Adv. Oper. Theory 2 (2017), no. 3, 192–200
http://doi.org/10.22034/aot.1612-1078
ISSN: 2538-225X (electronic)
http://aot-math.org

ON THE WEAK COMPACTNESS OF WEAK* DUNFORD–PETTIS OPERATORS ON BANACH LATTICES

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Communicated by T. Schlumprecht

Abstract. We characterize Banach lattices on which each positive weak* Dunford–Pettis operator is weakly (resp., M-weakly, resp., order weakly) compact. More precisely, we prove that if F is a Banach lattice with order continuous norm, then each positive weak* Dunford–Pettis operator $T : E \longrightarrow F$ is weakly compact if, and only if, the norm of E' is order continuous or F is reflexive. On the other hand, when the Banach lattice F is Dedekind σ-complete, we show that every positive weak* Dunford–Pettis operator $T : E \longrightarrow F$ is M-weakly compact if, and only if, the norms of E' and F are order continuous or E is finite-dimensional.

1. Introduction and preliminaries

Recall from [1] that an operator T from a Banach space X into a Banach space Y is said to be weak Dunford–Pettis (wDP) if the sequence $f_n(T(x_n))$ converges to 0 whenever (x_n) converges weakly to 0 in X and (f_n) converges weakly to 0 in Y', equivalently, T carries relatively weakly compact subsets of X onto Dunford–Pettis subsets of Y.

Recently in [9], we have defined a new class of operators that we called weak* Dunford–Pettis operators. This class of operators is essentially based on the concept of limited sets introduced in [7]. We have characterized this class of operators and studied some of its properties in [9]. Let us recall that an operator T from a Banach space X into a Banach space Y is called weak* Dunford–Pettis

Date: Received: Dec. 12, 2016; Accepted: Mar. 17, 2017.
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2010 *Mathematics Subject Classification.* Primary 46B42; Secondary 47B60, 47B65.
Key words and phrases. Weak* Dunford–Pettis operator, weakly compact operator, M-weakly compact operator, order weakly compact operator, DP* property.

whenever $x_n \longrightarrow 0$ for $\sigma(X, X')$ in X and $f_n \longrightarrow 0$ for $\sigma(Y', Y)$ in Y' imply $f_n(T(x_n)) \longrightarrow 0$, equivalently, T carries relatively weakly compact subsets of X onto limited subsets of Y. Furthermore, if Y is a Grothendieck space then, the notions of weak Dunford–Pettis and weak* Dunford–Pettis operators coincide.

Note that there exists an operator which is weak* Dunford–Pettis but not weakly compact. In fact, the identity operator of the Banach lattice ℓ^1 is weak* Dunford–Pettis but it is not weakly compact. Conversely, there exists an operator which is weakly compact but fails to be weak* Dunford–Pettis. In fact, the identity operator of the Banach lattice ℓ^2 is weakly compact but it is not weak* Dunford–Pettis.

In [2] the authors studied the weak compactness of Dunford–Pettis (resp., weak Dunford–Pettis) operators. Also, in [5] the authors studied the M-weak compactness of positive Dunford-Pettis (resp., semi-compact) operators. On the other hand, the class of weak* Dunford–Pettis operators is bigger than the class of Dunford–Pettis operators and is included in that of weak Dunford–Pettis operators. So, it is natural to study the weak compactness of weak* Dunford–Pettis operators and the connection between weak* Dunford–Pettis and M-weakly compact (resp., order weakly compact) operators on Banach lattices.

The article is organized as follows, after a preliminary, we study the weak compactness of weak* Dunford–Pettis operators (Theorem 2.1 and Theorem 2.2). As consequences, we will obtain a characterization of reflexive Banach lattices (Corollary 2.3). Further, we characterize Banach lattices for which each weak* Dunford–Pettis operator is M-weakly compact (Theorem 2.4) and we finish the paper by characterizing Banach lattices on which each weak* Dunford–Pettis operator is order weakly compact (Theorem 2.5).

Throughout this paper X, Y will denote Banach spaces and E, F will denote Banach lattices. The positive cone of E will be denoted by E^+. B_X is the closed unit ball of X.

