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SOME LOWER BOUNDS FOR THE NUMERICAL RADIUS OF HILBERT SPACE OPERATORS

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Abstract. We show that if T is a bounded linear operator on a complex Hilbert space, then

$$\frac{1}{2}\|T\| \leq \sqrt{\frac{w^2(T)}{2} + \frac{w(T)}{2}\sqrt{w^2(T) - c^2(T)}} \leq w(T),$$

where $w(\cdot)$ and $c(\cdot)$ are the numerical radius and the Crawford number, respectively. We then apply it to prove that for each $t \in [0, \frac{1}{2})$ and natural number k,

$$\frac{(1+2t)^{\frac{1}{2k}}}{2^{\frac{1}{k}}} m(T) \leq w(T),$$

where $m(T)$ denotes the minimum modulus of T. Some other related results are also presented.

1. Introduction and preliminaries

Let $\mathbb{B}(H)$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space H with an inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. If $\dim H = n$, we identify $\mathbb{B}(H)$ with the space $\mathcal{M}_n$ of all $n \times n$ matrices with entries in the complex field. For $T \in \mathbb{B}(H)$, let $\|T\|$ and $m(T)$ denote the usual operator norm and the minimum modulus of T, respectively. Here $m(T)$ is defined to be the largest number $\alpha \geq 0$ such that $\|Tx\| \geq \alpha \|x\|$ $(x \in H)$. The numerical radius

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and the Crawford number of $T \in \mathbb{B}(H)$ are defined by

$$w(T) = \sup\{|\langle Tx, x\rangle| : x \in H, \|x\| = 1\}$$

and

$$c(T) = \inf\{|\langle Tx, x\rangle| : x \in H, \|x\| = 1\},$$

respectively. These concepts are useful in studying linear operators and have attracted the attention of many authors in the last few decades (e.g., see [4, 8], and their references). It is well known that $w(\cdot)$ defines a norm on $\mathbb{B}(H)$ such that for all $T \in \mathbb{B}(H)$,

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|. \tag{1.1}$$

The inequalities in (1.1) are sharp. The first inequality becomes an equality if $T^2 = 0$. The second inequality becomes an equality if T is normal. Any operator $T \in \mathbb{B}(H)$ can be represented as $T = H + iK$, the so-called Cartesian decomposition, where $H = \mathrm{Re}(T) = \frac{T+T^*}{2}$ and $K = \mathrm{Im}(T) = \frac{T-T^*}{2i}$ are called the real and imaginary parts of T. It has been shown in [7] that,

$$\sup\left\{\|\alpha H + \beta K\| : \alpha, \beta \in \mathbb{R}, \alpha^2 + \beta^2 = 1\right\} = w(T).$$

In particular, $\|H\| \leq w(T)$ and $\|K\| \leq w(T)$.

Concerning the inequality (1.1), Kittaneh [6] has shown the following precise estimate of $w(T)$ by using norm inequalities:

$$\frac{1}{\sqrt{2}}\sqrt{\|H^2 + K^2\|} \leq w(T) \leq \sqrt{\|H^2 + K^2\|}. \tag{1.2}$$

Obviously, (1.2) is sharper than the inequality of (1.1). Yamazaki [9] has used the Aluthge transform to improve the second inequality (1.1) so that

$$w(T) \leq \frac{1}{2}\left(\|T\| + w(\widetilde{T})\right).$$

Here $\widetilde{T}$ (the Aluthge transform of T) is defined as $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, where U is a partial isometry of the polar decomposition of T and $|T| = (T^*T)^{\frac{1}{2}}$ means the absolute value of T.

Further, it has been shown in [1] that,

$$\frac{1}{2}\sqrt{\left\||T|^2 + |T^*|^2\right\| + 2c(T^2)} \leq w(T) \leq \frac{1}{2}\sqrt{\left\||T|^2 + |T^*|^2\right\| + 2w(T^2)}.$$

For more material about the numerical radius and other results on numerical radius inequality, see, e.g., [3], [5], and the references therein.

