is $\|.\|_{\sup}$-dense in $\mathcal{C}(G/H)$.*

Proof. (1) Let $p \geq 1$ and $\phi \in L^p(G/H, \mu)$. Let $f \in L^p(G)$ with $T_H(f) = \phi$. By $\|.\|_{L^p(G)}$-density of $\mathrm{Trig}(G)$ in $L^p(G)$ we can pick a sequence $\{f_n\}$ in $\mathrm{Trig}(G)$ such that $f = \|.\|_{L^p(G)} - \lim_n f_n$. Then, continuity of the linear map $T_H : L^p(G) \to L^p(G/H, \mu)$, implies

$$\phi = T_H(f) = \|.\|_{L^p(G/H,\mu)} - \lim_n T_H(f_n),$$

which completes the proof.
(2) Invoking uniformly boundedness of T_H, uniformly density of $\mathrm{Trig}(G)$ in $\mathcal{C}(G)$, and the same argument as used in (1), we get $\|.\|_{\sup}$-density of $\mathrm{Trig}(G/H)$ in $\mathcal{C}(G/H)$. □

The following theorem guarantees $\widehat{G/H}$-separation property of the homogeneous space G/H.

Theorem 4.11. *Let H be a closed subgroup of a compact group G. Then $\widehat{G/H}$ separates points of G/H.*

Proof. Let $x, y \in G$ with $xH \neq yH$. Let $\psi \in \mathcal{C}(G/H)$ such that $\psi(xH) \neq \psi(yH)$. Since $\mathrm{Trig}(G/H)$ is $\|.\|_{\sup}$-dense in $\mathcal{C}(G/H)$, we can uniformly approximate ψ with elements of $T_H(\mathrm{Trig}(G)) = \mathrm{Trig}(G/H)$. Thus, there exists an irreducible representation $(\pi, \mathcal{H}_\pi)$ of G and non-zero vectors $\zeta, \xi \in \mathcal{H}_\pi$ such that

$$T_H(\pi_{\zeta,\xi})(xH) \neq T_H(\pi_{\zeta,\xi})(yH).$$

This automatically guarantees that $T_H^\pi \neq 0$, and $\pi(x) \neq \pi(y)$ as well. □

Next, as an application of our results, we shall show that the reverse inclusion of (4.6) holds, if and only if H is a normal subgroup of G.

Corollary 4.12. *Let H be a closed subgroup of a compact group G. Then, H is normal in G if and only if $\widehat{G/H} = H^\perp$.*

Proof. Let $\widehat{G/H} = H^\perp$. Theorem 4.11 guarantees that G/H satisfies the $H^\perp$-separation property. Then, Theorem 4.2 implies that H is normal in G. Conversely, let H be a closed normal subgroup of the compact group G. Then Theorem 4.2 of [4] implies that $\widehat{G/H} = H^\perp$.

□

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