Adv. Oper. Theory 2 (2017), no. 3, 215–227
http://doi.org/10.22034/aot.1612-1070
ISSN: 2538-225X (electronic)
http://aot-math.org

# PROPERTIES OF $J$-FUSION FRAMES IN KREIN SPACES

SHIBASHIS KARMAKAR[1], SK. MONOWAR HOSSEIN[2]* and KALLOL PAUL[1]

Communicated by P. Aiena

Abstract. In this article we introduce the notion of $J$-Parseval fusion frames in a Krein space $\mathbb{K}$ and characterize 1-uniform $J$-Parseval fusion frames with $\zeta = \sqrt{2}$. We provide some results regarding construction of new $J$-tight fusion frame from given $J$-tight fusion frames. We also characterize an uniformly $J$-definite subspace of a Krein space $\mathbb{K}$ in terms of $J$-fusion frame. Finally we generalize the fundamental identity of Hilbert space frames in the setting of Krein spaces.

## 1. Introduction

Frames for Hilbert space is a well known concept. The theory was introduced by Duffin and Schaeffer [1] in the year 1952. Now a days frame theory has applications in almost every areas of applied mathematics. With the development of Hilbert space frame theory and its continuous application in Data transmission, Networking and Signal Processing some new research areas are continuously emerging. Filter bank theory, packet-based communication system are some of the examples of new research areas in frame theory. Hilbert space frame theory set-up can hardly be modeled naturally by one single system. Furthermore, it is often difficult to handle a large amount of numerical data in a single frame system. In these cases it is highly beneficial to split a large frame system into a set of overlapping smaller systems so that we can be able to process each local system efficiently. Interestingly our brain stores and processes information by


*Date*: Received: Dec. 7, 2016; Accepted: Mar. 21, 2017.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 42C15; Secondary 46C05, 46C20.
*Key words and phrases.* Krein Space, fusion frame, $J$-fusion frame, Gramian operator, regular subspace.

using distributed processing. A distributed frame theory relating to a set of local frame systems is clearly in demand. This led to the development of fusion frames, which provides exactly the framework to model these applications. A beautiful approach was introduced by Casazza and Kutyniok [3] and Casazza et al. [4] that gives a general method for adding together local frames to get global frames. Different aspects and applications of fusion frames can be seen in [6, 7, 8].

Since fusion frame in Hilbert space has huge applications so it is natural to extend these ideas in indefinite inner product spaces such as Krein spaces. Krein spaces have some interesting applications in modern analysis. Some known areas of application are in high energy physics, quantum cosmology, Krein space filtering etc. The theory of frames in Krein space can be found in [10, 11, 12, 15, 16]. Acosta-Humánez et al. first defined fusion frames for Krein spaces in [13]. They obtained a correspondence between fusion frames in Hilbert spaces and fusion frames in Krein spaces. But their definition involves fundamental symmetry of Krein space which is not unique. In [18] we defined Fusion frame in Krein spaces in a more geometric setting motivated by the work of Giribet et al. [11], and we called them *$J$-fusion frames*. Defining $J$-fusion frame in Krein spaces could reveal some structural properties of fusion frames and also can be used as a tool for problems in Krein spaces. We also showed that $J$-fusion frame contains conventional $J$-frames as a very special case, thereby going beyond $J$-frame theory in Krein spaces.

In this article we investigate the properties of $J$-fusion frame in Krein spaces. In Section 2 we briefly review the notion of *$J$-fusion frame* in Krein spaces along with some definitions and properties. In Section 3 we define *$J$-Parseval fusion frame* in Krein spaces as a generalization of Parseval frame in Hilbert spaces and we relate this concept with *$J$-orthonormal basis of subspaces* in a Krein space $\mathbb{K}$ by introducing a real number $\zeta \in [\sqrt{2}, 2)$. This real number $\zeta$ act as a correlation between positive frame vectors and negative frame vectors. We also prove that every Krein space is richly supplied with $J$-Parseval fusion frames. Then we generalize the fundamental identity of Hilbert space frames in the setting of Krein spaces. Our final result characterize an *uniformly $J$-definite subspace* of $\mathbb{K}$ in terms of $J$-fusion frame.

## 2. Preliminary Notes

In this section we briefly recall some basic notations, definitions and some important properties useful for our study. For more detailed information one can see [10, 15, 11, 3, 5, 16, 17].

Let $\pi_M$ denote the *orthogonal projection* from the Hilbert space $\mathbb{H}$ onto the closed subspace $M$ of $\mathbb{H}$.

**Definition 2.1.** [3] Let $I$ be some index set and $\{W_i : i \in I\}$ be a family of closed subspaces of $\mathbb{H}$. Also let $\{v_i : i \in I\}$ be a family of weights *i.e.* $v_i > 0$ $\forall i \in I$. Then $\{(W_i, v_i) : i \in I\}$ is said to be a *fusion frame* for $\mathbb{H}$ if there exist constants $0 < C \leq D < \infty$ such that

$$C\|f\|^2 \leq \sum_{i\in I} v_i^2 |[\pi_{W_i} f, f]| \leq D\|f\|^2 \quad \text{for every } f \in \mathbb{H} \tag{2.1}$$

$C$ and $D$ are known as lower and upper *fusion frame bounds*, respectively, for the fusion frame $\{(W_i, v_i) : i \in I\}$. If $C = D = A$ (*let*), then the fusion frame is known as *A-tight fusion frame* and if $C = D = 1$, then the fusion frame is called *Parseval fusion frame*. Moreover, a fusion frame is called *v-uniform*, if $v := v_i = v_j$ for all $i,\ j \in I$.

The family of subspaces $\{W_i : i \in I\}$ is said to be an *orthonormal basis of subspaces* if $\mathbb{H} = \oplus_{i\in I} W_i$.

Let $\pi_M$ be an *orthogonal projection* in a Krein space $\mathbb{K}$ onto $M$ and $Q_M$ be a *J-orthogonal projection* from $\mathbb{K}$ onto $M$. We know that the *J-orthogonal projection* $Q_M$ exists if $M$ is a *projectively complete* subspace of $\mathbb{K}$. So we do not have any relation between $\pi_M$ and $Q_M$ unless $M$ is projectively complete or regular. But we have the following lemma.

**Lemma 2.2.** [18] *Let $M$ be an uniformly $J$-definite subspace for the Krein space $\mathbb{K}$. If $W$ is a closed subspace of $M$, then $Q_W|_M = \pi_W|_M$.*

*Proof.* Let $x \in M$ and $x_1, x_2 \in W$. Let $Q_W|_M(x) = x_1$ and $\pi_W|_M(x) = x_2$. Then $x - x_1[\perp]W$ which implies that $[x - x_1, w] = 0\ \forall w \in W$. Since $M$ is uniformly $J$-definite hence $[x - x_1, w]_J = 0\ \forall w \in W$. Then by the definition of orthogonal projection we have $\pi_W|_M(x) = x_1$. So we have $x_1 = x_2$. Hence $Q_W|_M = \pi_W|_M$. □

Let us assume that $M$ is uniformly $J$-positive. Then $(M, [\cdot,\cdot])$ is itself a Hilbert space. So let $P_W$ be the orthogonal projection from $M$ onto $W$. The above lemma states that $P_W = Q_W|_M = \pi_W|_M$.

Let $\pi_M^{\#}$ be the *J-adjoint* of $\pi_M$. Then we have $\pi_M^{\#} = J\pi_M J$. Also $\pi_{JM} = J\pi_M J$. Hence we have the following equation $\pi_{JM} = \pi_M^{\#}$.

