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ON THE BEHAVIOR AT INFINITY OF CERTAIN INTEGRAL OPERATOR WITH POSITIVE KERNEL

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ABSTRACT. Let $\alpha > 0$ and $\gamma > 0$. We consider integral operator of the form

$$\mathcal{G}_{\phi_\gamma} f(x) := \frac{1}{\Psi_\gamma(x)} \int_0^x (1 - \frac{y}{x})^{\alpha-1} \phi_\gamma(y) f(y) dy, \quad x > 0.$$

This paper is devoted to the study of the infinity behavior of $\mathcal{G}_{\phi_\gamma}$. We also provide separately result on the similar problem in the weighted Lebesgue space.

1. INTRODUCTION

Let $\alpha > 0$, $\gamma > 0$ and

$$\mathcal{G}_{\phi_\gamma} f(x) := \frac{1}{\Psi_\gamma(x)} \int_0^x (1 - \frac{y}{x})^{\alpha-1} \phi_\gamma(y) f(y) dy \quad x > 0, \tag{1.1}$$

where

$$\Psi_\gamma(x) := \int_0^x (1 - \frac{y}{x})^{\alpha-1} \phi_\gamma(y) dy.$$

We denote by $\mathfrak{L}$ the family of positive nondecreasing functions $\{\phi_\gamma(y)\}$ with respect to y such that

$$\int_{I \subset \mathbb{R}_+ := (0, +\infty)} \phi_\gamma(y) dy < \infty.$$

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If $\alpha > 0$ and $\phi_\gamma(x) = 1$, the operator (1.1) coincides with the classical Riemann–Liouville fractional integral operator ([9]). Also, we will see (1.1) as well-known Hardy operator denoting $\alpha = 1$ and $\phi_\gamma(x) = 1$ ([3]). In the last decades a considerable attention of researchers was attracted to the study of the mapping properties of integral operators such as Hardy operators, Riemann–Liouville operators etc, in weighted Lebesgue spaces (see, e.g., monographs [4], [5], [11] and papers [1], [2], [6], [7], [8], [10], [12]). Hardy inequality is one of the main tools to study other integral operators from the boundedness viewpoint (see e.g., [4], [11]). In this paper, the problem of the approximation of the identity for (1.1) have been studied in the L^p sense and in the almost everywhere sense i.e. how can we write the following equality?

$$\lim_{\gamma\to\infty} \mathcal{G}_{\phi_\gamma} f(x) = f(x). \tag{1.2}$$

When $\alpha \in (0, 1)$, we illustrate the convergence of (1.2) is not established. The other sections of our work are devoted to the proof of (1.2) for $\alpha \geq 1$ and the similar problem in the weighted Lebesgue space setting. It seems that the results of this work can be applied to a wider class of integral operators including much broader class of kernels. We assume throughout the paper $\mathbb{R}_+ := (0, +\infty)$ and $\{\phi_\gamma(x)\} \in \mathfrak{L}$. The symbol $p' := \frac{p}{p-1}, p \neq 1$ denotes the conjugate numbers of p, and the symbol $\square$ marks the end of a proof.

2. Main results

2.1. **Divergence of $\mathcal{G}_{\phi_\gamma}$ for $\alpha \in (0, 1)$.** The following example illustrate this fact. Let us begin with a few basic definitions: The **gamma function** is defined for $\{z \in \mathbb{C}, z \neq 0, -1, -2, ...\}$ to be:

$$\Gamma(z) = \int_0^\infty s^{z-1} e^{-s} ds.$$

Remember some important characteristics of the gamma function:
1. For $z \in \{\mathbb{N} \setminus 0\}, \Gamma(z) = z!$,
2. $\Gamma(z + 1) = z\Gamma(z)$.

