Adv. Oper. Theory 2 (2017), no. 2, 108–113
http://doi.org/10.22034/aot.1612-1067
ISSN: 2538-225X (electronic)
http://aot-math.org

ON MAPS COMPRESSING THE NUMERICAL RANGE BETWEEN C^*-ALGEBRAS

ASHWAQ FAHAD ALBIDEEWI[1] and MOHAMED MABROUK[1,2*]

Communicated by A. Sims

ABSTRACT. In this paper, we deal with the problem of characterizing linear maps compressing the numerical range. A counterexample is given to show that such a map need not be a Jordan $*$-homomorphism in general even if the C^*-algebras are commutative. Next, under an auxiliary condition we show that such a map is a Jordan $*$-homomorphism.

1. INTRODUCTION

Let $\mathcal{A}$ and $\mathcal{B}$ be unital complex Banach algebras. Denote by $\mathbf{1}_{\mathcal{A}}$ and $\mathbf{1}_{\mathcal{B}}$ the units of $\mathcal{A}$ and $\mathcal{B}$ respectively (or simply $\mathbf{1}$ if no confusion can arise). Define the set of normalized states

$$S(\mathcal{A}) = \{f \in \mathcal{A}' : f(\mathbf{1}) = \|f\| = 1\},$$

where $\mathcal{A}'$ denotes the dual space. For any element $a \in \mathcal{A}$, the algebraic *numerical range* $V(a)$ and *numerical radius* $v(a)$ of a are defined by

$$V(a) = \{f(a) : f \in S(\mathcal{A})\} \text{ and } v(a) = \sup_{z\in V(a)} |z|.$$

It is well known that V is a compact and convex set of the complex plane, $v(.)$ is a norm on $\mathcal{A}$ and this norm is equivalent to the usual operator norm. The suggested references on numerical ranges are [2, 10]. A linear map $T : \mathcal{A} \longrightarrow \mathcal{B}$ is said to be numerical range (resp. numerical radius) preserving if $V(T(a)) = V(a)$

Date: Received: Dec. 2, 2016; Accepted: Feb. 16, 2017.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 15A86, 47A12; Secondary 46L05, 47B49.
Key words and phrases. Numerical Range; C^*-algebra, compressing the numerical range, Jordan $*$-homomorphism.

(resp. $v(T(a)) = v(a)$) for every $a \in \mathcal{A}$. Also, we shall say that T compresses the numerical range if $V(T(a)) \subset V(a)$ for every $a \in \mathcal{A}$.

There has been considerable interest in studying maps between C^*-algebras leaving invariant the numerical range or the numerical radius. A nice survey of earlier known results relating to the preserving problem can be found in [4, 14]. In 1975, Pellegrini [16] studied numerical range preserving operators on a Banach algebra. Particularly, when $\mathcal{A}$ and $\mathcal{B}$ are two C^*-algebras, it was shown that a linear isomorphism $T : \mathcal{A} \longrightarrow \mathcal{B}$ is a Jordan $*$-isomorphism if and only if it is numerical range preserving. Later, Chan [5] showed that a linear isomorphism $T : \mathcal{A} \longrightarrow \mathcal{A}$ is numerical radius preserving if and only if cT is a Jordan $*$-isomorphism for some central and unitary element $c \in \mathcal{A}$. Surjective nonlinear maps $T : \mathcal{A} \longrightarrow \mathcal{B}$ between unital C^*-algebras that satisfy $v(T(a) - T(b)) = v(a - b)$ for all $a, b \in \mathcal{A}$ were characterized in [1] under a mild condition that $T(\mathbf{1}) - T(0)$ belongs to the center of $\mathcal{B}$. Recently, in [3], the assumption $T(\mathbf{1}) - T(0)$ belongs to the center of $\mathcal{B}$ is successfully removed.

The aim of this paper, is to study maps between C^*-algebras compressing the numerical range. Firstly, we shall give an example showing that such a map need not to be a Jordan $*$-homomorphism. Next, We will show that under some supplementary condition such a map is a Jordan $*$-homomorphism.