Let $W$ be a subspace of a Krein space $\mathbb{K}$. Let $\mathbb{P}^{++}$ denote the set of all *positive J-definite subspaces* of $\mathbb{K}$, $\mathbb{P}^{+}$ denote the set of all *non-negative subspaces* of $\mathbb{K}$. Similarly $\mathbb{P}^{--}$ and $\mathbb{P}^{-}$ denote the set of all *negative J-definite and non-positive subspaces* of $\mathbb{K}$, respectively. Also let $\tilde{\mathbb{P}}$ be the set of all *indefinite subspaces* of $\mathbb{K}$. The set of all *neutral subspaces* sometimes referred as $\mathbb{P}^0$. We have $\mathbb{P}^0 \subset \mathbb{P}^+$ or $\mathbb{P}^0 \subset \mathbb{P}^-$. Then $W \in \mathbb{P}^+ \cup \mathbb{P}^- \cup \tilde{\mathbb{P}}$. Throughout in our work we consider either $W \in \mathbb{P}^+ \cup \mathbb{P}^{--}$ or $W \in \mathbb{P}^{++} \cup \mathbb{P}^-$. Without any loss of generality we assume $W \in \mathbb{P}^+ \cup \mathbb{P}^{--}$ to establish our results.

We consider the space $\left(\sum_{i\in I} \oplus W_i\right)$, where $\{W_i : i \in I\}$ is a collection of subspaces in the Krein space $\mathbb{K}$ such that $W_i \in \mathbb{P}^+ \cup \mathbb{P}^{--}\ \forall i \in I$ . If $f \in \left(\sum_{i\in I} \oplus W_i\right)$, then $f = \{f_i\}_{i\in I}$, where $f_i \in W_i$ for each $i \in I$. Let $I_+ = \{i \in I : [f_i, f_i] \geq 0 \text{ for all } f_i \in W_i\}$ and $I_- = \{i \in I : [f_i, f_i] < 0 \text{ for all } f_i \in W_i\}$. We define $[f, g] = \sum_{i\in I}[f_i, g_i]$, where $f, g \in \left(\sum_{i\in I} \oplus W_i\right)$. If the series is *unconditionally convergent*, then $[\cdot,\cdot]$ defines an inner product on $\left(\sum_{i\in I} \oplus W_i\right)$.

2.1. **Definition of $J$-fusion frame.** Let $\mathbb{F} = \{(W_i, v_i) : i \in I\}$ be a *Bessel family* of closed subspaces in a Krein space $\mathbb{K}$ with *synthesis operator* $T_{W,v} \in L\Big(\left(\sum_{i\in I} \oplus W_i\right)_{\ell_2}, \mathbb{K}\Big)$ such that $W_i \in \mathbb{P}^+ \cup \mathbb{P}^{--}\ \forall i \in I$. Let $I_+ = \{i \in I : [f_i, f_i] \geq 0 \text{ for all } f_i \in W_i\}$ and $I_- = \{i \in I : [f_i, f_i] < 0 \text{ for all } f_i \in W_i\}$. Now consider the orthogonal decomposition of $\big(\sum_{i\in I}\oplus W_i\big)_{\ell_2}$ given by

$$\big(\sum_{i\in I}\oplus W_i\big)_{\ell_2}=\big(\sum_{i\in I_+}\oplus W_i\big)_{\ell_2}\bigoplus\big(\sum_{i\in I_-}\oplus W_i\big)_{\ell_2},$$

and denote by $P_\pm$ the *orthogonal projection* onto $(\sum_{i\in I_\pm}\oplus W_i)_{\ell_2}$. Also, let $T_{W,v_\pm}=T_{W,v}P_\pm$. If $M_\pm=\overline{\sum_{i\in I_\pm}W_i}$, notice that $\sum_{i\in I_\pm}W_i\subseteq R(T_{W,v_\pm})\subseteq M_\pm$ and

$$R(T_{W,v})=R(T_{W,v_+})+R(T_{W,v_-})$$

**Definition 2.3.** The Bessel family $\mathbb{F}=\{(W_i,v_i):i\in I\}$ is a *$J$-fusion frame* for $\mathbb{K}$ if $R(T_{W,v_+})$ is a *maximal uniformly $J$-positive subspace* of $\mathbb{K}$ and $R(T_{W,v_-})$ is a *maximal uniformly $J$-negative subspace* of $\mathbb{K}$.

If $\{(W_i,v_i):i\in I\}$ is a *$J$-fusion frame* for $\mathbb{K}$, then $\big(\sum_{i\in I}\oplus W_i,[\cdot,\cdot]\big)$ is a Krein space. *The fundamental symmetry*, let, $J_2$ is defined by $J_2(f)=\{f_i:i\in I_+\}\cup\{-f_i:i\in I_-\}$ for all $f\in\mathbb{K}$. Also $[f,g]_{J_2}=\sum_{i\in I_+}[f_i,g_i]-\sum_{i\in I_-}[f_i,g_i]$. Now consider the space $\big(\sum_{i\in I}\oplus W_i\big)_{\ell_2}=\Big\{f:\big(\sum_{i\in I}\oplus W_i\big):\sum_{i\in I}\|f_i\|_J^2<\infty\Big\}$. We use this space frequently in our work.

The following theorems studied by Karmakar et al. [18] to characterize $J$-fusion frame and $J$-fusion frame operator in Krein spaces.

**Theorem 2.4.** [18] *Let $\mathbb{F}=\{(W_i,v_i)\}_{i\in I}$ be a $J$-fusion frame for $\mathbb{K}$. Then $\mathbb{F}_\pm=\{(W_i,v_i)\}_{i\in I_\pm}$ is fusion frame for the Hilbert space $(M_\pm,\pm[\cdot,\cdot])$ i.e. there exist constants $B_-\leq A_-<0<A_+\leq B_+$ such that*

$$A_\pm[f,f]\leq\sum_{i\in I_\pm}v_i^2|[\pi_{W_i}|_{M_\pm}(f),f]|\leq B_\pm[f,f]\ \textit{for every}\ f\in M_\pm \tag{2.2}$$

**Theorem 2.5.** [18] *For each $i\in I$, let $v_i>0$ and let $\{f_{ij}\}_{j\in J_i}$ be a $J$-frame sequence in $\mathbb{K}$ with $J$-frame bounds $B_{i-}$, $A_{i-}$, $A_{i+}$ and $B_{i+}$ for each $i\in I$, such that $-\infty<sup_iB_{i-}\leq inf_iA_{i-}<0<inf_iA_{i+}\leq sup_iB_{i+}<\infty$. Define $W_i=\overline{span_{j\in J_i}\{f_{ij}\}}$ for each $i\in I$. If $W_i$ is definite for each $i\in I$, then the following conditions are equivalent:*