Let us recall from [1] that an operator $T : X \longrightarrow Y$ is called a Dunford–Pettis operator if T carries weakly convergent sequences into norm convergent sequences. Alternatively, $T : X \longrightarrow Y$ is a Dunford–Pettis operator if, and only if, T carries relatively weakly compact sets into norm totally bounded sets. Also, $T : E \longrightarrow X$ is said to be M-weakly compact if for every disjoint sequence (x_n) in B_E we have $\|T(x_n)\| \longrightarrow 0$ and $T : E \longrightarrow X$ is said to be order weakly compact if it carries order bounded intervals of E to relatively weakly compact sets in X. Equivalently, for every disjoint order bounded sequence (x_n) in E we have $\|T(x_n)\| \longrightarrow 0$.

Following [7], a norm bounded subset A of X is said to be a limited set if every weak* null sequence (f_n) of X' converges uniformly on A. It is easy to check that every relatively norm compact set is limited but the converse is not true in general. In fact, the set $\{e_n : n \in N\}$ of unit coordinate vectors is a limited set in ℓ^∞ which is not relatively compact.

A Banach space X has the Dunford–Pettis (DP) property if, and only if, $f_n(x_n) \longrightarrow 0$ for every weakly null pair of sequences $((x_n), (f_n))$ in $X \times X'$.

Next Borwein et al. [4] introduced a stronger version of DP property. A Banach space X has the Dunford–Pettis* (DP*) property if, $f_n(x_n) \longrightarrow 0$ for every weakly null sequence (x_n) in X and every weak* null sequence (f_n) in X'.

To state our results, we need to fix some notations and recall some definitions. A Banach lattice is a Banach space $(E, \|.\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. A norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, (x_α) converges to 0 for the norm $\|\cdot\|$ where the notation $x_\alpha \downarrow 0$ means that (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$.

Note that if E is a Banach lattice, its topological dual E', endowed with the dual norm and the dual order, is also a Banach lattice. Also, a vector lattice E is Dedekind σ-complete if every majorized countable nonempty subset of E has a supremum. We will use the term operator $T : E \longrightarrow F$ to mean a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. Note that each positive linear mapping on a Banach lattice is continuous. If an operator $T : E \longrightarrow F$ is positive, then its adjoint $T' : F' \longrightarrow E'$ is likewise positive, where T' is defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$. For terminologies concerning Banach lattice theory and positive operators we refer the reader to the excellent book of Aliprantis-Burkinshaw [1].

2. Main results

Our first result gives a characterization of Banach lattices E for which each positive weak* Dunford–Pettis operator $T : E \longrightarrow E$ is weakly compact (resp., M-weakly compact).

Theorem 2.1. *Let E be Dedekind σ-complete. Then, the following assertions are equivalent:*

(1) *the norms of E and E' are order continuous;*
(2) *every positive weak* Dunford–Pettis operator from E into E is M-weakly compact;*
(3) *for all operators $S, T : E \longrightarrow E$ such that $0 \leq S \leq T$ and T is weak* Dunford–Pettis, S is M-weakly compact;*
(4) *every positive weak* Dunford–Pettis operator from E into E is weakly compact;*
(5) *for every positive weak* Dunford–Pettis operator $T : E \longrightarrow E$, the operator product T^2 is weakly compact.*

Proof. (1) $\Longrightarrow$ (2) Let $T : E \longrightarrow E$ be a positive weak* Dunford–Pettis operator, and let $(x_n) \subset B_E$ be a disjoint sequence. We shall show that $\|T(x_n)\| \longrightarrow 0$. By [8, Corollary 2.6], it suffices to prove that $|T(x_n)| \longrightarrow 0$ in the $\sigma(E, E')$-topology of E and $f_n(Tx_n) \longrightarrow 0$ for every disjoint and norm bounded sequence $(f_n) \subset (E')^+$. Indeed,

- Let $f \in (E')^+$. As the norm of E' is order continuous then, $x_n \longrightarrow 0$ and $|x_n| \longrightarrow 0$ in the $\sigma(E, E')$-topology of E (because (x_n) is disjoint). On the other hand, it follows from [1, Theorem 1.23] that for each n there exists some

$g_n \in [-f, f]$ with $f|T(x_n)| = g_n(T(x_n))$. Now, since $|x_n| \longrightarrow 0$ in the $\sigma(E, E')$-topology of E and T is positive then, $0 \leq f|Tx_n| = g_n(Tx_n) = T'(g_n)(x_n) \leq |T'(g_n)||x_n| \leq T'(f)|x_n| \longrightarrow 0$ and hence $|T(x_n)| \longrightarrow 0$ in the $\sigma(E, E')$-topology of E.