For $T \in \mathbb{B}(H)$, let us recall the abbreviated notations

$$|\cos|T = \inf\left\{\frac{|\langle Tx, x\rangle|}{\|Tx\|\|x\|} : x \in H, \|Tx\| \neq 0\right\}$$

and

$$|\sin|T = \sqrt{1 - |\cos|^2 T}.$$

In the next section, we establish some considerable improvement of the first inequality (1.1). More precisely, we prove that

$$\frac{1}{2}\|T\| \leq \sqrt{\frac{w^2(T)}{2} + \frac{w(T)}{2}\sqrt{w^2(T) - c^2(T)}} \leq w(T)$$

and

$$\frac{1}{2}\|T\| \leq \max\left\{|\sin|T, \frac{\sqrt{2}}{2}\right\} w(T) \leq w(T).$$

Next, we will give some applications. Particularly, for each $t \in [0, \frac{1}{2})$ and natural number k, we show that

$$\frac{(1+2t)^{\frac{1}{2k}}}{2^{\frac{1}{k}}} m(T) \leq w(T).$$

2. Main results

In this section we present some lower bounds for the numerical radii of Hilbert space operators. We start our work with the following result.

Theorem 2.1. *Let $T \in \mathbb{B}(H)$. Then*

$$\frac{1}{2}\|T\| \leq \sqrt{\frac{w^2(T)}{2} + \frac{w(T)}{2}\sqrt{w^2(T) - c^2(T)}} \leq w(T).$$

Proof. Clearly, $\sqrt{\frac{w^2(T)}{2} + \frac{w(T)}{2}\sqrt{w^2(T) - c^2(T)}} \leq w(T)$. On the other hand, let $x \in H$ with $\|x\| \leq 1$. Let $\langle Tx, x\rangle = \lambda_x |\langle Tx, x\rangle|$ for some unit $\lambda_x \in \mathbb{C}$. Hence $\langle \overline{\lambda}_x Tx, x\rangle = |\langle Tx, x\rangle| \geq 0$. Let $H + iK$ be the Cartesian decomposition of $\overline{\lambda}_x T$. Then $\langle Hx, x\rangle + i\langle Kx, x\rangle = \langle \overline{\lambda}_x Tx, x\rangle \geq 0$. Hence

$$\langle \overline{\lambda}_x Tx, x\rangle = \langle Hx, x\rangle, \qquad \langle Kx, x\rangle = 0.$$

We have

$$\begin{aligned}
\frac{1}{4}\|Tx\|^2 &= \frac{1}{4}\left(\left\|\overline{\lambda}_x Tx - \langle \overline{\lambda}_x Tx, x\rangle x\right\|^2 + |\langle Tx, x\rangle|^2\right) \\
&= \frac{1}{4}\left(\|Hx - \langle Hx, x\rangle x + iKx\|^2 + |\langle Tx, x\rangle|^2\right) \qquad (\text{since } \langle Kx, x\rangle = 0) \\
&\leq \frac{1}{4}\left((\|Hx - \langle Hx, x\rangle x\| + \|Kx\|)^2 + |\langle Tx, x\rangle|^2\right) \\
&\leq \frac{1}{4}\left(\left(\sqrt{\|Hx\|^2 - |\langle Hx, x\rangle|^2} + \|Kx\|\right)^2 + |\langle Tx, x\rangle|^2\right) \\
&\leq \frac{1}{4}\left(\left(\sqrt{w^2(T) - |\langle Tx, x\rangle|^2} + w(T)\right)^2 + |\langle Tx, x\rangle|^2\right) \qquad (2.1) \\
&\qquad\qquad (\text{since } \|Hx\|, \|Kx\| \leq w(T) \text{ and } |\langle Tx, x\rangle| = |\langle Hx, x\rangle|) \\
&= \frac{w^2(T)}{2} + \frac{w(T)}{2}\sqrt{w^2(T) - |\langle Tx, x\rangle|^2}.
\end{aligned}$$