We close this Introduction with some definitions and properties of the numerical range needed in the sequel. In the case of C^*-algebra, a linear functional $f \in \mathcal{A}'$ is said to be *positive* ($f \geq 0$) if $f(xx^*) \geq 0$ for all $x \in \mathcal{A}$. Note that the set of normalized states $S(\mathcal{A})$ is nothing but

$$S(\mathcal{A}) = \{f \in \mathcal{A}' : f \geq 0 \text{ and } f(\mathbf{1}) = 1\}.$$

Recall also that a positive linear functional f on $\mathcal{A}$ is said to be *pure* if for every positive functional g on $\mathcal{A}$ satisfying $g(xx^*) \leq f(xx^*)$ for all $x \in \mathcal{A}$, there is a scalar $0 \leq \lambda \leq 1$ such that $g = \lambda f$. The set of pure states on $\mathcal{A}$ is denoted by $P(\mathcal{A})$. It is well known that $P(\mathcal{A})$ coincides with the set of all extremal points of $S(\mathcal{A})$.

For any element $a \in \mathcal{A}$ and any scalars $\alpha, \beta \in \mathbb{C}$, we have: $V(a) \subset \mathbb{R}$ (resp. $V(a) \subset [0, +\infty)$) if and only if $a = a^*$ (resp. $a \geq 0$). Also $V(\alpha 1 + \beta a) = \alpha + \beta V(a)$ and $V(a) = \{\alpha\} \iff a = \alpha 1$. The numerical radius v is a norm and satisfies $\frac{1}{e}\|a\| \leq v(a) \leq \|a\|$, where $e = \exp(1)$. See [2] and [11] for further details.

2. Main result

Let $\mathcal{A}$ and $\mathcal{B}$ be two unital complex C^*-algebras. Let $T : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear map. Recall that T is numerical range compressing if

$$V(T(a)) \quad \subset \quad V(a), \quad \forall a \in \mathcal{A}, \tag{2.1}$$

Note that if T compresses the numerical range then $T(\mathbf{1}) = \mathbf{1}$, since the numerical range is a nonempty set of $\mathbb{C}$ and $V(T(\mathbf{1})) \subset V(\mathbf{1}) = \{1\}$. Let us begin by the following example, which shows that a linear map which compresses the numerical range need not to be a Jordan $*$-homomorphism.

Example 2.1. Consider the C^*-algebra $\mathcal{A} = \mathcal{M}_2(\mathbb{C})$ and define the map $T : \mathcal{A} \longrightarrow \mathcal{A}$ for any matrix $A = (a_{ij})_{1 \leq i,j \leq 2} \in \mathcal{A}$ by

$$T(A) = \frac{1}{2}\, A + \frac{1}{4}\mathrm{tr}(A)\mathbf{1}$$

where tr denotes the usual function trace. Clearly, we have $f \circ T \in S(\mathcal{A})$ whenever $f \in S(\mathcal{A})$. Hence according to [16, Theorem 2.2], T satisfies condition (2.1). Consider the two matrices $A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$. An easy calculation will convince the reader that B is unitary, but $T(A^2) \neq T(A)^2$ and $T(B)$ is not unitary. This shows that T is not neither a Jordan $*$-homomorphism nor a unitary preserving map.