- Let $(f_n) \subset (E')^+$ be a disjoint and norm bounded sequence. As the norm of E is order continuous then, it follows from [10, Corollary 2.4.3] that $f_n \longrightarrow 0$ in the $\sigma(E', E)$-topology of E'. Now, since T is weak* Dunford–Pettis then $f_n(Tx_n) \longrightarrow 0$.

(2) $\Longrightarrow$ (3) Let $S, T : E \longrightarrow E$ be two operators such that $0 \leq S \leq T$ and T is weak* Dunford–Pettis. By [6, Theorem 3.1], S is likewise weak* Dunford–Pettis and by our hypothesis S is M-weakly compact.

(3) $\Longrightarrow$ (4) Let $T : E \longrightarrow E$ be a positive weak* Dunford–Pettis operator. Since $0 \leq T \leq T$, it follows that T is M-weakly compact, and hence it is weakly compact.

(4) $\Longrightarrow$ (5) Obvious.

(5) $\Longrightarrow$ (1) **Step** 1: We prove that the norm of E is order continuous. If not, it follows from the proof of [12, Theorem 1] that E contains a closed sublattice isomorphic to ℓ^∞ and there is a positive projection $P : E \longrightarrow \ell^\infty$. Let $i : \ell^\infty \longrightarrow E$ be the canonical injection of ℓ^∞ into E. Since ℓ^∞ has the DP* property, the operator $T = i \circ P : E \longrightarrow \ell^\infty \longrightarrow E$ is weak* Dunford–Pettis. But, its second power T^2 which coincides with T, is not weakly compact. Otherwise, the operator

$$P \circ T \circ i : \ell^\infty \longrightarrow E \longrightarrow \ell^\infty \longrightarrow E \longrightarrow \ell^\infty$$

would be weakly compact. But $P \circ T \circ i$, which is just the identity operator Id_{ℓ^∞}, is not weakly compact. This presents a contradiction and hence E has an order continuous norm.

Step 2: We show that the norm of the topological dual E' is order continuous. If not, [10, Proposition 2.3.11 and Theorem 2.4.14] affirm the existence of a sublattice H of E isomorphic to ℓ^1 and a positive projection $P_2 : E \longrightarrow \ell^1$. Now, let T_2 be the operator defined by

$$T_2 = i_2 \circ P_2 : E \longrightarrow \ell^1 \longrightarrow E$$

where $i_2 : \ell^1 \longrightarrow E$ is the canonical injection of ℓ^1 into E. Since ℓ^1 has the DP* property, T_2 is weak* Dunford–Pettis but $(T_2)^2 = T_2$ is not weakly compact. Otherwise, the operator $P_2 \circ T_2 \circ i_2 = Id_{\ell^1}$ would be weakly compact, and this gives a contradiction. □

In the following result we give necessary and sufficient conditions on E and F under which every weak* Dunford–Pettis operator $T : E \longrightarrow F$ is weakly compact.

Theorem 2.2. *Let F be a Banach lattice with order continuous norm. Then the following assertions are equivalent:*

(1) *every positive weak* Dunford–Pettis operator $T : E \longrightarrow F$ is weakly compact;*
(2) *one of the following is valid:*

(a) *the norm of E' is order continuous;*
(b) *F is reflexive .*

Proof. (1) $\Longrightarrow$ (2) Assume that the norm of E' is not order continuous. It follows from [10, Proposition 2.3.11 and Theorem 2.4.14] that E contains a sub-lattice isomorphic to ℓ^1 and there exists a positive projection $P : E \longrightarrow \ell^1$. To finish the proof we have to show that F is reflexive. By the Eberlein-Smulian's Theorem it suffices to show that every sequence (x_n) in the closed unit ball of F has a sub-sequence converges weakly to an element of F.