Hence

$$\frac{1}{2}\|Tx\| \leq \sqrt{\frac{w^2(T)}{2} + \frac{w(T)}{2}\sqrt{w^2(T) - |\langle Tx, x\rangle|^2}} \qquad (\|x\| \leq 1). \tag{2.2}$$

If we replace x by $\frac{x}{\|x\|}$ in the above inequality, then we obtain

$$\begin{aligned}
\frac{1}{2}\|Tx\| &\leq \|x\|\sqrt{\frac{w^2(T)}{2} + \frac{w(T)}{2}\sqrt{w^2(T) - \left|\langle T(\frac{x}{\|x\|}), \frac{x}{\|x\|}\rangle\right|^2}} \\
&\leq \sqrt{\frac{w^2(T)}{2} + \frac{w(T)}{2}\sqrt{w^2(T) - \left|\langle T(\frac{x}{\|x\|}), \frac{x}{\|x\|}\rangle\right|^2}} \\
&\leq \sqrt{\frac{w^2(T)}{2} + \frac{w(T)}{2}\sqrt{w^2(T) - c^2(T)}}.
\end{aligned}$$

Thus

$$\frac{1}{2}\|Tx\| \leq \sqrt{\frac{w^2(T)}{2} + \frac{w(T)}{2}\sqrt{w^2(T) - c^2(T)}}.$$

Taking the supremum over $x \in H$ with $\|x\| \leq 1$ in the above inequality we deduce the desired inequality. □

Remark 2.2. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $\|A\| = w(A) = c(A) = 1$. Thus

$$\frac{1}{2}\|A\| = \frac{1}{2} < \sqrt{\frac{w^2(A)}{2} + \frac{w(A)}{2}\sqrt{w^2(A) - c^2(A)}} = \frac{\sqrt{2}}{2} < w(A) = 1.$$

Hence the inequalities in Theorem 2.1 can be strict.

Corollary 2.3. *Let $T \in \mathbb{B}(H)$. Then*

$$\|Tx\|^2 + |\langle Tx, x\rangle|^2 \leq 4w^2(T) \qquad (x \in H,\ \|x\| \leq 1).$$

Proof. Let $x \in H$ with $\|x\| \leq 1$. By (2.1) it follows that

$$\begin{aligned}
\frac{1}{4}\|Tx\|^2 &\leq \frac{1}{4}\left(\left(\sqrt{w^2(T) - |\langle Tx, x\rangle|^2} + w(T)\right)^2 + |\langle Tx, x\rangle|^2\right) \\
&\leq \frac{1}{4}\left(2\left(\sqrt{w^2(T) - |\langle Tx, x\rangle|^2}\right)^2 + 2w^2(T) + |\langle Tx, x\rangle|^2\right) \\
&\qquad\qquad \text{(by the arithmetic geometric mean inequality)} \\
&= \frac{1}{4}\left(4w^2(T) - |\langle Tx, x\rangle|^2\right),
\end{aligned}$$

which gives $\|Tx\|^2 + |\langle Tx, x\rangle|^2 \leq 4w^2(T)$. □

Corollary 2.4. *Let $A = [a_{ij}] \in \mathcal{M}_n$. Then*

$$\frac{\sum_{k=1}^{n} |a_{ki}|^2}{2} \leq w^2(A) + w(A)\sqrt{w^2(A) - |a_{ii}|^2} \qquad (1 \leq i \leq n).$$

Proof. Let $x = [0, \cdots, 0, 1, 0, \cdots, 0]^t$ with 1 in place of i. Then $Ax = [a_{1i}, a_{2i}, \cdots, a_{ni}]^t$ and $\langle Ax, x\rangle = a_{ii}$. So, by (2.2) we obtain

$$\frac{1}{2}\sqrt{\sum_{k=1}^{n}|a_{ki}|^2} = \frac{1}{2}\|Ax\| \leq \sqrt{\frac{w^2(A)}{2} + \frac{w(A)}{2}\sqrt{w^2(A) - |\langle Ax, x\rangle|^2}}$$
$$= \sqrt{\frac{w^2(A)}{2} + \frac{w(A)}{2}\sqrt{w^2(A) - |a_{ii}|^2}}.$$