*(i) $\{v_if_{ij}\}_{i\in I,j\in J_i}$ is a $J$-frame for $\mathbb{K}$.*

*(ii) $\{(W_i,v_i):i\in I\}$ is a $J$-fusion frame for $\mathbb{K}$.*

Now let $\mathbb{F}=\{(W_i,v_i):i\in I\}$ be a *$J$-fusion frame* for the Krein space $\mathbb{K}$. Then $\{W_i:i\in I_+\}$ is a collection of *uniformly $J$-positive subspaces* of $\mathbb{K}$ and $\{W_i:i\in I_-\}$ is a collection of *uniformly $J$-negative subspaces* of $\mathbb{K}$. The *$J$-adjoint operator* of the *synthesis operator* $T_{W,v}$ is denoted by $T_{W,v}^{\#}$ and is called the *analysis operator* of the *$J$-fusion frame* in $\mathbb{K}$. Now $T_{W,v}^{\#}=(T_{W,v_+}^{\#}+T_{W,v_-}^{\#})$. We have $N(T_{W,v_+}^{\#})^{[\perp]}=R(T_{W,v_+})=M_+$. The Analysis operators are defined as $T_{W,v_+}^{\#}(f)=\{v_i\pi_{JW_i}(f)\}_{i\in I_+}$ and $T_{W,v_-}^{\#}(f)=-\{v_i\pi_{JW_i}(f)\}_{i\in I_-}$ for all $f\in\mathbb{K}$. So $T_{W,v}^{\#}(f)=\{\sigma_iv_i\pi_{JW_i}(f)\}_{i\in I}$ for all $f\in\mathbb{K}$. Also we have $T_{W,v_+}^{\#}(f)=\{v_i\pi_{W_i}(f)\}_{i\in I_+}$ for all $f\in M_+$. Here $\sigma_i=1$ if $i\in I_+$ and $\sigma_i=-1$ if $i\in I_-$. Note that $[f, g_i] = [f, Q_{W_i}|_{M_+}(g_i)] = [f, Q_{W_i}(g_i)] = [Q_{W_i}(f), g_i]$. Hence $T^{\#}_{W,v_+}(f) = \{v_i Q_{W_i}(f)\}_{i\in I_+}$. We also know that $Q_{W_i}(f) \in W_i$, so $T^{\#}_{W,v_+} \in \big(\sum_{i\in I_+} \oplus W_i\big)_{\ell_2}$.

**Definition 2.6.** The linear operator $S_{W,v} : \mathbb{K} \to \mathbb{K}$ defined by $S_{W,v}(f) = \sum_{i\in I} \sigma_i v_i^2 \pi_{J(W_i)}(f)$ is said to be the *$J$-fusion frame operator* for the *$J$-fusion frame* $\{(W_i, v_i\}_{i\in I}$.

We next mention the following two theorems proved by Karmakar et al. in [18], which describe *$J$-fusion frame operator* in a Krein space $\mathbb{K}$.

**Theorem 2.7.** [18] *If $\{(W_i, v_i)\}_{i\in I}$ is a $J$-fusion frame for the Krein space $\mathbb{K}$ with synthesis operator $T_{W,v} \in L\Big(\big(\sum_{i\in I} \oplus W_i\big)_{\ell_2}, \mathbb{K}\Big)$, then the $J$-frame operator $S_{W,v}$ is bijective and $J$-selfadjoint.*

**Theorem 2.8.** [18] *If $\{(W_i, v_i)\}_{i\in I}$ is a $J$-fusion frame for the Krein space $\mathbb{K}$ with $J$-fusion frame operator $S_{W,v}$, then $\{(S^{-1}_{W,v}(W_i), v_i)\}_{i\in I}$ is a $J$-fusion frame for $\mathbb{K}$ with $J$-fusion frame operator $S^{-1}_{W,v}$.*

## 3. Main results

The idea of tight/Parseval frames has many important applications in Hilbert space frame theory as well as in Krein space frame theory. In finite dimensional cases this concept has some beautiful geometric interpretations. So in this section we define $J$-tight fusion frame in Krein spaces as a generalization of Parseval frame in Hilbert spaces and study some important properties relating to $J$-fusion frames.

From the definition of *$J$-fusion frame* we know that every $J$-fusion frame in $\mathbb{K}$ decompose the Krein space into two parts. The positive part $M_+$ is the maximal uniformly $J$-positive subspace and the negative part $M_-$ is the maximal uniformly $J$-negative subspace. The cone of neutral vectors in $\mathbb{K}$ is denoted by $\mathtt{C}$ and is defined by, $\mathtt{C} = \{n \in \mathbb{K} : [n, n] = 0\}$. In [11] we have a concept of angle between an uniformly $J$-definite subspace of $\mathbb{K}$ and the cone of neutral vectors $\mathtt{C}$. Using the results of [11], we have the following two equations. $c_0(M_+, \mathtt{C}) = \frac{1}{\sqrt{2}}\Big(\sqrt{\frac{1+\alpha^+}{2}} + \sqrt{\frac{1-\alpha^+}{2}}\Big)$ and $c_0(M_-, \mathtt{C}) = \frac{1}{\sqrt{2}}\Big(\sqrt{\frac{1+\beta^+}{2}} + \sqrt{\frac{1-\beta^+}{2}}\Big)$, where $\alpha^+ = \gamma(G_{M_+})$ and $\beta^+ = \gamma(G_{M_-})$. $\gamma$ is the *reduced minimum modulus* of the respective *Gramian operators* $G_{M_+}$ and $G_{M_-}$.

Now every $J$-fusion frame is associated with a positive real number $\zeta$, where $\zeta = c_0(M_+, \mathtt{C}) + c_0(M_-, \mathtt{C})$. We also have $\zeta \in [\sqrt{2}, 2)$. We prove that $\zeta$ plays a positive role in determining when a *$J$-fusion frame* is a *$J$-orthonormal basis of subspaces* in $\mathbb{K}$. The motivation for introducing the number $\zeta$ comes from the concept of *frame potential* in Hilbert spaces. In 2001, Benedetto and Fickus [9] developed the notion of frame potential which is analog to a potential energy in the physical world. It is a function which is maximized exactly when the vectors form a tight frame. Tight frames seem to occur when vectors are as close to orthogonal as they can be. In Hilbert space the pattern of the frame vectors are arbitrarily distributed in the space. But for any given $J$-frame in a Krein space we have something more. The positive set of frame vectors lie in the subspace

$M_+$ and the negative set of frame vectors lie in the subspace $M_-$, although in $M_+$ and $M_-$ the distribution of the frame vectors may be arbitrary. So to define *$J$-frame force* between positive and negative vectors we introduced the notion $\zeta$ in [17]. Also the $J$-frame force between positive and negative vectors are not conservative in nature. So to deal with both conservative and non-conservative forces in a single frame system, in [17] we defined the *$J$-potential* between a positive and a negative vector in the following way,

$$P_J(f_i, f_j) = \frac{1}{2}(\zeta^2 - 1), \quad \text{for } i \in I_+, j \in I_-$$

To emphasize on the pattern of the subspaces $\{W_i : i \in I\}$ in $\mathbb{K}$, we defined $\zeta$ to give a correlation between vectors of $M_+$ and $M_-$.

Next we introduce the notion of *$J$-tight fusion frame*.

**Definition 3.1.** Let $(\mathbb{K}, [\cdot,\cdot], J)$ be a Krein space and $\mathbb{F} = \{(W_i, v_i) : i \in I\}$ be a $J$-fusion frame for the Krein space $\mathbb{K}$. Then $\mathbb{F}$ is said to be a *$J$-tight fusion frame* if and only if

$$A_\pm[f, f] = \sum_{i \in I_\pm} v_i^2 |[\pi_{W_i}|_{M_\pm}(f), f]|, \quad \text{for all } f \in M_\pm \tag{3.1}$$

Moreover, $\mathbb{F}$ is said to be a *$J$-Parseval fusion frame* if it is a $J$-tight frame for the Krein space $\mathbb{K}$ and in addition $A_\pm = \pm 1$.

