Adv. Oper. Theory 3 (2018), no. 1, 53–60
http://doi.org/10.22034/aot.1702-1129
ISSN: 2538-225X (electronic)
http://aot-math.org

POSITIVE MAP AS DIFFERENCE OF TWO COMPLETELY POSITIVE OR SUPER-POSITIVE MAPS

TSUYOSHI ANDO

This paper is dedicated to the memory of the late Professor Uffe Haagerup

Communicated by M. Tomforde

Abstract. For a linear map from $\mathbb{M}_m$ to $\mathbb{M}_n$, besides the usual positivity, there are two stronger notions, complete positivity and super-positivity. Given a positive linear map φ we study a decomposition $\varphi = \varphi^{(1)} - \varphi^{(2)}$ with completely positive linear maps $\varphi^{(j)}$ $(j = 1, 2)$. Here $\varphi^{(1)} + \varphi^{(2)}$ is of simple form with norm small as possible. The same problem is discussed with super-positivity in place of complete positivity.

1. Introduction and problems

Let $\mathbb{M}_k$ denote the space of $k \times k$ (complex) matrices. Each matrix in $\mathbb{M}_k$ is considered as a linear map from $\mathbb{C}^k$ to itself. An element x of $\mathbb{C}^k$ is treated as a *column* k-vector, correspondingly x^* is a *row* k-vector. Then given $a, b \in \mathbb{C}^k$, according to the rule of matrix multiplication, a^*b is the inner product of a and b, that is, $a^*b = \langle a|b\rangle$ while ba^* is a matrix of rank-one in $\mathbb{M}_k$. Be careful about that the inner product is linear in b and anti-linear in a.

For selfadjoint $X, Y \in \mathbb{M}_k$, the order relation $X \geq Y$ or equivalently $Y \leq X$ is defined as $X - Y$ is positive semi-definite. Therefore $X \geq 0$ or $0 \leq X$ simply means that X is positive semi-definite. The norm $\|X\|$ denotes the *operator norm*

$$\|X\| := \sup_{\|a\|=1} \|Xa\|.$$

Date: Received: Feb. 25, 2017; Accepted: Mar. 11, 2017.

2010 *Mathematics Subject Classification.* Primary 47C15; Secondary 47A30, 15A69.

Key words and phrases. Positive map, completely positive map, super-positive map, norm, tensor product.

Throughout this paper, we assume $2 \le m \le n$. There are canonical identifications:

$$\mathbb{M}_m \otimes \mathbb{M}_n \sim \mathbb{M}_m(\mathbb{M}_n) \sim \mathbb{M}_{mn}.$$

Here $\mathbb{M}_m(\mathbb{M}_n)$ denotes the space of $m \times m$ block-matrices with entries in $\mathbb{M}_n$ and the first identification is in the following way:

$$X \otimes Y \sim [\xi_{jk} Y]_{j,k} \quad \text{for } X = [\xi_{jk}]_{j,k} \in \mathbb{M}_m,\ Y \in \mathbb{M}_n.$$

Here, for simplicity of notations, an $m \times m$ (numerical) matrix with (j,k)-entry $\xi_{j,k}$ is written as $[\xi_{jk}]_{j,k}$. In analogy, an $m \times m$ block-matrix with (j,k)-block-entry $S_{j,k}$ is denoted by $[S_{jk}]_{j,k}$.

Therefore a block matrix $\mathbf{S} = [S_{jk}]_{j,k} \in \mathbb{M}_m(\mathbb{M}_n)$ is uniquely assigned as

$$[S_{jk}]_{j,k} \ \sim \ \sum_{j,k} E_{jk} \otimes S_{jk},$$

where $E_{jk} (j,k = 1,2,\dots,m)$ are *matrix-units* in $\mathbb{M}_m$, that is, $E_{jk} = e_j e_k^*$ where e_j $(j = 1,\dots,m)$ is the canonical orthonormal basis of $\mathbb{C}^m$.

In the following, $\mathbb{M}_{(m,n)}$ denotes the real subspace of $\mathbb{M}_m(\mathbb{M}_n)$, consisting of selfadjoint elements, that is, the subspace of $\mathbf{S} = [S_{jk}]_{j,k}$ with $S_{jk} = S_{kj}^*$ $(j,k = 1,\dots,m)$.

The cone of positive semi-definite (block) matrices in $\mathbb{M}_{(m,n)}$ will be denoted by $\mathfrak{P}_0$. The order relation based on this cone is denoted by $\ge$ as usual. Therefore $\mathbf{S} \ge 0$ means that $\mathbf{S}$ is positive semi-definite.