This yields

$$\frac{\sum_{k=1}^{n}|a_{ki}|^2}{2} \leq w^2(A) + w(A)\sqrt{w^2(A) - |a_{ii}|^2}.$$

□

Theorem 2.5. *Let $T \in \mathbb{B}(H)$. Then*

$$\frac{1}{2}\|T\| \leq \max\left\{|\sin|T, \frac{\sqrt{2}}{2}\right\} w(T) \leq w(T).$$

Proof. Clearly, $\max\left\{|\sin|T, \frac{\sqrt{2}}{2}\right\} w(T) \leq w(T)$. On the other hand, let $x \in H$ with $\|x\| \leq 1$. By (2.2) we have

$$\frac{1}{2}\|Tx\| \leq \sqrt{\frac{w^2(T)}{2} + \frac{w(T)}{2}\sqrt{w^2(T) - |\langle Tx, x\rangle|^2}}.$$

Hence

$$\frac{1}{2}\|Tx\| \leq \sqrt{\frac{w^2(T)}{2} + \frac{w(T)}{2}\sqrt{w^2(T) - \|Tx\|^2|\cos|^2 T}},$$

or equivalently,

$$\|Tx\|^2 - 2w^2(T) \leq 2w(T)\sqrt{w^2(T) - \|Tx\|^2|\cos|^2 T}. \tag{2.3}$$

We consider two cases.

Case 1. $\|Tx\|^2 - 2w^2(T) \leq 0$. So we get $\|Tx\| \leq \sqrt{2}w(T)$ and hence

$$\frac{1}{2}\|T\| \leq \frac{\sqrt{2}}{2}w(T). \tag{2.4}$$

Case 2. $\|Tx\|^2 - 2w^2(T) > 0$. It follows from (2.3) that

$$\|Tx\|^4 - 4\|Tx\|^2 w^2(T) + 4w^4(T) \leq 4w^4(T) - 4w^2(T)\|Tx\|^2|\cos|^2 T.$$

This implies

$$\|Tx\|^2 \leq 4\left(1 - |\cos|^2 T\right) w^2(T)$$

which yields

$$\frac{1}{2}\|Tx\| \leq |\sin|T w(T).$$

Taking the supremum over $x \in H$ with $\|x\| \leq 1$ in the above inequality we get

$$\frac{1}{2}\|T\| \leq |\sin|T w(T). \tag{2.5}$$

Finally, by (2.4) and (2.5) we conclude the desired inequality. □

Remark 2.6. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1+i \end{bmatrix}$. Simple computations show that $\|A\| = w(A) = \sqrt{2}$ and $|\sin|A = \sqrt{2} - 1$. Thus

$$\frac{1}{2}\|A\| = \frac{\sqrt{2}}{2} < \max\left\{|\sin|A, \frac{\sqrt{2}}{2}\right\} w(A) = \frac{\sqrt{2}}{2} \times \sqrt{2} = 1 < w(A) = \sqrt{2}.$$

Hence the inequalities in Theorem 2.5 can be strict.

As a consequence of Theorem 2.5 we have the following result.

Corollary 2.7. *Let $T, S \in \mathbb{B}(H)$. Then*

$$w(TS) \leq 4\max\left\{|\sin|T, \frac{\sqrt{2}}{2}\right\} \max\left\{|\sin|S, \frac{\sqrt{2}}{2}\right\} w(T)w(S) \leq 4w(T)w(S).$$

Proof. Applying the second inequality of (1.1) and Theorem 2.5, we get

$$\begin{aligned} w(TS) &\leq \|TS\| \\ &\leq \|T\|\|S\| \\ &\leq 2\max\left\{|\sin|T, \frac{\sqrt{2}}{2}\right\} w(T) \times 2\max\left\{|\sin|S, \frac{\sqrt{2}}{2}\right\} w(S) \\ &= 4\max\left\{|\sin|T, \frac{\sqrt{2}}{2}\right\} \max\left\{|\sin|S, \frac{\sqrt{2}}{2}\right\} w(T)w(S) \leq 4w(T)w(S). \end{aligned}$$

□

A fundamental inequality for the numerical radius is the power inequality, which says that for $T \in \mathbb{B}(H)$,

$$w(T^k) \leq w^k(T)$$

for $k = 1, 2, \cdots$ (see, e.g., [5]). We are now in a position to establish one of our main results.