At the 4[th] Seminar on Functional analysis and its applications, which was held in University of Mashhad in March 2016 it is shown that in [9] that if $\mathcal{A}$ and $\mathcal{B}$ are commutative and T is a numerical range compressing, then T is a unital $*$-homomorphism, see [9, Theorem 2.5 & 2.6]. In fact, in his proof, the author shows that such a map is completely positive and preserves unitary elements. But we remark that this proof is based on the fact that if an element u is unitary in $\mathcal{A}$, then $|f(u)| = 1$ for any $f \in S(\mathcal{A})$. But this fails to be true even if the C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are commutative. It is in fact true only when f is a pure state, see for instance [5, Proposition 1]. To see why this, let $\mathcal{A} = C(\mathbb{T})$ be the C^*-algebra of all continuous functions on the unit circle $\mathbb{T}$ and let m be the normalised arc length measure on $\mathbb{T}$. Then the linear functional φ, defined by $\varphi(f) = \int f\, dm = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\, dt$ is a state of $\mathcal{A}$. The element $u \in \mathcal{A}$ defined by $u(z) = z, \forall z \in \mathbb{T}$ is unitary but $\varphi(u) = 0 < 1$. Finally observe that φ is not a $*$-homomorphism although that $V(\varphi(a)) \subset V(a)$ for all $a \in \mathcal{A}$, since φ is a state. Therefore, the main result [9, Theorem 2.5 & 2.6] is wrong in general.

Based on the aforesaid a natural question arises. Namely, what additional condition on a linear map T compressing the numerical range which forces T to be a Jordan $*$-homomorphism? To that end, we shall impose the following additional requirement on the map T.

Assumption 2.2. For any $a, b \in \mathcal{A}^+$ such that $ab = 0$, we have $T(a) \geq T(b)$ implies that $T(a)T(b) = 0$.

We establish the following.

Theorem 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital C^*-algebras. Any surjective linear map $T : \mathcal{A} \longrightarrow \mathcal{B}$ compressing the numerical range and satisfying Assumption 2.2 is a unital Jordan $*$-homomorphism.*

Before turning to the proof of Theorem 2.3, few remarks can be made.

Remark 2.4. If T preserves the numerical range then Assumption 2.2 is already satisfied. Indeed, let $a, b \in \mathcal{A}^+$ such that $ab = 0$ and $T(a) \geq T(b)$. Since $V(T(a-b)) = V(a-b)$ and $T(a-b) \geq 0$, then $a - b \geq 0$. By [15, Theorem 2.2.5], $0 \leq b^3 \leq bab = 0$. Accordingly $b^3 = b = 0$. Therefore $T(a) = T(b) = 0$.

Remark 2.5. Conditions (2.1) and Assumption 2.2 do not imply in general that T is linear as the following example quoted from [12] shows. Let $\mathcal{A} = \mathcal{B} = \mathcal{M}_2(\mathbb{C})$. Consider the mapping $T : \mathcal{A} \longrightarrow \mathcal{A}$ defined as

$$T(A) = \begin{cases} A & \text{if } A \text{ is invertible} \\ 0 & \text{otherwise.} \end{cases}$$

Straightforward computations show that T satisfies assumptions (2.1) and Assumption 2.2 but is not additive.

Remark 2.6. In [6], it was been shown that if $T : \mathcal{A} \longrightarrow \mathcal{B}$ is a bounded linear map between unital C^*-algebras preserving the zero products of self-adjoint elements in $\mathcal{A}$ then $T = T(\mathbf{1})J$ for a Jordan $*$-homomorphism J from $\mathcal{A}$ into the bidual B^{**} of $\mathcal{B}$. Note that Assumption 2.2 does not imply in general that T preserves the zero product of self-adjoint elements or a Jordan $*$-homomorphism. To see why this consider the C^*-algebra $\mathcal{A} = C([0,1])$ and the map $T : \mathcal{A} \longrightarrow \mathcal{A}$ given by $T(f) = 2f - f(1)$. Clearly, T is surjective and unital. But then $(T(f))^2 - T(f^2) = 2f^2 - 4f(1)f + 2f(1)^2$ is not always zero. Hence T is not a Jordan $*$-homomorphism. Next, let $f, g \in \mathcal{A}^+$ be such that $fg = 0$ and $T(f) \geq T(g)$. Since $T(f) \geq T(g)$, then $f(1) \geq g(1)$ and $f(x) \geq g(x) + \frac{1}{2}(f(1) - g(1))$, for any $x \in [0,1]$. This together with the fact $fg = 0$ yields that $g = 0$. Therefore $T(f)T(g) = 0$. Accordingly T satisfies Assumption 2.2. On the other hand, one can check easily that T does not preserve the zero product of self-adjoint elements.