In the tensor product theory a fact of key importance is the following (see [3, Chapter I-4]):

$$0 \le X \in \mathbb{M}_m,\ 0 \le Y \in \mathbb{M}_n \quad \Longrightarrow \quad 0 \le X \otimes Y.$$

The cone generated by $X \otimes Y$ with $0 \le X \in \mathbb{M}_m$ and $0 \le Y \in \mathbb{M}_n$ will be denoted by $\mathfrak{P}_+$. Because of finite dimensionality of $\mathbb{M}_{(m,n)}$ it is known (see [2, p.8]) that $\mathfrak{P}_+$ is a (topologically) closed cone, contained in $\mathfrak{P}_0$. A (block) matrix in $\mathfrak{P}_+$ is said to be *separable*.

The space $\mathbb{M}_{(m,n)}$ becomes a real Hilbert space with inner product

$$\langle \mathbf{T} | \mathbf{S} \rangle := \operatorname{Tr}(\mathbf{TS}),$$

and we can consider the dual cone $\mathfrak{P}_-$ of the cone $\mathfrak{P}_+$ defined by

$$\mathbf{S} \in \mathfrak{P}_- \quad \Longleftrightarrow \quad \langle \mathbf{S} | \mathbf{T} \rangle \ge 0 \quad \forall\ \mathbf{T} \in \mathfrak{P}_+. \tag{1.1}$$

The cone $\mathfrak{P}_-$ is (topologically) closed by definition. In view of the closedness of $\mathfrak{P}_+$, according to a general theory of convexity, $\mathfrak{P}_+$ is the dual cone of $\mathfrak{P}_-$.

It is well-known that the cone $\mathfrak{P}_0$ is selfdual, that is,

$$\mathbf{S} \in \mathfrak{P}_0 \quad \Longleftrightarrow \quad \langle \mathbf{S} | \mathbf{T} \rangle \ge 0 \quad \forall\ \mathbf{T} \in \mathfrak{P}_0.$$

As a consequence we have the inclusion relations:

$$\mathfrak{P}_+ \subset \mathfrak{P}_0 \subset \mathfrak{P}_-.$$

Notice the algebraic relations:

$$\mathfrak{P}_0 - \mathfrak{P}_0 = \mathfrak{P}_+ - \mathfrak{P}_+ = \mathbb{M}_{(m,n)}. \tag{1.2}$$

Given a linear map $\varphi : \mathbb{M}_m \to \mathbb{M}_n$, its Choi matrix $\mathbf{C}_\varphi$ [7, p.49] is defined by

$$\mathbf{C}_\varphi := [\varphi(E_{jk})]_{j,k} \in \mathbb{M}_m(\mathbb{M}_n).$$

On the basis of the relation

$$\varphi(X) = \sum_{j,k} \xi_{jk}\varphi(E_{jk}) \quad \forall\ X = [\xi_{jk}]_{j,k} \in \mathbb{M}_m,$$

the original map φ is uniquely recaptured from its Choi matrix.

Further $\varphi \longleftrightarrow \mathbf{C}_\varphi$ is a linear bijection between the space of selfadjoint linear maps φ, that is,

$$\varphi(X^*) = \varphi(X)^* \quad \forall\ X \in \mathbb{M}_m,$$

and the space $\mathbb{M}_{(m,n)}$. This bijection is usually called the Jamiolkowski isomorphism (see [7, p.49]).

A linear map $\varphi : \mathbb{M}_m \to \mathbb{M}_n$ is said to be *positive* if $\varphi(X) \geq 0$ whenever $X \geq 0$. Our starting point is the following relation, deduced from (1.1) and the definition of $\mathfrak{P}_+$ (see [2, Theorem 2.1]):

$$\varphi \ \text{ positive } \iff \mathbf{C}_\varphi \in \mathfrak{P}_- \tag{1.3}$$

$$\iff \Big[\langle x|S_{jk}x\rangle\Big]_{j,k} \geq 0 \quad \text{in } \mathbb{M}_m \quad \forall\ x \in \mathbb{C}^n.$$

There is a welll-known notion, stronger than positivity. A linear map $\varphi : \mathbb{M}_m \to \mathbb{M}_n$ is said to be *completely positive* if the linear map $id_N \otimes \varphi : \mathbb{M}_N \otimes \mathbb{M}_m \equiv \mathbb{M}_N(\mathbb{M}_m) \to \mathbb{M}_N(\mathbb{M}_n)$ defined by

$$(id_N \otimes \varphi)([T_{jk}]_{j,k}) := [\varphi(T_{jk})]_{j,k} \quad \forall\ T_{jk} \in \mathbb{M}_N$$

is positive for all $N = 1, 2, \ldots$.