Theorem 2.8. *Let $T \in \mathbb{B}(H)$. For each $t \in [0, \frac{1}{2})$ and natural number k,*

$$\frac{(1+2t)^{\frac{1}{2k}}}{2^{\frac{1}{k}}} m(T) \leq w(T).$$

Proof. Let $t \in [0, \frac{1}{2})$ and $k \in \mathbb{N}$. Let $x \in H$ with $\|x\| \leq 1$. We consider two cases.

Case 1. $\|Tx\|^2 - 2w^2(T) \leq 0$. So we have

$$\begin{aligned} &w^2(T) - 2tw(T)\mathrm{Re}\langle Tx, x\rangle + (t^2 - \frac{1}{4})\|Tx\|^2 \\ &\geq w^2(T) - 2tw(T)\mathrm{Re}\langle Tx, x\rangle + 2(t^2 - \frac{1}{4})w^2(T) \\ &= 2w^2(T)\left|t - \frac{\langle Tx, x\rangle}{2w(T)}\right|^2 + \frac{w^2(T) - |\langle Tx, x\rangle|^2}{2} \geq 0. \end{aligned}$$

Hence

$$w^2(T) - 2tw(T)\mathrm{Re}\langle Tx, x\rangle + (t^2 - \frac{1}{4})\|Tx\|^2 \geq 0. \tag{2.6}$$

Case 2. $\|Tx\|^2 - 2w^2(T) > 0$. It follows from (2.2) that

$$\frac{1}{2}\|Tx\| \leq \sqrt{\frac{w^2(T)}{2} + \frac{w(T)}{2}\sqrt{w^2(T) - |\langle Tx, x\rangle|^2}}.$$

This implies

$$\left(\frac{1}{4}\|Tx\|^2 - \frac{w^2(T)}{2}\right)^2 \leq \frac{w^2(T)}{4}\left(w^2(T) - |\langle Tx, x\rangle|^2\right)$$

which yields

$$4w^2(T)\|Tx\|^2 - \|Tx\|^4 - 4w^2(T)|\langle Tx, x\rangle|^2 \geq 0. \tag{2.7}$$

By (2.7), we get

$$\begin{aligned} & w^2(T) - 2tw(T)\mathrm{Re}\langle Tx, x\rangle + (t^2 - \frac{1}{4})\|Tx\|^2 \\ & = \|Tx\|^2 \left| t - \frac{w(T)\langle Tx, x\rangle}{\|Tx\|^2}\right|^2 + \frac{4w^2(T)\|Tx\|^2 - \|Tx\|^4 - 4w^2(T)|\langle Tx, x\rangle|^2}{4\|Tx\|^2} \geq 0, \end{aligned}$$

whence

$$w^2(T) - 2tw(T)\mathrm{Re}\langle Tx, x\rangle + (t^2 - \frac{1}{4})\|Tx\|^2 \geq 0. \tag{2.8}$$

By (2.6) and (2.8), we obtain

$$2tw(T)\mathrm{Re}\langle Tx, x\rangle \leq w^2(T) + (t^2 - \frac{1}{4})\|Tx\|^2.$$

If we replace T by $\frac{\mathrm{Re}\langle Tx,x\rangle}{|\mathrm{Re}\langle Tx,x\rangle|}T$ in the above inequality, then we get

$$2tw(T)\,|\mathrm{Re}\langle Tx, x\rangle| \leq w^2(T) + (t^2 - \frac{1}{4})\|Tx\|^2 \qquad (\|x\| \leq 1). \tag{2.9}$$