**Example 3.2.** Consider the vector space $\ell_2(\mathbb{R})$. Let us define a inner product on $\ell_2(\mathbb{R})$ by $[x, y] = x_1y_1 + x_2y_2 - x_3y_3 + x_4y_4 + \ldots$, where $x = (x_1, x_2, x_3, x_4, \ldots), y = (y_1, y_2, y_3, y_4, \ldots)$ and $x_i, y_i \in \mathbb{R}$ for $i \in \mathbb{N}$. Consider the subspaces $W_1 = span\{(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0, 0, \ldots)\}$, $W_2 = span\{(\frac{\sqrt{3}}{2}, -\frac{1}{2}, 0, 0, \ldots)\}$, $W_3 = span\{(0, 1, 0, 0, \ldots)\}$ and $W_4 = span\{(\frac{1}{\sqrt{2}}, 0, \frac{\sqrt{3}}{\sqrt{2}}, \ldots)\}$. Let $v_1 = v_2 = v_3 = \frac{\sqrt{2}}{\sqrt{3}}$ and $v_4 = 1$. Then the collection $\big\{(W_i, v_i) : i \in \{1, 2, 3, 4\}\big\}$ is a $J$-fusion frame for the above Krein space $(\ell_2(\mathbb{R}), [\cdot,\cdot])$. It is also a $J$-Parseval fusion frame with $\zeta = \frac{3}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}$. By numerical calculation we have $\alpha^+ = 1$ and $\beta^+ = \frac{1}{2}$.

3.1. **Some results on $J$-tight fusion frames.** In this section we define *$J$-orthonormal basis of subspaces* for a Krein space $\mathbb{K}$. We also find a relation between *$J$-orthonormal basis of subspaces* and 1-*uniform $J$-Parseval fusion frames* in $\mathbb{K}$.

**Definition 3.3.** Let $\mathbb{F} = \{W_i : i \in I\}$ be a family of closed subspaces in a Krein space $\mathbb{K}$. Then the family is said to be an *$J$-orthonormal basis of subspaces* if $\mathbb{K}^+ = \dot{\oplus}_{i \in I_+} W_i$ and $\mathbb{K}^- = \dot{\oplus}_{i \in I_-} W_i$, where $\mathbb{K} = \mathbb{K}^+[\oplus]\mathbb{K}^-$ is the *cannonical decomposition* of the Krein space.

**Example 3.4.** Let us consider the Krein space $\big(\mathbb{R}^3(\mathbb{R}), [\cdot,\cdot]\big)$, where the inner product is defined by $[(x_1, x_2, x_3), (y_1, y_2, y_3)] = x_1y_1 + x_2y_2 - x_3y_3$. Here $(x_1, x_2, x_3)$ and $(y_1, y_2, y_3) \in \mathbb{R}^3$. Now consider the collection $\mathbb{F} = \{(W_i, 1)\}_{i=1}^3$, where $W_1 = span\{(1, \frac{1}{\sqrt{3}}, 0)\}$, $W_2 = span\{(1, -\sqrt{3}, 0)\}$ and $W_3 = span\{(\frac{1}{2}, \frac{1}{2}, 1)\}$.

Then the collection $\mathbb{F}$ is a 1-uniform $J$-Parseval fusion frame for $\mathbb{R}^3$. But it is not an orthonormal basis of subspaces in $\mathbb{R}^3$. Also in this case $\zeta \neq \sqrt{2}$.

**Definition 3.5.** Let $(\mathbb{K}, [\cdot,\cdot], J)$ be a Krein space. Let us consider a collection of subspaces $\mathbb{F} = \{W_i : i \in I\}$ such that $W_i \in \mathbb{P}^+ \cup \mathbb{P}^{--}$ $\forall i \in I$. Then the collection $\mathbb{F}$ is said to be a *disjoint family of subspaces* in $\mathbb{K}$ if $\overline{\sum_{i\in I_+} W_i} \cap \overline{\sum_{i\in I_-} W_i} = \{0\}$.

**Example 3.6.** Let $\{(W_i, v_i) : i \in I\}$ be a $J$-fusion frame in a Krein space $\mathbb{K}$. Then the collection $\{W_i : i \in I\}$ is a disjoint family of subspaces in $\mathbb{K}$.

**Definition 3.7.** Let $\mathbb{F} = \{W_i : i \in I\}$ be a collection of subspaces in the Krein space $\mathbb{K}$ such that $W_i \in \mathbb{P}^+ \cup \mathbb{P}^{--}$ $\forall i \in I$. Then $\mathbb{F}$ is said to be a *strictly disjoint family of subspaces* if and only if $\overline{\sum_{i\in I_+} W_i}\ [\perp]\ \overline{\sum_{i\in I_-} W_i}$.

The next theorem establishes a relation between *1-uniform $J$-Parseval fusion frames* with $\zeta = \sqrt{2}$ and *$J$-orthonormal basis of subspaces* for a given Krein space.

**Theorem 3.8.** *Let $\mathbb{K}$ be a Krein space. Then a $J$-fusion frame is a $J$-orthonormal basis of subspaces in $\mathbb{K}$ if and only if $\zeta = \sqrt{2}$ and it is a 1-uniform $J$-Parseval fusion frame for $\mathbb{K}$.*

*Proof.* Let $\{W_i : i \in I\}$ be a $J$-orthonormal basis of subspaces in $\mathbb{K}$. So let $I_+ = \{i \in I : [f_i, f_i] > 0 \text{ for all } f_i \in W_i\}$ and $I_- = \{i \in I : [f_i, f_i] < 0 \text{ for all } f_i \in W_i\}$. Then $\mathbb{K}^+ = \dot{\oplus}_{i\in I_+} W_i$ and $\mathbb{K}^- = \dot{\oplus}_{i\in I_-} W_i$, where $\mathbb{K} = \mathbb{K}^+[\dot{\oplus}]\mathbb{K}^-$. Now $(\mathbb{K}^+, [\cdot,\cdot])$ is a Hilbert space and $\gamma(G_{\mathbb{K}^+}) = 1$. Hence $c_0(\mathbb{K}^+, \mathtt{C}) = \frac{1}{\sqrt{2}}$. By similar arguments we have $c_0(\mathbb{K}^-, \mathtt{C}) = \frac{1}{\sqrt{2}}$. Hence $\zeta = \sqrt{2}$. Also it is easy to see that $\{W_i : i \in I_+\}$ is an orthonormal basis of subspaces in $\mathbb{K}^+$. Hence it is a 1-uniform Parseval fusion frame for $\mathbb{K}^+$. Similarly we can say that $\{W_i : i \in I_-\}$ is also a 1-uniform Parseval fusion frame for $(\mathbb{K}^-, -[\cdot,\cdot])$. So, $\{W_i : i \in I_+\} \cup \{W_i : i \in I_-\} = \{W_i : i \in I\}$ is a 1-uniform $J$-Parseval fusion frame for $\mathbb{K}$ with $\zeta = \sqrt{2}$.