Consider the operator $S : \ell^1 \longrightarrow F$ defined by $S((\alpha_i)) = \sum_{i=1}^{\infty} \alpha_i x_i$ for each $(\alpha_i) \in \ell^1$. The composed operator

$$T = S \circ P : E \longrightarrow \ \ell^1 \longrightarrow F$$

is weak* Dunford–Pettis and hence by our hypothesis T is weakly compact. So, the sequence $(x_n) = (T(e_n))$ has a sub-sequence which converges weakly to an element of F, where (e_n) is the canonical basis of ℓ^1.

$(2; a) \Longrightarrow (1)$ The same proof as the implication (1) $\Longrightarrow$ (2) of Theorem 2.1.

$(2; b) \Longrightarrow (1)$ Obvious. □

Remark 1. *We need the condition "the norm of F is order continuous" just for the proof of the first sufficient condition. Indeed, the identity operator of the Banach lattice ℓ^∞ is weak* Dunford–Pettis but fails to be weakly compact. However, the norm of $(\ell^\infty)'$ is order continuous.*

A Banach lattice E is said to be an AM-space if for each $x, y \in E$ such that $\inf(x, y) = 0$, we have $\|x + y\| = \max\{\|x\|, \|y\|\}$. The Banach lattice E is an AL-space if its topological dual E' is an AM-space.

As a consequence of Theorem 2.2 and [1, Theorem 5.24], we obtain the following characterization of reflexive Banach lattices.

Corollary 2.3. *Let E be an infinite-dimensional $AL-$space and F with order continuous norm. Then, the following assertions are equivalent:*

(1) *F is reflexive;*
(2) *every operator from E into F is weakly compact;*
(3) *every positive weak* Dunford–Pettis operator from E into F is weakly compact.*

Proof. (1) $\Longrightarrow$ (2) Follows from [1, Theorem 5.24].

(2) $\Longrightarrow$ (3) Obvious.

(3) $\Longrightarrow$ (1) The norm of E' is not order continuous. Indeed, if the norm of E' is order continuous, as E is an $AL-$space then, E' is an $AM-$space with unit and hence it follows from [11, Theorem 5.10] that there exists some $f \in (E')^+$ such that $B_{E'} = [-f, f]$ is weakly compact, this implies that E' is reflexive and hence E is reflexive. Now, by [1, Exercise 14 page 253] the Banach lattice E is finite-dimensional. This gives a contradiction. Finally, since the norm of E' is not order continuous, it follows from Theorem 2.2 that F is reflexive. □

Recall from Aliprantis-Burkinshaw [1, page 222], that E is said to be lattice embeddable into F whenever there exists a lattice homomorphism $T : E \longrightarrow F$ and there exist two positive constants K and M satisfying

$$K\|x\| \leq \|T(x)\| \leq M\|x\| \text{ for all } x \in E.$$

T is called a lattice embedding from E into F. In this case $T(E)$ is a closed sub-lattice of F which can be identified with E.

Note that there exist weak* Dunford–Pettis operators that are not M-weakly compact. Indeed, the operator $T : \ell^1 \longrightarrow \ell^\infty$ defined by

$$T((\alpha_n)) = (\sum_{n=1}^{\infty} \alpha_n)_{n=1}^{\infty} = \sum_{n=1}^{\infty} \alpha_n (1,1,1,\ldots)$$

is weak* Dunford–Pettis but fails to be M-weakly compact.

Theorem 2.4. *Let F be a Dedekind σ-complete Banach lattices. Then the following assertions are equivalent:*

(1) *every positive weak* Dunford–Pettis operator $T : E \longrightarrow F$ is M-weakly compact;*
(2) *one of the following is valid:*
  (a) *the norms of E' and F are order continuous;*
  (b) *E is finite-dimensional.*

Proof. (1) $\Longrightarrow$ (2) **Step 1.** We prove that if the norm of F is not order continuous, then E is finite-dimensional.