Furthermore, if we replace T by $e^{i\theta}T$ in (2.9), then we deduce

$$2tw(T)\left|\mathrm{Re}\left(e^{i\theta}\langle Tx, x\rangle\right)\right| \leq w^2(T) + (t^2 - \frac{1}{4})\|Tx\|^2.$$

Since $\sup\left\{\left|\mathrm{Re}\left(e^{i\theta}\langle Tx, x\rangle\right)\right| : \theta \in \mathbb{R}\right\} = |\langle Tx, x\rangle|$, by taking the supremum over $\theta \in \mathbb{R}$ in the above inequality we reach

$$2tw(T)|\langle Tx, x\rangle| \leq w^2(T) + (t^2 - \frac{1}{4})\|Tx\|^2. \tag{2.10}$$

By (2.10), we get

$$2tw(T)|\langle Tx, x\rangle| \leq w^2(T) + (t^2 - \frac{1}{4})\|Tx\|^2 \leq w^2(T) + (t^2 - \frac{1}{4})m^2(T).$$

Thus

$$2tw(T)|\langle Tx, x\rangle| \leq w^2(T) + (t^2 - \frac{1}{4})m^2(T). \tag{2.11}$$

By taking the supremum over $x \in H$ with $\|x\| = 1$ in (2.11), we obtain

$$2tw^2(T) \leq w^2(T) + (t^2 - \frac{1}{4})m^2(T),$$

or equivalently,

$$\frac{(1+2t)^{\frac{1}{2}}}{2} m(T) \leq w(T).$$

Replacing T by T^k in the last inequality gives

$$\frac{(1+2t)^{\frac{1}{2}}}{2} m(T^k) \leq w(T^k).$$

Since $m^k(T) \leq m(T^k)$ and $w(T^k) \leq w^k(T)$, the above inequality becomes

$$\frac{(1+2t)^{\frac{1}{2}}}{2} m^k(T) \leq w^k(T).$$

Thus $\frac{(1+2t)^{\frac{1}{2k}}}{2^{\frac{1}{k}}} m(T) \leq w(T)$. □

Remark 2.9. Recall that an operator $T \in \mathbb{B}(H)$ is said to be idempotent if $T^2 = T$ and an involution if $T^2 = I$. It is well-known that, if T is idempotent such that $T \neq 0$, then $w(T) = \frac{1}{2}(1 + \|T\|)$ and if T is involution then, $w(T) = \frac{1}{2}(\|T\| + \|T\|^{-1})$ (see, e.g., [1]). So, by Theorem 2.8 for each $t \in [0, \frac{1}{2})$ and $k \in \mathbb{N}$, the following statements hold:

(i) If T is an idempotent operator such that $T \neq 0$, then

$$2^{1-\frac{1}{k}}(1+2t)^{\frac{1}{2k}} m(T) \leq 1 + \|T\|.$$

(ii) If T is an involution operator, then

$$2^{1-\frac{1}{k}}(1+2t)^{\frac{1}{2k}} m(T) \leq \|T\| + \|T\|^{-1}.$$

Corollary 2.10. *Let $T \in \mathbb{B}(H)$. For each $t \in [0, \frac{1}{2})$,*

$$\frac{\|T\|}{2} \leq \sqrt{\frac{w^2(T) - 2tw(T)\mu(T)}{1 - 4t^2}},$$

where $\mu(T) = \inf\{|Re\langle Tx, x\rangle| : x \in H, \|x\| \leq 1\}$.