3. Proof of Theorem 2.3:

We present now the proof of Theorem 2.3. Our arguments are influenced by ideas from the proof of [7, Theorem 5] but by using properties of the numerical range. We divide the proof into three steps.

Step 1. T is unital and positive. Moreover, for each $b \geq 0$ in $\mathcal{B}$ there is an $a \geq 0$ in $\mathcal{A}$ such that $T(a) = b$.

Firstly, note that $T(\mathbf{1}) = \mathbf{1}$, since $V(T(\mathbf{1})) \subset V(\mathbf{1}) = \{1\}$. Now, let $a \in \mathcal{A}^+$. Then $V(a) \subset [0, \infty)$. Since $V(T(a)) \subset V(a)$ we infer that $T(a) \in \mathcal{B}^+$ and in particular T is self adjoint, that is $T(a)^* = T(a), \forall a = a^* \in \mathcal{A}$. Now, let $b \geq 0$ and $a \in \mathcal{A}$ such that $T(a) = b$. Without loss of generality we may assume that $a = a^*$ (otherwise take $\frac{a+a^*}{2}$ instead of a). By [8, Proposition 12.5], there exist $a_+, a_- \geq 0$ such that $a = a_+ - a_-$ and $a_+ a_- = a_- a_+ = 0$. Then $b = T(a_+) - T(a_-)$ with $T(a_+) \geq 0$ and $T(a_-) \geq 0$. Assumption 2.2 entails that $T(a_+)T(a_-) = T(a_-)T(a_+) = 0$. Since every self adjoint element in a C^*-algebra can be uniquely written as the difference of two positive elements with zero product, we infer that $T(a_-) = 0$. This completes the proof of the first step.

Step 2. The kernel of T is a closed ideal of $\mathcal{A}$.

Firstly observe that by the proof of Step 1, we have that if $T(a) = 0$ and $a = a_+ - a_-$ with $a_+ a_- = a_- a_+ = 0$ and $a_\pm \geq 0$, then $T(a_+) = T(a_-) = 0$. Thus each element in ker$T$ is a linear combination of positive elements in ker$T$. Now, Lemma 5.1 of [17] can be used to deduce that ker$T$ is a two sided ideal. However, for the sake of completeness we sketch a different proof of this fact. To that end it suffices to show that $a \geq 0$ and $T(a) = 0$ imply that $T(ax) = T(xa) = 0$ for all positive element $x \in \mathcal{A}$. Fix such an $a \in \mathcal{A}$, a similar reasoning to that of [7, Theorem 5] entails that $T(ax)^* = T(xa) = -T(ax)$. By keeping in mind that $T(x) \geq 0$ for any $x \in \mathcal{A}^+$, we infer that the linear functional $f \circ T$ is positive and unital, for any $f \in S(\mathcal{B})$. Accordingly $f \circ T \in S(\mathcal{A})$. Hence, applying the Cauchy-Schwarz inequality to $f \circ T$ yields

$$|f \circ T(ax)|^2 = |f \circ T(a^{\frac{1}{2}} a^{\frac{1}{2}} x)|^2 \leq f \circ T(a)\ f \circ T(xax) = 0.$$

Accordingly, $f(T(ax)) = 0$, for all $f \in S(\mathcal{B})$ and so $T(ax) = T(xa) = 0$ as desired. The kernel of T is therefore an ideal. Since $v(T(a)) \leq v(a), \forall a \in \mathcal{A}$, v and $\|.\|$ are two equivalent norms, then T is bounded and so the kernel of T is closed.

Step 3. T is a Jordan $*$-homomorphism.