The **beta function** is defined for $\{x, y \in \mathbb{C}, Re(x) > 0, Re(y) > 0\}$ to be:

$$\beta(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} dt.$$

Additionally,

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}.$$

We have

$$\Psi_\gamma(x) := \int_0^x (1 - \frac{y}{x})^{\alpha-1} \phi_\gamma(y) dy. \quad \alpha \in (0, 1), \gamma > 0, x > 0.$$

Let $\phi_\gamma(x) = x^\gamma$, then by the substitution $y := ux$

$$\int_0^x (1 - \frac{y}{x})^{\alpha-1} y^\gamma dy = \int_0^1 (1 - u)^{\alpha-1} u^\gamma x^{\gamma+1} du$$

$$= x^{\gamma+1}\int_0^1 (1-u)^{\alpha-1}u^{\gamma}du = x^{\gamma+1}\beta(\gamma+1,\alpha) = x^{\gamma+1}\frac{\Gamma(\gamma+1)\Gamma(\alpha)}{\Gamma(\gamma+\alpha+1)} := P_*$$

Stirling's formula $\lim_{n\to\infty}\frac{n!}{\sqrt{2\pi n}(\frac{n}{e})^n} = 1$ allows to replace factorials with their approximation, so when $\gamma\to\infty$,

$$\Gamma(\gamma+1) \approx \sqrt{2\pi\gamma}\gamma^{\gamma}e^{-\gamma}$$

$$\Gamma(\gamma+\alpha+1) \approx \sqrt{2\pi(\gamma+\alpha)}(\gamma+\alpha)^{\gamma+\alpha}e^{-\gamma-\alpha}$$

then

$$\frac{\Gamma(\gamma+1)\Gamma(\alpha)}{\Gamma(\gamma+\alpha+1)} \approx \frac{\sqrt{2\pi\gamma}\gamma^{\gamma}e^{-\gamma}\Gamma(\alpha)}{\sqrt{2\pi(\gamma+\alpha)}(\gamma+\alpha)^{\gamma+\alpha}e^{-\gamma-\alpha}} \approx \left(\frac{1}{\gamma}\right)^{\alpha}\Gamma(\alpha).$$

Hence when $\gamma\to\infty$

$$P_* \approx x^{\gamma+1}\left(\frac{1}{\gamma}\right)^{\alpha}\Gamma(\alpha).$$

Remark 2.1. Let $\gamma=\alpha$. Assume that $0<\delta\le\frac{1}{2}$. then

$$\liminf_{\gamma\to\infty}\frac{1}{\Psi_\gamma(x)}\int_0^{x-\delta}(1-\frac{y}{x})^{\gamma-1}y^{\gamma}dy \ge \frac{1}{2}.$$

2.2. **convergence almost everywhere of $\mathcal{G}_{\phi_\gamma}$ for $\alpha\ge 1$.** We capitalize on the fact that any nondecreasing function has only a countable number of discontinuities, and they are all jump discontinuities. So we can change any such function into such a function that is also right-continuous by changing its values at a countable number of points. For all $\gamma>0$, let ϕ'_γ represent ϕ_γ changed in such a way to make ϕ'_γ right-continuous. We do this by letting $\phi'_\gamma(x_1) = \lim_{x\to x_1^+}\phi_\gamma(x)$ for all $x_1>0$. We claim that ϕ'_γ then satisfies every hypothesis we make for ϕ_γ.

1. Certainly, ϕ'_γ remains positive and nondecreasing on $I\subset\mathbb{R}_+ := (0,+\infty)$, remains in $L^1(I)$ for any bounded subinterval I of $(0,+\infty)$, and $\int\phi'_\gamma = \int\phi_\gamma$.
2. Let $u_0\in(0,1), x>0$. Assume $u_0<u_1<1$. Then $4\phi'_\gamma(u_0x)\le\phi_\gamma(u_1x)$ so

$$0 \le \limsup_{\gamma\to\infty}\frac{\phi'_\gamma(u_0x)}{\Psi_\gamma(x)} \le \lim_{\gamma\to\infty}\frac{\phi_\gamma(u_1x)}{\Psi_\gamma(x)} = 0.$$

Therefore we assume also

$$\lim_{\gamma\to\infty}\frac{\phi_\gamma(u_0x)}{\Psi_\gamma(x)} = 0. \tag{2.1}$$

In the next theorem we will prove the equality 1.2 at any Lebesgue point of f. So we need some preliminaries.