Proof. Let $t \in [0, \frac{1}{2})$ and let $x \in H$ with $\|x\| \leq 1$. By (2.9), we have

$$2tw(T)\,|\mathrm{Re}\langle Tx, x\rangle| \leq w^2(T) + (t^2 - \frac{1}{4})\|Tx\|^2.$$

Since $\mu(T) = \inf\{|\mathrm{Re}\langle Tx, x\rangle| : x \in H, \|x\| \leq 1\}$, so by the above inequality we obtain

$$w^2(T) - 2tw(T)\mu(T) \geq w^2(T) - 2tw(T)\,|\mathrm{Re}\langle Tx, x\rangle| \geq (\frac{1}{4} - t^2)\|Tx\|^2.$$

Hence

$$(\frac{1}{4} - t^2)\|Tx\|^2 \leq w^2(T) - 2tw(T)\mu(T).$$

By taking the supremum over $x \in H$ with $\|x\| = 1$ in the above inequality, we wet

$$(\frac{1}{4} - t^2)\|T\|^2 \leq w^2(T) - 2tw(T)\mu(T).$$

Now, by the last inequality, we deduce the desired inequality. □

Let us recall that by [2, Lemma 2.1] we have

$$w(x \otimes y) = \frac{1}{2}\left(|\langle x, y\rangle| + \|x\|\|y\|\right),$$

for all $x, y \in H$. Here, $x \otimes y$ denotes the rank one operator in $\mathbb{B}(H)$ defined by $(x \otimes y)(z) := \langle z, y\rangle x$ for all $z \in H$. The following result is a reverse the Cauchy-Schwarz inequality in the setting of Hilbert spaces.

Corollary 2.11. *Let $x, y \in H$. For each $t \in [0, \frac{1}{2})$ and $k \in \mathbb{N}$, the following statements hold.*

(i) $\left(\frac{1}{\max\left\{\sqrt{1-\inf\left\{\frac{|\langle x,z\rangle|^2}{\|x\|^2\|z\|^2}: z\in H, \langle z,y\rangle\neq 0\right\}}, \frac{\sqrt{2}}{2}\right\}} - 1\right)\|x\|\|y\| \leq |\langle x, y\rangle|.$

(ii) $\left(2^{1-\frac{1}{k}}(1+2t)^{\frac{1}{2k}} \inf\{|\langle z, y\rangle| : z \in H, \|z\| = 1\} - \|y\|\right)\|x\| \leq |\langle x, y\rangle|.$

Proof. Simple computations show that

$$|\sin|(x \otimes y) = \sqrt{1 - \inf\left\{\frac{|\langle x, z\rangle|^2}{\|x\|^2\|z\|^2} : z \in H, \langle z, y\rangle \neq 0\right\}} \tag{2.12}$$

and

$$m(x \otimes y) = \|x\| \inf\{|\langle z, y\rangle| : z \in H, \|z\| = 1\}. \tag{2.13}$$

So, by Theorem 2.5 and (2.12), we obtain

$$\frac{1}{2}\|x\|\|y\| \leq \max\left\{|\sin|(x \otimes y), \frac{\sqrt{2}}{2}\right\}\frac{1}{2}\left(|\langle x, y\rangle| + \|x\|\|y\|\right),$$

or equivalently,

$$\left(\frac{1}{\max\left\{\sqrt{1 - \inf\left\{\frac{|\langle x,z\rangle|^2}{\|x\|^2\|z\|^2} : z \in H, \langle z, y\rangle \neq 0\right\}}, \frac{\sqrt{2}}{2}\right\}} - 1\right)\|x\|\|y\| \leq |\langle x, y\rangle|.$$

Furthermore, for each $t \in [0, \frac{1}{2})$ and $k \in \mathbb{N}$, by Theorem 2.8 and (2.13) we get

$$\frac{(1+2t)^{\frac{1}{2k}}}{2^{\frac{1}{k}}}\|x\| \inf\{|\langle z, y\rangle| : z \in H, \|z\| = 1\} \leq \frac{1}{2}\left(|\langle x, y\rangle| + \|x\|\|y\|\right),$$

or equivalently,

$$\left(2^{1-\frac{1}{k}}(1+2t)^{\frac{1}{2k}} \inf\{|\langle z, y\rangle| : z \in H, \|z\| = 1\} - \|y\|\right)\|x\| \leq |\langle x, y\rangle|.$$

□

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