In fact, assume that the norm of F is not order continuous and E is infinite-dimensional. We will construct a positive operator $T : E \longrightarrow F$ which is weak* Dunford–Pettis but is not M-weakly compact. Since, E is infinite-dimensional then, [3, Lemma 2.3] implies that there exist a positive disjoint sequence (x_n) of E^+ such that $\|x_n\| = 1$ for all n. So, by [3, Lemma 2.5], there exists a positive disjoint sequence (g_n) of E' with $\|g_n\| \leq 1$ such that $g_n(x_m) = 1$ for all $n = m$ and $g_n(x_m) = 1$ for $n \neq m$.

Now, consider the positive operator $P : E \longrightarrow \ell^\infty$ defined by $P(x) = (g_n(x))_n$.

On the other hand, since the norm of F is not order continuous, it follows from [1, Theorem 4.51] that ℓ^∞ is lattice embeddable in F. i.e., there exists a lattice homomorphism $S : \ell^\infty \longrightarrow F$ and there exist two positive constants M and m satisfying

$$m \left\|(\lambda_k)_k\right\|_\infty \leq \left\|S\left((\lambda_k)_k\right)\right\| \leq M \left\|(\lambda_k)_k\right\|_\infty$$

for all $(\lambda_k)_k \in \ell^\infty$.
Put $T = S \circ P : E \longrightarrow \ell^\infty \longrightarrow F$ and note that T is a positive weak* Dunford–Pettis (Because ℓ^∞ has DP* property), but is not M-weakly compact. Indeed, since (x_n) is a disjoint norm bounded sequence in E^+ and

$$\begin{aligned} \|T(x_n)\| = \|S \circ P(x_n)\| &= \|S((g_k(x_n))_k)\| \\ &\geq m\|(g_k(x_n))\|_\infty = m|g_n(x_n)| = m \end{aligned}$$

for every n then, T is not M-weakly compact.

Step 2. We prove that if the norm of E' is not order continuous then, $F = \{0\}$. Assume that the norm of E' is not order continuous and $F \neq \{0\}$. There exist $0 < y \in F^+$ and it follows from [10, Theorem 2.4.14 and Proposition 2.3.11] the existence of a sub-lattice H of E isomorphic to ℓ^1 and a positive

projection $P_2 : E \longrightarrow \ell^1$.
On the other hand, we define the positive operator $S_2 : \ell^1 \longrightarrow F$ defined by

$$S_2((\lambda_n)_n) = (\sum_{n=1}^{\infty} \lambda_n)y$$

for all $((\lambda_n)_n) \subset \ell^1$. Put $T_2 = S_2 \circ P_2 : E \longrightarrow \ell^1 \longrightarrow F$, and note that T_2 is a positive weak* Dunford–Pettis. On the other hand, consider the sequence (e_n) in ℓ^1, where e_n is the sequence with nth entry equals to 1 and others are zero. From $T_2(e_n) = y$ for each $n \in \mathbb{N}$, it follows that $\|T_2(e_n)\| \nrightarrow 0$. Hence T_2 is not M-weakly compact.

$(2; a) \Longrightarrow (1)$ The same as the proof of $(1) \Longrightarrow (2)$ in Theorem 2.1.

$(2; b) \Longrightarrow (1))$ In this case, every operator $T : E \longrightarrow F$ is M-weakly compact. In fact, if E is finite-dimensional then for every norm bounded disjoint sequence (x_n) of E there exists some n_0 such that $x_n = 0$ for all $n \geq n_0$. So, $T(x_n) = 0$ for all $n \geq n_0$. Then $\|T(x_n)\| \longrightarrow 0$ and hence T is M-weakly compact. □

Remark 2. *The condition "F is Dedekind σ-complete" is not an accessory in the above theorem. Indeed, each positive operator $T : \ell^\infty \longrightarrow c$ is M-weakly compact, but neither the assertion (a) nor the assertion (b) was valid.*

Finally, we characterize Banach lattices on which each weak* Dunford–Pettis operator is order weakly compact. Note that a weak* Dunford–Pettis operator is not necessary order weakly compact. In fact, the identity operator $Id_\infty : \ell^\infty \longrightarrow \ell^\infty$ is weak* Dunford–Pettis but fails to be order weakly compact.