Firstly, note that by Step 1 we have $T(\mathbf{1}) = \mathbf{1}$ and T is positive. By Step 2, $\ker T$ is a closed ideal of $\mathcal{A}$. Then T induces the unital and positive bijective linear map $\widetilde{T} : \mathcal{A}/\mathrm{ker}T \longrightarrow \mathcal{B}$ defined by $\widetilde{T}(a + \mathrm{ker}T) = T(a)$. Again Step 1, entails that $\widetilde{T}^{-1}$ is also positive. So, by [13, Corollary 5] we have $\widetilde{T}$ is a Jordan $*$-isomorphism. Thus T, the composition of the natural quotient map and $\widetilde{T}$, is a Jordan $*$-homomorphism.

Acknowledgments. The authors wish to express their thanks to the referee for carefully reading the paper and for giving valuable suggestions.

References

1. Z. Bai, J. Hou, and Z. Xu, *Maps preserving numerical radius distance on C^*-algebras*, Studia Math. **162** (2004), no. 2, 97–104.
2. F. F Bonsall and J. Duncan, *Numerical ranges of operators on normed spaces and elements of normed algebras*, Cambridge Univ. Press, London, 1971.
3. A. Bourhim and M. Mabrouk, *On maps preserving the numerical radius distance between C^*-algebras*, preprint.
4. A. Bourhim and J. Mashreghi, *A survey on preservers of spectra and local spectra*, Invariant Subspaces of the Shift Operator, Contemp. Math., Amer. Math. Soc., Providence, RI, 2015, pp. 45–98.
5. J. T. Chan, *Numerical radius preserving operators on C^*-algebras*, Arch. Math. (Basel) **70** (1998), no. 6, 486–488.
6. M. A. Chebotar, W.-F. Ke, P.-H. Lee, and N.-C. Wong, *Mappings preserving zero products*, Studia Math. **155**, no. 1, 77–94.
7. M. D. Choi, D. Hadwin, E. Nordgren, H. Radjavi, and P. Rosenthal, *On positive linear maps preserving invertibility*, J. Funct. Anal. **59** (1984), 462–469.
8. R. S. Doran and V. A. Belfi, *Characterizations of C^*-algebras. The Gelfand-Naimark theorems*, Monographs and Textbooks in Pure and Applied Mathematics, 101. Marcel Dekker, Inc., New York, 1986.
9. F. Golfarshchi, *Numerical range compressing linear maps on C^*-algebras*, The 4th Seminar on Functional Analysis and its Applications 2016, Mashhad, Iran, 1–4.
10. K. E. Gustafson and D. K. M. Rao, *Numerical range: the field of values of linear operators and matrices.*, Springer, New york, 1997.
11. P. R. Halmos, *A Hilbert space problem book 2nd ed.*, Springer, New York, 1982.

12. R. El Harti and M. Mabrouk, *Maps compressing and expanding the numerical range on C^*-algebras*, Linear Multilinear Algebra **63** (2015), no. 11, 2332–2339.
13. R. V. Kadison, *A generalized Schwarz inequality and algebraic invariants for operator algebras*, Ann. of Math (2) **56** (1952), 494–503.
14. C.-K. Li, *A survey on linear preservers of numerical ranges and radii*, Taiwanese J. Math. **5** (2001), no. 3, 477–496.
15. G. J. Murphy, *C^*-algebras and operator theory*, Academic Press, 1990.
16. V. Pellegrini, *Numerical range preserving operators on a Banach algebra*, Studia Math. **54** (1975), no. 2, 143–147.
17. E. Størmer, *On partially ordered vector spaces and their duals, with applications to simplexes and C^*-algebras*,Proc. London Math. Soc. (3) **18** (1968), no. 2, 245–265.

[1] Department of Mathematics, College of Applied Sciences, P. O. Box 715, Makkah 21955, Saudi Arabia.
E-mail address: Ashwaq.F.B@hotmail.com

[2] Department of Mathematics, Faculty of Sciences Cité Erriadh, 6072 Zrig, Gabès, Tunisia.
E-mail address: mohamed.mabrouk@fsg.rnu.tn