Conversely, let $\{(W_i, 1) : i \in I\}$ be a 1-uniform $J$-Parseval fusion frame for $\mathbb{K}$ with $\zeta = \sqrt{2}$. So let $I_+ = \{i \in I : [f_i, f_i] > 0 \text{ for all } f_i \in W_i\}$ and $I_- = \{i \in I : [f_i, f_i] < 0 \text{ for all } f_i \in W_i\}$. Consider $M_\pm = \overline{\sum_{i\in I_\pm} W_i}$. Now it is clear that $\{W_i : i \in I_+\}$ is a 1-uniform Parseval fusion frame for $(M_+, [\cdot,\cdot])$ and $\{W_i : i \in I_-\}$ is a 1-uniform Parseval fusion frame for $(M_-, -[\cdot,\cdot])$. Hence from [3] we know that $\{W_i : i \in I_+\}$ is an orthonormal basis of subspaces for $(M_+, [\cdot,\cdot])$ and $\{W_i : i \in I_-\}$ is an orthonormal basis of subspaces for $(M_-, -[\cdot,\cdot])$. Again we have $c_0(M_+, \mathtt{C}) + c_0(M_-, \mathtt{C}) = \sqrt{2}$, which implies that $c_0(M_+, \mathtt{C}) = \frac{1}{\sqrt{2}}$ and $c_0(M_-, \mathtt{C}) = \frac{1}{\sqrt{2}}$. Now by some simple numerical calculation we have $\gamma(G_{M_+}) = 1$ and also $\gamma(G_{M_-}) = 1$. Using all the above results we conclude that $\mathbb{K} = M_+[\dot{\oplus}]M_-$. Hence the proof. $\square$

Let $\mathbb{F} = \{W_i : i \in I\}$ be a collection of non-neutral definite subspaces in a Krein space $\mathbb{K}$. Consider $M_+ = \overline{\sum_{i\in I_+} W_i}$ and $M_- = \overline{\sum_{i\in I_-} W_i}$, where $I_+ = \{i \in I : [f_i, f_i] > 0 \text{ for all } f_i \in W_i\}$ and $I_- = \{i \in I : [f_i, f_i] < 0 \text{ for all } f_i \in W_i\}$. Now if $M_+$ is a maximal uniformly $J$-positive subspace of $\mathbb{K}$ and $M_-$ is a maximal uniformly $J$-negative subspace of $\mathbb{K}$, then $\{W_i : i \in I_+\}$ and $\{W_i : i \in I_-\}$ will be fusion frames for $(M_+, [\cdot,\cdot])$ and $(M_-, -[\cdot,\cdot])$, respectively. Let $T_{W,v_1}$ be the synthesis operator for the fusion frame $\{W_i : i \in I_+\}$ and $T_{W,v_2}$ be the synthesis operator for the frame $\{W_i : i \in I_-\}$. Let $T_{W,v_1}^*$ and $T_{W,v_2}^*$ be the adjoint operators of $T_{W,v_1}$ and $T_{W,v_2}$, respectively in Hilbert space sense.

Now we derive an useful result regarding $J$-Parseval fusion frames for a Krein space $\mathbb{K}$. The following theorem guarantees that a Krein space is richly supplied with $J$-Parseval fusion frames.

**Theorem 3.9.** *Let $\mathbb{K}$ be a Krein space. Assume that $M_1$ and $M_2$ are closed, $J$-definite subspaces of $\mathbb{K}$, respectively. Let $\{(X_i, u_i)\}_{i\in I_1}$ and $\{(Y_i, v_i)\}_{i\in I_2}$ are Parseval fusion frames for $M_1$ and $M_2$, respectively. Then $\{X_i\}_{i\in I_1} \cup \{Y_i\}_{i\in I_2}$ is a strictly disjoint family in $\mathbb{K}$ only if $\{(X_i, u_i)\}_{i\in I_1} \cup \{(Y_i, v_i)\}_{i\in I_2}$ is a $J$-Parseval fusion frame for $\mathbb{K}$ with $\zeta \in [\sqrt{2}, 2)$.*

*Proof.* Without any loss of generality we assume that $M_1$ is positive $J$-definite. Hence it is intrinsically complete and so $(M_1, [\cdot,\cdot])$ is a Hilbert space. Now $\{(X_i, u_i) : i \in I_1\}$ is a Parseval fusion frame for $(M_1, [\cdot,\cdot])$ and so $\overline{\sum_{i\in I_+} X_i} = M_1$ and also $X_i \in \mathbb{P}^{++}$.

Again $\{X_i\} \cup \{Y_i\}$ is a strictly disjoint family in $\mathbb{K}$, so $\overline{\sum_{i\in I_1} X_i}\ [\perp]\ \overline{\sum_{i\in I_2} Y_i} = \{0\}$. Hence $\overline{\sum_{i\in I_2} Y_i}$ is a closed $J$-negative subspace of $\mathbb{K}$. Now $\{(Y_i, v_i) : i \in I_2\}$ is a Parseval fusion frame for $M_2$. Therefore, $\overline{\sum_{i\in I_2} Y_i} = M_2$. Therefore $M_2$ is a negative $J$-definite subspace of $\mathbb{K}$. Since $M_1\ [\perp]\ M_2$, so we have a fundamental decomposition of $\mathbb{K}$ *i.e.* $\mathbb{K} = M_1\ [\dot{+}]\ M_2$ (see [15]). Since both $M_1$ and $M_2$ are closed $J$-definite and also intrinsically complete, so both $M_1$ and $M_2$ are uniformly $J$-definite (see [10]). Let $\zeta = c_0(M_+, \mathtt{C}) + c_0(M_-, \mathtt{C})$, then $\{(X_i, u_i)\}_{i\in I_1} \cup \{(Y_i, v_i)\}_{i\in I_2}$ is a $J$-fusion frame for $\mathbb{K}$ with $\zeta$. As both $\{(X_i, u_i)\}_{i\in I_1}$ and $\{(Y_i, v_i)\}_{i\in I_2}$ are Parseval fusion frames for $M_1$ and $M_2$, respectively, so $\{(X_i, u_i)\}_{i\in I_1} \cup \{(Y_i, v_i)\}_{i\in I_2}$ is also a $J$-Parseval fusion frame for $\mathbb{K}$ with $\zeta$. Hence the proof. □

*Remark* 3.10. The statement of the above theorem is sufficient but not necessary. Since $\{(X_i, u_i)\}_{i\in I_1} \cup \{(Y_i, v_i)\}_{i\in I_2}$ is a $J$-Parseval fusion frame for $\mathbb{K}$ with $\zeta\in[\sqrt{2}, 2)$, so there exists uniformly $J$-positive definite subspace $M_+$ and uniformly $J$-negative subspace $M_-$ such that $\{(X_i, u_i)\}_{i\in I_1}$ is a Parseval fusion frame for $(M_+, [\cdot,\cdot])$ and $\{(Y_i, v_i)\}_{i\in I_2}$ is a Parseval fusion frame for the Hilbert space $(M_-, -[\cdot,\cdot])$. But $M_+$ may not be $J$-perpendicular to $M_-$. Hence $\{X_i\}_{i\in I_1} \cup \{Y_i\}_{i\in I_2}$ may not be a strictly disjoint family in $\mathbb{K}$.

The following result for $J$-Parseval fusion frames describes how $J$-Parseval fusion frames can be combined to form a new $J$-Parseval fusion frame under some restrictions.