Usefulness of use of the Choi matrix is seen in the following theorem of Choi [4] (see [2, Theorem 2.2])

$$\varphi \quad \text{completely positive} \iff \mathbf{C}_\varphi \in \mathfrak{P}_0. \tag{1.4}$$

In accordance with (1.3) and (1.4), a positive linear map $\varphi : \mathbb{M}_m \to \mathbb{M}_n$ will be said to be *super-positive* [2, p.11] when

$$\mathbf{C}_\varphi \ \in \ \mathfrak{P}_+. \tag{1.5}$$

Therefore a positive linear map φ is completely positive if and only if all eigenvalues of its Choi matrix are non-negative. On the contrary, there is no simple test to check super-positivity of φ. An obvious condition, which guarantees its super-positivity, is *block-diagonality* of the Choi matrix $\mathbf{C}_\varphi = [S_{jk}]_{j,k}$, that is,

$$S_{jk} = 0 \quad \text{for } j \neq k.$$

In this case $S_{jj} \geq 0$ $(j = 1, \ldots, m)$ is guaranteed by the positivity of φ.

Though not used in the subsequent discussion, we notice that the following intrinsic characterization of super-positivity of φ was established by Horodecki's [5, Theorem 2]

$$\varphi \quad \text{super} - \text{positive} \iff$$
$$\psi \circ \varphi \quad \text{completely positive} \quad \forall \text{ positive } \psi : \mathbb{M}_n \to \mathbb{M}_N.$$

As usual, the (mapping) norm of a linear map $\varphi : \mathbb{M}_m \to \mathbb{M}_n$ is defined by

$$\|\varphi\| = \sup\{\|\varphi(X)\|;\ \|X\| \le 1,\ X \in \mathbb{M}_m\}.$$

Here advantage of positivity of φ is seen in the following fact, a consequence of a theorem of Russo-Dye [6] (see [7, Theorem 1.3.3]):

$$\varphi \quad \text{positive} \quad \Longrightarrow \quad \|\varphi\| = \|\varphi(I_m)\|. \tag{1.6}$$

In view of (1.2), it is seen from (1.4) and (1.5) that every selfadjoint linear map $\varphi : \mathbb{M}_m \to \mathbb{M}_n$ is written as difference of two completely positive (or even super-positive) linear maps $\varphi^{(j)}$ $(j = 1, 2)$;

$$\varphi = \varphi^{(1)} - \varphi^{(2)}. \tag{1.7}$$

Of course, such decomposition is never unique.

In this paper, which is a continuation of [2], we study the problem how to construct a decomposition (1.7) of positive φ, for which the Choi matrix of $\varphi^{(1)} + \varphi^{(2)}$ is block-diagonal and its norm is small as possible.

2. Case of complete positivity

For notational convenience, let us define the *partial trace* $\chi(\mathbf{S})$ of $\mathbf{S} = [S_{jk}]_{j,k} \in \mathbb{M}_{(m,n)}$ by

$$\chi(\mathbf{S}) := \sum_j S_{jj} \in \mathbb{M}_n.$$

Then (1.6) says that

$$\varphi \quad \text{positive} \quad \Longrightarrow \quad \|\varphi\| = \|\chi(\mathbf{C}_\varphi)\|. \tag{2.1}$$

For selfadjoint $\mathbf{S}$, its *modulus* $|\mathbf{S}| \in \mathfrak{P}_0$ is defined as the positive (semi-definite) square root of $\mathbf{S}^2$. Further its *positive part* $\mathbf{S}^+$ and the *negative part* $\mathbf{S}^-$ are defined as

$$\mathbf{S}^+ := \tfrac{1}{2} \cdot \{|\mathbf{S}| + \mathbf{S}\} \quad \text{and} \quad \mathbf{S}^- := \tfrac{1}{2} \cdot \{|\mathbf{S}| - \mathbf{S}\}.$$

All $|\mathbf{S}|,\ \mathbf{S}^+$ and $\mathbf{S}^-$ belong to the cone $\mathfrak{P}_0$ and the decomposition

$$\mathbf{S} = \mathbf{S}^+ - \mathbf{S}^-$$

is called the *Jordan decomposition* of $\mathbf{S}$. (See [3, p. 99].)