Theorem 2.5. *Let F be a Dedekind σ-complete Banach lattices. Then the following assertions are equivalent:*

(1) *every positive weak* Dunford–Pettis operator $T : E \longrightarrow F$ is order weakly compact;*
(2) *one of the following is valid:*
   (a) *the norm of E are order continuous;*
   (b) *the norm of F are order continuous.*

Proof. $(1) \Longrightarrow (2)$ Assume by way of contradiction that neither E nor F has an order continuous norm. To finish the proof, we have to construct a positive weak* Dunford–Pettis operator $T : E \longrightarrow F$ which is not order weakly compact.
Since the norm of E is not order continuous, it follows from [1, Theorem 4.14] that there exists some $y \in E^+$ and a disjoint sequence $(x_n) \subset [0, y]$ which does not converge to zero in norm. We may assume that $\|x_n\| = 1$ for all n. Hence, by [3, Lemma 2.5], there exists a positive disjoint sequence (g_n) of E' with $\|g_n\| \leq 1$ such that $g_n(x_m) = 1$ for all $n = m$ and $g_n(x_m) = 1$ for $n \neq m$.

We define the positive operator P as follows:

$$P : E \longrightarrow \ell^\infty, \quad P(x) = (g_n(x))_n.$$

On the other hand, since the norm of F is not order continuous, it follows from [1, Theorem 4.51] that ℓ^∞ is lattice embeddable in F. i.e., there exists a lattice homomorphism $S : \ell^\infty \longrightarrow F$ and there exists two positive constants M and m

satisfying

$$m \left\| (\lambda_k)_k \right\|_\infty \leq \left\| S\left((\lambda_k)_k \right) \right\| \leq M \left\| (\lambda_k)_k \right\|_\infty$$

for all $(\lambda_k)_k \in \ell^\infty$. Put $T = S \circ P : E \longrightarrow \ell^\infty \longrightarrow F$, and note that T is a positive weak* Dunford–Pettis (Because ℓ^∞ has DP*property), but it is not order weakly compact. Indeed, since (x_n) is an order bounded disjoint sequence in E and

$$\begin{aligned} \|T(x_n)\| = \|S \circ P(x_n)\| &= \|S((g_k(x_n))_k)\| \\ &\geq m\|(g_k(x_n))\|_\infty = m|g_n(x_n)| = m \end{aligned}$$

for every n then, T is not order weakly compact.

$(2;a) \Longrightarrow (1)$ In this case, each operator $T : E \longrightarrow F$ is order weakly compact.

$(2;b) \Longrightarrow (1)$ Let $T : E \longrightarrow F$ be a positive weak* Dunford–Pettis operator and let (x_n) be a positive disjoint order bounded sequence in E. We shall show that $\|T(x_n)\| \longrightarrow 0$. By [8, Corollary 2.6], it suffices to proof that $|T(x_n)| \longrightarrow 0$ in the $\sigma(F, F')$-topology of F and $f_n(T(x_n)) \longrightarrow 0$ for every disjoint and norm bounded sequence $(f_n) \subset (F')^+$. Indeed:

- As (x_n) is a positive disjoint order bounded sequence in E then, $x_n \longrightarrow 0$ in the $\sigma(E, E')$-topology of E (see [1, Remark in page 192]) hence $0 \leq T(x_n) \longrightarrow 0$ for $\sigma(F, F')$.

- Let $(f_n) \subset (F')^+$ be a disjoint and norm bounded sequence. As the norm of F is order continuous, then by [10, Corollary 2.4.3], $f_n \longrightarrow 0$ in the $\sigma(F', F)$-topology of F'. Now, since T is weak* Dunford–Pettis then, $f_n(T(x_n)) \longrightarrow 0$. This show that T is order weakly compact. $\square$

Remark 3. *The some example in the above remark affirm that the condition "F Dedekind σ-complete" is not an accessory.*

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