**Theorem 3.11.** *Let $\{(X_i, v_i) : i \in I\}$ and $\{(Y_i, v_i) : i \in I\}$ are $J$-Parseval fusion frames for $\mathbb{K}$ such that $M_+ = \overline{\sum_{i\in I_+} Y_i} = \overline{\sum_{i\in I_+} X_i}$ and $M_- = \overline{\sum_{i\in I_-} Y_i} = \overline{\sum_{i\in I_-} X_i}$. Also let $\{X_i : i \in I\}$ and $\{Y_i : i \in I\}$ are strictly disjoint family of subspaces in $\mathbb{K}$. Then $\{(X_i + Y_i, v_i) : i \in I\}$ is a $J$-Parseval fusion frame for $\mathbb{K}$ if and only if ${T^*_{X,v}}^+ {T_{Y,v}}^+ + {T^*_{Y,v}}^+ {T_{X,v}}^+ = 0$ and ${T^*_{X,v}}^- {T_{Y,v}}^- + {T^*_{Y,v}}^- {T_{X,v}}^- = 0$, where ${T_{X,v}}^+$, ${T_{Y,v}}^+$, ${T_{X,v}}^-$ and ${T_{Y,v}}^-$ are synthesis operators of $\{(X_i, v_i) : i \in I_+\}$, $\{(Y_i, v_i) : i \in I_+\}$, $\{(X_i, v_i) : i \in I_-\}$ and $\{(Y_i, v_i) : i \in I_-\}$, respectively.*

*Proof.* Since $\{X_i : i \in I\}$ and $\{Y_i : i \in I\}$ are strictly disjoint family of subspaces in $\mathbb{K}$, so $X_i + Y_i$ are closed subspaces for each $i \in I$. Let $\{(X_i + Y_i, v_i) : i \in I\}$ be a $J$-Parseval fusion frame for $\mathbb{K}$. It is given that $\{(X_i, v_i) : i \in I\}$ and $\{(Y_i, v_i) : i \in I\}$ are also $J$-Parseval fusion frames for $\mathbb{K}$. Hence $X_i \subset M_+$ for all $i \in I_+$. Similarly $Y_i \subset M_+$ for all $i \in I_+$. According to our assumption for each $i \in I_+$, $X_i$ is uniformly $J$-positive subspace of $\mathbb{K}$. Hence regular. Since $X_i + Y_i$ is a closed subspace of $M_+$, so it is also regular. Similarly for each $i \in I_-$, $X_i + Y_i$ is also regular. Therefore $X_i + Y_i$ is uniformly $J$-definite for $i \in I$.

Since $\{(X_i, v_i) : i \in I\}$ is a $J$-Parseval fusion frame for $\mathbb{K}$, therefore $\{(X_i, v_i) : i \in I_+\}$ is a Parseval fusion frame for $(M_+, [\cdot,\cdot])$. Let ${T_{X,v}}^+$ be the synthesis operator for $\{(X_i, v_i) : i \in I_+\}$ in $(M_+, [\cdot,\cdot])$. The synthesis operator ${T_{X,v}}^+$ is defined by ${T_{X,v}}^+(\{f_i\}) = \sum_{i\in I_+} v_i f_i$, where $f_i \in X_i$. Now ${T^*_{X,v}}^+$, the analysis operator is defined by ${T^*_{X,v}}^+(f) = \{v_i \pi_{W_i}|_{M_+}(f)\}_{i\in I_+}$.

Similarly $\{(Y_i, v_i) : i \in I\}$ is also a $J$-Parseval fusion frame for $\mathbb{K}$. Hence $\{(Y_i, v_i) : i \in I_+\}$ is a Parseval frame for $(M_+, [\cdot,\cdot])$. So, ${T_{Y,v}}^+$ and ${T^*_{Y,v}}^+$ are defined as above. Also $\{(X_i, v_i) : i \in I_-\}$ and $\{(Y_i, v_i) : i \in I_-\}$ are Parseval frames for $(M_-, -[\cdot,\cdot])$, so we can define the operators ${T_{X,v}}^-$, ${T^*_{X,v}}^-$, ${T_{Y,v}}^-$ and ${T^*_{Y,v}}^-$ as usual.

Now $\{(X_i+Y_i, v_i) : i \in I\}$ is a $J$-Parseval fusion frame for $\mathbb{K}$. So $\{(X_i+Y_i, v_i) : i \in I_+\}$ is a Parseval fusion frame for $(M_+, [\cdot,\cdot])$. Similarly $\{(X_i+Y_i, v_i) : i \in I_-\}$ is a Parseval fusion frame for $(M_-, -[\cdot,\cdot])$. Let ${T_{X+Y,v}}^+$ be the synthesis operator for the fusion frame $\{(X_i + Y_i, v_i) : i \in I_+\}$ in $(M_+, [\cdot,\cdot])$ and ${T_{X+Y,v}}^-$ be the synthesis operator for the fusion frame $\{(X_i+Y_i, v_i) : i \in I_-\}$ in $(M_-, -[\cdot,\cdot])$. Also let us assume that ${T^*_{X+Y,v}}^+$ and ${T^*_{X+Y,v}}^-$ be the adjoint operators of ${T_{X+Y,v}}^+$ and ${T_{X+Y,v}}^-$, respectively. Now let us define ${\mathfrak{T}_{X+Y,v}}^+ := {T_{X,v}}^+ + {T_{Y,v}}^+$. Then we have ${\mathfrak{T}^*_{X+Y,v}}^+ = {T^*_{X,v}}^+ + {T^*_{Y,v}}^+$. A direct calculation shows that ${\mathfrak{T}_{X+Y,v}}^+ = {T_{X+Y,v}}^+$ is the synthesis operator for the frame $\{(X_i + Y_i, v_i) : i \in I_+\}$ in $(M_+, [\cdot,\cdot])$.
Hence ${\mathfrak{T}^*_{X+Y,v}}^+ {\mathfrak{T}_{X+Y,v}}^+ = I$. Therefore, ${T^*_{X,v}}^+ {T_{Y,v}}^+ + {T^*_{Y,v}}^+ {T_{X,v}}^+ = 0$.
Similarly we can show that ${T^*_{X,v}}^- {T_{Y,v}}^- + {T^*_{Y,v}}^- {T_{X,v}}^- = 0$.

Conversely we have to show that $\{(X_i + Y_i, v_i) : i \in I\}$ is a $J$-Parseval fusion frame for $\mathbb{K}$. Since $\overline{\sum_{i\in I_+}(X_i + Y_i)} = M_+$, so it is sufficient to show that $\{(X_i + Y_i, v_i) : i \in I_+\}$ is a Parseval fusion frame for $(M_+, [\cdot,\cdot])$ and $\{(X_i+Y_i, v_i) : i \in I_-\}$ is a Parseval fusion frame for $(M_-, -[\cdot,\cdot])$. Let ${\mathfrak{T}_{X+Y,v}}^+$ be the synthesis operator for the Bessel family $\{(X_i+Y_i, v_i) : i \in I_+\}$. Then ${\mathfrak{T}_{X+Y,v}}^+(\{h_i\}) = \sum_{i\in I_+} v_i h_i = \sum_{i\in I_+} v_i(f_i + g_i) = {T_{X,v}}^+ + {T_{Y,v}}^+$. Now it is easy to show that the condition ${T^*_{X,v}}^+ {T_{Y,v}}^+ + {T^*_{Y,v}}^+ {T_{X,v}}^+ = 0$ implies that $\{(X_i + Y_i, v_i) : i \in I_+\}$ is a Parseval fusion frame for $M_+$. Similarly as above we can show that $\{(X_i+Y_i, v_i) : i \in I_-\}$ is a Parseval fusion frame for $M_-$. We know that $\zeta = c_0(M_+, \mathtt{C}) + c_0(M_-, \mathtt{C})$. Now since the quantities $c_0(M_+, \mathtt{C})$ and $c_0(M_-, \mathtt{C})$ are fixed throughout the proof and $\{(X_i, v_i) : i \in I\}$ is a given $J$-Parseval frame with $\zeta \in [\sqrt{2}, 2)$, so we already have that required $\zeta$. Hence the proof. □

**Definition 3.12.** Let $\{(W_i, v_i)\}_{i\in I}$ be a $J$-fusion frame for the Krein space $\mathbb{K}$ with $J$-fusion frame operator $S_{W,v}$, then the collection $\{(S^{-1}_{W,v}(W_i), v_i)\}_{i\in I}$ is called the *cannonical $J$-dual fusion frame* for $\{(W_i, v_i)\}_{i\in I}$ in $\mathbb{K}$.