Lemma 2.1. *If φ is a selfadjoint linear map : $\mathbb{M}_m \to \mathbb{M}_n$ with Choi matrix $\mathbf{C}_\varphi$,*

$$\|\chi(|\mathbf{C}_\varphi|)\| \ \le\ m \cdot \|\varphi\|.$$

A proof is found in [2, Theorem 6.2].

Theorem 2.2. *Let φ be a selfadjoint linear map : $\mathbb{M}_m \to \mathbb{M}_n$ with Choi matrix $\mathbf{C}_\varphi$. Define completely positive linear maps $\varphi^{(1)}$ and $\varphi^{(2)}$ by*

$$\mathbf{C}_{\varphi^{(1)}} := \mathbf{C}_\varphi^+ \quad \textit{and} \quad \mathbf{C}_{\varphi^{(2)}} := \mathbf{C}_\varphi^-.$$

Then $\varphi = \varphi^{(1)} - \varphi^{(2)}$ and $\|\varphi^{(1)} + \varphi^{(2)}\| \le m \cdot \|\varphi\|$.

Proof. By (2.1) and Lemma 2.1

$$\|\varphi^{(1)} + \varphi^{(2)}\| = \|\chi(|\mathbf{C}_{\varphi}|)\| \le m \cdot \|\varphi\|.$$

□

When φ is positive, a decomposition (1.7) with completely positive $\varphi^{(1)}$ and $\varphi^{(2)}$, for which $\mathbf{C}_{\varphi^{(1)}} + \mathbf{C}_{\varphi^{(2)}}$ is block-diagonal and

$$\varphi^{(1)}(I_m) + \varphi^{(2)}(I_m) = m \cdot \varphi(I_m),$$

can be constructed rather easily.

We need a result in $\mathbb{M}_{(2,n)}$ for its proof.

Lemma 2.3.

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathfrak{P}_- \implies \begin{bmatrix} X & \pm B \\ \pm B^* & Y \end{bmatrix} \in \mathfrak{P}_0 \quad \exists\, X, Y \ge 0, X + Y = A + C.$$

Proof. By (1.3), $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathfrak{P}_-$ means that $A, C \ge 0$ and

$$\langle x|Ax\rangle \cdot \langle x|Cx\rangle \;\ge\; |\langle x|Bx\rangle|^2 \quad \forall\, x \in \mathbb{C}^n,$$

which implies that

$$\langle x|\tfrac{1}{2}(A + C)x\rangle \;\ge\; |\langle x|Bx\rangle| \quad \forall\, x \in \mathbb{C}^n. \tag{2.2}$$

We may assume here that $A + C$ is invertible. Then, with $D := \{\frac{1}{2}(A + C)\}^{\frac{1}{2}}$, (2.2) means that the *numerical radius* of $D^{-1}BD^{-1}$ is ≤ 1, that is,

$$\|x\|^2 \ge |\langle x|(D^{-1}BD^{-1})x\rangle| \quad \forall\, x \in \mathbb{C}^n.$$

Then by [1, Theorem 1] there are $R, T \ge 0$ such that $R + T = 2I_n$ and

$$\begin{bmatrix} R & \pm D^{-1}BD^{-1} \\ \pm D^{-1}B^*D^{-1} & T \end{bmatrix} \ge 0.$$

Let $X := DRD$ and $Y := DTD$. Then

$$X + Y = A + C \quad \text{and} \quad \begin{bmatrix} X & \pm B \\ \pm B^* & Y \end{bmatrix} \ge 0.$$

□

To apply some results of $\mathbb{M}_{(2,n)}$ to the case of $\mathbb{M}_{(m,n)}$ the following trivial facts will be used without any mention.

(1) $\quad A_j \ge 0 \;\; (j = 1, 2, \ldots, m) \implies \operatorname{diag}(A_1, \ldots, A_m) \in \mathfrak{P}_+ \subset \mathfrak{P}_0.$

(2) $\quad \mathbf{S} = [S_{jk}]_{j,k} \in \mathfrak{P}_- \implies \begin{bmatrix} S_{pp} & S_{pq} \\ S_{qp} & S_{qq} \end{bmatrix} \in \mathfrak{P}_- \;\; (\text{in } \mathbb{M}_{(2,n)}) \quad \forall\, p < q.$

(3) If $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathfrak{P}_0$ (resp. $\in \mathfrak{P}_+$) in $\mathbb{M}_{(2,n)}$ then, for any $1 \leq j < k \leq m$, the (block) matrix $\mathbf{S} \in \mathfrak{P}_0$ (resp. $\in \mathfrak{P}_+$) where

$$S_{jj} = A, S_{jk} = B, S_{kj} = B^*, S_{kk} = C, \text{ and } S_{pq} = 0 \text{ if } p \neq j \text{ or } \ q \neq k.$$

Theorem 2.4. *Let φ be a positive linear map : $\mathbb{M}_m \to \mathbb{M}_n$. Then there are completely positive linear maps $\varphi^{(j)}$ $(j = 1, 2) : \mathbb{M}_m \to \mathbb{M}_n$ such that $\varphi = \varphi^{(1)} - \varphi^{(2)}$, the Choi matrix of $\varphi^{(1)} + \varphi^{(2)}$ is block-diagonal and*

$$\varphi^{(1)}(I_m) + \varphi^{(2)}(I_m) \ = \ m \cdot \varphi(I_m).$$

Proof. Let $\mathbf{C}_\varphi = [S_{jk}]_{j,k}$ be the Choi matrix of φ. Since

$$\begin{bmatrix} S_{jj} & S_{jk} \\ S_{kj} & S_{kk} \end{bmatrix} \in \mathfrak{P}_- \quad \text{in } \mathbb{M}_{(2,n)} \quad \forall\ j < k$$

by Lemma 2.3 there are $0 \leq X_{j,k},\ X_{k,j} \in \mathbb{M}_n$ such that

$$X_{j,k} + \ X_{k,j} = S_{jj} + S_{kk} \quad \text{and} \quad \begin{bmatrix} X_{j,k} & \pm S_{jk} \\ \pm S_{kj} & X_{k,j} \end{bmatrix} \geq 0. \tag{2.3}$$

Let $\varphi^{(j)}$ $(j = 1, 2)$ be the selfadjoint linear maps: $\mathbb{M}_m \to \mathbb{M}_n$ with respective Choi matrix $\mathbf{C}_{\varphi^{(j)}}$ $(j = 1, 2)$ given by

$$\mathbf{C}_{\varphi^{(1)}} := \tfrac{1}{2}\Big\{\mathrm{Diag}(\mathbf{C}_\varphi) + \mathrm{diag}(A_1, \ldots, A_m) + \mathbf{C}_\varphi\Big\}$$

and

$$\mathbf{C}_{\varphi^{(2)}} := \tfrac{1}{2}\Big\{\mathrm{Diag}(\mathbf{C}_\varphi) + \mathrm{diag}(A_1, \ldots, A_m) - \mathbf{C}_\varphi\Big\},$$

where

$$\mathrm{Diag}(\mathbf{C}_\varphi) := \mathrm{diag}(S_{11}, \ldots, S_{mm})$$

and

$$A_j := \sum_{1 \leq k < j} X_{k,j} + \sum_{j < k \leq m} X_{j,k} \quad (j = 1, 2, \ldots, m).$$

Then it is clear that $\varphi = \varphi^{(1)} - \varphi^{(2)}$, and by (2.3)

$$\chi\Big(\mathbf{C}_{\varphi^{(1)}} + \mathbf{C}_{\varphi^{(2)}}\Big) = m \cdot \chi(\mathbf{C}_\varphi),$$

and that the Choi matrix of $\varphi^{(1)} + \varphi^{(2)}$ is block-diagonal. That $\mathbf{C}_{\varphi^{(j)}} \in \mathfrak{P}_0$ $(j = 1, 2)$ comes also from (2.3). Therefore both $\varphi^{(j)}$ $(j = 1, 2)$ are completely positive by (1.4). □

Optimality of the constant m in Theorem 2.4 is pointed out in [2, p.28]. In fact, when $m = n$, for the positive linear map $\varphi_0(X) := X^T$ (*transpose map*) any decomposition $\varphi_0 = \varphi^{(1)} - \varphi^{(2)}$ with completely positive $\varphi^{(j)}$ $(j = 1, 2)$ satisfies necessarily

$$\|\varphi^{(1)} + \varphi^{(2)}\| \geq m \cdot \|\varphi_0\|.$$

3. Case of super-positivity

In the case of decomposition with super-positive linear maps, there is no canonical decomposition as Jordan decomposition in Section 2. However, the same idea as in the proof of Theorem 2.4 can be used to find a suitable decomposition.