The following theorem is an identity between $J$-fusion frame for the Krein space $\mathbb{K}$ and the *cannonical $J$-dual fusion frame* for $\mathbb{K}$.

**Theorem 3.13.** *Let $\{(W_i, v_i) : i \in I\}$ be $J$-fusion frame for the Krein space $\mathbb{K}$ with cannonical $J$-dual fusion frame $\{(S_{W,v}^{-1}(W_i), v_i) : i \in I\}$. Then $\forall\ I_1 \subset I$ and $\forall\ f \in \mathbb{K}$ we have $\sum_{i\in I_1} \sigma_i v_i^2[\pi_{W_i} Jf, Jf] - \sum_{i\in I} \sigma_i v_i^2[\pi_{S_{W,v}^{-1}(W_i)} JS_{W,v}^{I_1}, JS_{W,v}^{I_1}]$ $= \sum_{i\in I_1^c} \sigma_i v_i^2[\pi_{W_i} Jf, Jf] - \sum_{i\in I} \sigma_i v_i^2[\pi_{S_{W,v}^{-1}(W_i)} JS_{W,v}^{I_1^c}, JS_{W,v}^{I_1^c}]$, where $\sigma_i = 1$ if $i \in I_+$ and $\sigma_i = -1$ if $i \in I_-$.*

*Proof.* Let $S_{W,v}$ denote the frame operator for $\{(W_i, v_i) : i \in I\}$. Then we have $S_{W,v}(f) = \sum_{i\in I} \sigma_i v_i^2 \pi_{J(W_i)}(f)$. Also $S_{W,v} = S_{W,v}^{I_1} + S_{W,v}^{I_1^c}$. Then $I = S_{W,v}^{-1} S_{W,v}^{I_1} + S_{W,v}^{-1} S_{W,v}^{I_1^c}$. Therefore from operator inequality we have $S_{W,v}^{-1} S_{W,v}^{I_1} - S_{W,v}^{-1} S_{W,v}^{I_1^c} = S_{W,v}^{-1} S_{W,v}^{I_1} S_{W,v}^{-1} S_{W,v}^{I_1} - S_{W,v}^{-1} S_{W,v}^{I_1^c} S_{W,v}^{-1} S_{W,v}^{I_1^c}$. Then for every $f, g \in \mathbb{K}$ we have

$$
\begin{aligned}
&[S_{W,v}^{-1} S_{W,v}^{I_1}(f), g] - [S_{W,v}^{-1} S_{W,v}^{I_1} S_{W,v}^{-1} S_{W,v}^{I_1}(f), g] \\
&\qquad = [S_{W,v}^{I_1}(f), S_{W,v}^{-1} g] - [S_{W,v}^{-1} S_{W,v}^{I_1}(f), S_{W,v}^{I_1} S_{W,v}^{-1} g]
\end{aligned}
$$

Now if we choose $g = S_{W,v}(f)$, then the above equation reduces to

$$
\begin{aligned}
&= [S_{W,v}^{I_1}(f), f] - [S_{W,v}^{-1} S_{W,v}^{I_1}(f), S_{W,v}^{I_1}(f)] \\
&\qquad = \sum_{i\in I_1} \sigma_i v_i^2[\pi_{W_i} Jf, Jf] - \sum_{i\in I} \sigma_i v_i^2[\pi_{S_{W,v}^{-1}(W_i)} JS_{W,v}^{I_1}, JS_{W,v}^{I_1}]
\end{aligned}
$$

Now replacing $I_1$ by $I_1^c$ we can have the other part of the equality. Combining we finally get

$$
\begin{aligned}
&\sum_{i\in I_1} \sigma_i v_i^2[\pi_{W_i} Jf, Jf] - \sum_{i\in I} \sigma_i v_i^2[\pi_{S_{W,v}^{-1}(W_i)} JS_{W,v}^{I_1}, JS_{W,v}^{I_1}] \\
&\qquad = \sum_{i\in I_1^c} \sigma_i v_i^2[\pi_{W_i} Jf, Jf] - \sum_{i\in I} \sigma_i v_i^2[\pi_{S_{W,v}^{-1}(W_i)} JS_{W,v}^{I_1^c}, JS_{W,v}^{I_1^c}]
\end{aligned}
$$

□

We next state the following theorem by Douglas [14] which we will use in our study.

**Theorem 3.14.** [14] *Let $S$ and $T$ be bounded linear operators on a Hilbert space $\mathbb{H}$. Then the following conditions are equivalent*
*(i) $R(S) \subset R(T)$*
*(ii) there exists $\lambda \geq 0$ such that $SS^* \leq \lambda^2 TT^*$*
*(iii) there exists a closed linear operator $S_1$ such that $SS_1 = T$*

We characterize uniformly $J$-definite subspaces of a Krein space $\mathbb{K}$ in terms of an inequality regarding $J$-fusion frames.

Let $M$ be a non-negative subspace of a Krein space $\mathbb{K}$, then $M = M^0[\dot{+}]M^+$ where $M^+$ is the positive part of $M$ and $M^0$ is the *isotropic part* of $M$, precisely $M^+ = M \ominus M^0$, where $M \ominus M^0 = M \cap (M \cap M^0)^\perp$. Let $G_M$ be the *Gramian operator* of $M$. Then $N(G_M) = M^0$ and $R(G_M) = M^+$. So we have $G_M =$

$G_{M^+}$. The decomposition of $M$ is guaranteed by the *spectral decomposition* of the *Gramian operator*.

**Theorem 3.15.** *Let $\mathbb{F} = \{(W_i, v_i\}_{i\in I}$ be a Bessel family in a Krein space $\mathbb{K}$, also let $M = \overline{\sum W_i : i \in I}$ and $M^0 = M \cap M^{[\perp]}$. If there exist constants $0 < A \le B < \infty$ such that*

$$A[f,f] \le \sum_{i\in I} v_i^2 |[\pi_{JM}\pi_{W_i} J\pi_M(f), f]| \le B[f,f] \text{ for every } f \in M, \tag{3.2}$$

*then the deficiency subspace $M \ominus M^0$ is a (closed) uniformly $J$-positive subspace of $M$. Moreover, if $\mathbb{F}$ is a frame for the Hilbert space $(M, [\cdot,\cdot]_J)$, the converse holds.*

*Proof.* First, let us assume that the equation (3.2) holds. Then $[f,f] \ge 0$ for all $f \in M$. So, $M$ is a $J$-nonnegative subspace of $\mathbb{K}$, or equivalently, $(M, [\cdot,\cdot])$ is a non-negative inner product space.