This approach was used already in [2, Theorem 7.4]. Let me present the same result again to show how the difference of scalars, m and $2m-1$, appears.

Lemma 3.1.

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathfrak{P}_- \quad \Longrightarrow \quad \begin{bmatrix} A+C & \pm B \\ \pm B^* & A+C \end{bmatrix} \in \mathfrak{P}_+.$$

A proof is found in [2, Theorem 4.10]. This lemma corresponds to Lemma 2.3.

Theorem 3.2. *Let φ be a positive linear map : $\mathbb{M}_m \to \mathbb{M}_n$. Then there are super-positive linear maps $\varphi^{(j)}$ $(j=1,2) : \mathbb{M}_m \to \mathbb{M}_n$ such that $\varphi = \varphi^{(1)} - \varphi^{(2)}$, the Choi matrix of $\varphi^{(1)} + \varphi^{(2)}$ is block-diagonal and*

$$\varphi^{(1)}(I_m) + \varphi^{(2)}(I_m) \;=\; (2m-1)\cdot\varphi(I_m).$$

Proof. Let $\mathbf{C}_\varphi = [S_{jk}]_{j,k}$ be the Choi matrix of φ, and let $\varphi^{(j)}$ $(j=1,2)$ be the linear maps with respective Choi matrix $\mathbf{C}_{\varphi^{(j)}}$ $(j=1,2)$ given by

$$\mathbf{C}_{\varphi^{(1)}} := \tfrac{1}{2}\Big\{(m-1)\cdot \mathrm{Diag}(\mathbf{C}_\varphi) + I_m \otimes \chi(\mathbf{C}_\varphi) + \mathbf{C}_\varphi\Big\}$$

and

$$\mathbf{C}_{\varphi^{(2)}} := \tfrac{1}{2}\Big\{(m-1)\cdot \mathrm{Diag}(\mathbf{C}_\varphi) + I_m \otimes \chi(\mathbf{C}_\varphi) - \mathbf{C}_\varphi\Big\}.$$

It is clear that

$$\chi\Big(\mathbf{C}_{\varphi^{(1)}} + \mathbf{C}_{\varphi^{(2)}}\Big) \;=\; (2m-1)\cdot\chi(\mathbf{C}_\varphi),$$

and that the Choi matrix of $\varphi^{(1)} + \varphi^{(2)}$ is block-diagonal.

It remains to show that $\mathbf{C}_{\varphi^{(j)}} \in \mathfrak{P}_+$ $(j=1,2)$. As in the proof of Theorem 2.4, this follows principally from Lemma 3.1:

$$\begin{bmatrix} S_{jj}+S_{kk} & \pm S_{jk} \\ \pm S_{kj} & S_{jj}+S_{kk} \end{bmatrix} \in \mathfrak{P}_+ \quad \forall\, j<k.$$

□

Optimality of the constant $2m-1$ in Theorem 3.2 is pointed out in [2, Theorem 7.6]. In fact, when $m=n$, for the (completely) positive map $\varphi_0(X) = X$ (*identity map*), any decomposition $\varphi_0 = \varphi^{(1)} - \varphi^{(2)}$ with super-positive $\varphi^{(j)}$ $(j=1,2)$ satisfies necessarily

$$\|\varphi^{(1)} + \varphi^{(2)}\| \;\geq\; (2m-1)\cdot\|\varphi_0\|.$$

References

1. T. Ando, *Structure of operators with numerical radius one*, Acta Sci. Math. (Szeged), **34** (1972), 169–178.
2. T. Ando, *Cones and norms in the tensor product of matrix spaces*, Linear Algebra Appl., **379** (2004), 3–41.
3. R. Bhatia, *Matrix Analysis*, Springer, New York, 1997.
4. M.-D. Choi, *Completely positive linear map on complex matrices*, Linear Algebra Appl. **10** (1975), 285–290.
5. M. Horodecki, P. Horodecki, and R. Horodecki, *Separability of mixed states: necessary and sufficient conditions*, Phys. Lett. A **233** (1996), no. 1-2, 1–8.
6. B. Russo and H. A. Dye, *A note on unitary operators in C^*-algebra*, Duke Math. J. **33** (1966), 413–416.
7. E. Størmer, *Positive Linear Maps of Operator Algebras*, Springer, New York, 2013.

Hokkaido University (Emeritus), Japan.
E-mail address: ando@es.hokudai.ac.jp