Now given that $\mathbb{F} = \{(W_i, v_i\}_{i\in I}$ is a Bessel family in $\mathbb{K}$. So it is a Bessel family in the associated Hilbert space $(\mathbb{K}, [\cdot,\cdot]_J)$. Then $T^*_{W,v}(f) = \{v_i \pi_{W_i}(f)\}_{i\in I},\ f \in \mathbb{K}$. Now for all $f \in M$,

$$\sum_{i\in I} v_i^2 [\pi_{W_i}, f]_J \le C[f,f]_J$$

where $C = \|T^*_{W,v}\|_J^2 > 0$, then $\sum_{i\in I} v_i^2 \pi_{W_i} \le C\pi_M$ since $\pi_M \pi_{W_i} = \pi_{W_i}\pi_M = \pi_{W_i}$. So, using equation (3.2) it is easy to see that

$$\begin{aligned} A[G_M f, f]_J &\le \sum_{i\in I} v_i^2 |[\pi_{JM}\pi_{W_i} J\pi_M(f), f]| = \sum_{i\in I} v_i^2 |[\pi_M \pi_{JW_i} \pi_M(f), f]_J| \\ &= \sum_{i\in I} v_i^2 |[\pi_M J \pi_{W_i} J \pi_M(f), f]_J| = |[\pi_M J\{\sum_{i\in I} v_i^2 \pi_{W_i}\} J\pi_M(f), f]_J| \\ &\le C[(G_M)^2 f, f]_J,\ f \in \mathbb{K}. \end{aligned}$$

Thus, $0 \le G_M \le \frac{C}{A}(G_M)^2$. So applying Douglas theorem [14] we have

$$R((G_M)^{1/2}) \subseteq R(G_M)$$

Also we have $R(G_M) \subseteq R((G_M)^{1/2})$. Moreover, it follows that $R(G_M)$ is closed since $R(G_M) = R((G_M)^{1/2})$.

Let $M' = M \ominus M^0$. Since $R(G_M)$ is closed, so by reduced minimum modulus theorem there exists $\gamma > 0$ such that

$$[f,f] = [G_M f, f]_J = \|(G_M)^{1/2} f\|_J^2 \ge \gamma \|f\|_J^2 \ \text{ for every } f \in N(G_M)^\perp = M \ominus M^0.$$

Hence $M'$ is a closed uniformly $J$-positive subspace of $\mathbb{K}$.

Conversely, suppose that $W$ is a frame for $(M, [\cdot,\cdot]_J)$, i.e. there exist constants $B' \ge A' > 0$ such that

$$A' P_M \le \sum_{i\in I} v_i^2 \pi_{W_i} \le B' P_M,$$

$$A' P_M \le T_{W,v} T^*_{W,v} \le B' P_M,$$

where $T_{W,v} \in L\big((\sum_{i\in I} \oplus W_i)_{\ell_2}, \mathbb{K}\big)$ is the synthesis operator of $W$. If $M' = M \ominus M^0$ is a uniformly $J$-positive subspace of $\mathbb{K}$, then there exists some real number

$\alpha > 0$ such that $\alpha P_{M'} \leq G_{M'} \leq P_{M'}$. Then by Douglas theorem, $R(P_{M'}) \subseteq R((G_{M'})^{1/2}) \subseteq R(P_{M'})$. So we have $R(G_{M'}{}^{\frac{1}{2}}) = M' = R(G_{M'})$. Since $G_M = G_{M'}$ it is easy to see that

$$A'(G_M)^2 = A'(G_{M'})^2 \leq P_M J T_{W,v} T_{W,v}^* J P_M \leq B'(G_{M'})^2 = B'(G_M)^2.$$

Therefore again using Douglas theorem we have $R(P_M J T_{W,v}) = R(G_{M'}) = R((G_{M'})^{1/2})$ or equivalently, there exist $B \geq A > 0$ such that

$$AG_M = AG_{M'} \leq P_M J T_{W,v} T_{W,v}^* J P_M \leq BG_{M'} = BG_M$$

So from above we have

$$A[f,f] \leq [\pi_M J\{\sum_{i\in I} v_i^2 \pi_{W_i}\} J\pi_M(f), f]_J \leq B[f,f] \text{ for every } f \in M$$

$$i.e.\ A[f,f] \leq \sum_{i\in I} v_i^2 |[\pi_M \pi_{JW_i} \pi_M(f), f]_J| \leq B[f,f]$$

$$i.e.\ A[f,f] \leq \sum_{i\in I} v_i^2 |[\pi_{JM} \pi_{W_i} J\pi_M(f), f]| \leq B[f,f] \text{ for every } f \in M$$

$\square$

**Acknowledgments.** Shibashis Karmakar, Sk. Monowar Hossein and Kallol Paul gratefully acknowledge the support of Jadavpur University, Kolkata and Aliah University, Kolkata for providing all the facilities when the manuscript was prepared. Shibashis Karmakar also acknowledges the financial support of CSIR, Govt. of India.

## References

1. R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72** (1952), no.2, 341–366.
2. I. Daubechies, A. Grossmann, and Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys. **27** (1986), 1271–1283.
3. P. G. Casazza and G. Kutyniok, *Frames of subspaces*, Wavelets, Frames and Operator Theory (College Park, MD, 2003), 87–113, Contemp. Math. **345**, Amer. Math. Soc., Providence, RI, 2004.
4. P. G. Casazza, G. Kutyniok, and S. Li, *Fusion frames and distributed processing*, Appl. Comput. Harmon. Anal. **25** (2008), no. 1, 114–132.
5. M. S. Asgari and A. Khosravi, *Frames and bases of subspaces in Hilbert spaces*, J. Math. Anal. Appl. **308** (2005), no. 2, 541–553.
6. R. Calderbank, P. G. Casazza, A. Heinecke, and G. Kutyniok, *Sparse fusion frames: Existence and construction*, Adv. Comput. Math. **35** (2011), no. 1, 1–31.
7. P. G. Casazza, M. Fickus, D. Mixon, Y. Wang, and Z. Zhou, *Constructing tight fusion frames*, Appl. Comput. Harmon. Anal. **30** (2011), no. 2, 175–187.
8. G. Kutyniok, A. Pezeshki, R. Calderbank, and T. Liu, *Robust dimension reduction, fusion frames, and Grassmannian packings*, Appl. Comput. Harmon. Anal. **26** (2009), no. 1, 64–76.
9. J. Benedetto and M. Fickus, *Finite normalized tight frames*, Adv. in Comput. Math. **18** (2003), no. 2, 357–385.
10. J. Bognar, *Indefinite inner product spaces*, Springer Berlin, 1974.
11. J. I. Giribet, A. Maestripieri, F. Martínez Pería, and P. G. Massey, *On frames for Krein spaces*, J. Math. Anal. Appl. **393** (2012), no. 1, 122–137.
12. K. Esmeral, O. Ferrer, and E. Wagner, *Frames in Krein spaces arising from a non-regular W-metric*, Banach J. Math. Anal. **9** (2015), no. 1, 1–16.

13. P. Acosta-Humánez, K. Esmeral, and O. Ferrer, *Frames of subspaces in Hilbert spaces with $W$-metrics*, VERSITA **23** (2015), no. 2, 5–22.
14. R. G. Douglas, *On majorization, factorization and range inclusion of operators in Hilbert space*, Proc. Amer. Math. Soc. **17** (1966), 413–416.
15. I. S. Iokhvidov and T. Ya. Azizov, *Linear operators in spaces with an indefinite metric*, John Wiley & sons, 1989.
16. T. Ando, *Linear operators on Krein spaces*, Hokkaido University, Sapporo, Japan, 1989.
17. Sk. M. Hossein, S. Karmakar, and K. Paul, *Tight $J$-frames in Krein space and the Associated $J$-frame potential*, Int. J. Math. Anal. **20** (2016), no. 19, 917–931.
18. S. Karmakar, *$J$-fusion frame for Krein spaces*, arXiv:1611.01339.



[1]Department of Mathematics, Jadavpur University,
188, Raja Subodh Chandra Mallick Rd, Jadavpur University Campus Area, Jadavpur, Kolkata, West Bengal 700032, India.
*E-mail address*: shibashiskarmakar@gmail.com; kalloldada@gmail.com

[2]Department of Mathematics, Aliah University,
II-A/27, Action Area II, Newtown, Kolkata, West Bengal 700156, India.
*E-mail address*: sami_milu@yahoo.co.uk