Definition 2.2. A Lebesgue point of an integrable function f on $\mathbb{R}_+$ is a point $x\in\mathbb{R}_+$ satisfying

$$\forall\epsilon>0,\ \exists\delta_0>0\ :\ 0<\delta<\delta_0 \quad\Rightarrow\frac{1}{\delta}\int_{x-\delta}^{x}|f(y)-f(x)|\,dy<\epsilon.$$

Lemma 2.3. [13] *Let f be a monotone increasing function which is continuous on the right. Then there is a unique Borel measure μ such that for all a and b we have*

$$\mu(a,b] = f(b)-f(a).$$

Theorem 2.1. *If $f \in L^1_{loc}(\mathbb{R}_+)$. Let ϕ_γ be right-continuous. Then at any Lebesgue point $x \in \mathbb{R}_+$ of f we have*

$$\lim_{\gamma\to\infty} \frac{1}{\Psi_\gamma(x)} \int_0^x (1-\frac{y}{x})^{\alpha-1}\phi_\gamma(y)\,|f(y)-f(x)|\,dy = 0.$$

Proof. According to the introduction of the section 2.2 for each $\gamma > 0$ without a loss of generality we assume that $\phi_\gamma(y)$ is right-continuous on y. Then

$$\begin{aligned}
&\frac{1}{\Psi_\gamma(x)} \int_0^x (1-\frac{y}{x})^{\alpha-1}\phi_\gamma(y)\,|f(y)-f(x)|\,dy \\
=\ & \frac{1}{\Psi_\gamma(x)} \int_0^{x-\delta} (1-\frac{y}{x})^{\alpha-1}\phi_\gamma(y)\,|f(y)-f(x)|\,dy \\
+\ & \frac{1}{\Psi_\gamma(x)} \int_{x-\delta}^x (1-\frac{y}{x})^{\alpha-1}\phi_\gamma(y)\,|f(y)-f(x)|\,dy.
\end{aligned}$$

For $0< \delta < \delta_0$,

$$\begin{aligned}
&\frac{1}{\Psi_\gamma(x)} \int_{x-\delta}^x (1-\frac{y}{x})^{\alpha-1}\phi_\gamma(y)\,|f(y)-f(x)|\,dy \\
=\ & \frac{x^{1-\alpha}}{\Psi_\gamma(x)} \int_{x-\delta}^x (x-y)^{\alpha-1}\phi_\gamma(x-\delta)\,|f(y)-f(x)|\,dy \\
+\ & \frac{x^{1-\alpha}}{\Psi_\gamma(x)} \int_{x-\delta}^x (x-y)^{\alpha-1}\big(\phi_\gamma(y)-\phi_\gamma(x-\delta)\big)\,|f(y)-f(x)|\,dy := S_*.
\end{aligned}$$

Using Lemma 2.3,

$$\begin{aligned}
S_* \ =\ & \frac{x^{1-\alpha}\phi_\gamma(x-\delta)}{\Psi_\gamma(x)} \int_{x-\delta}^x (x-y)^{\alpha-1}\,|f(y)-f(x)|\,dy \\
+\ & \frac{x^{1-\alpha}}{\Psi_\gamma(x)} \int_{x-\delta}^x \left((x-y)^{\alpha-1}\int_{(x-\delta,y]} d\phi_\gamma(t)\right)|f(y)-f(x)|\,dy.
\end{aligned}$$

Since x is a Lebesgue point of f then

$$\forall\epsilon > 0,\ \exists\delta_0 > 0\ :\ 0 < \delta < \delta_0 \quad \Rightarrow \frac{\alpha}{\delta^\alpha}\int_{x-\delta}^x (x-y)^{\alpha-1}\,|f(y)-f(x)|\,dy \le \epsilon,$$

so

$$\begin{aligned}
S_* \ =\ & \frac{x^{1-\alpha}\phi_\gamma(x-\delta)}{\Psi_\gamma(x)} \int_{x-\delta}^x (x-y)^{\alpha-1}\,|f(y)-f(x)|\,dy \\
+\ & \frac{x^{1-\alpha}}{\Psi_\gamma(x)} \int_{x-\delta}^x \left(\int_t^x (x-y)^{\alpha-1}\,|f(y)-f(x)|\,dy\right)d\phi_\gamma(t) \\
\le\ & \frac{x^{1-\alpha}\phi_\gamma(x-\delta)\epsilon\delta^\alpha}{\alpha\Psi_\gamma(x)} + \frac{x^{1-\alpha}}{\alpha}\frac{\epsilon}{\Psi_\gamma(x)}\int_{(x-\delta,x]}(x-t)^\alpha d\phi_\gamma(t) \\
=\ & \frac{\epsilon x^{1-\alpha}}{\Psi_\gamma(x)}\left(\frac{\phi_\gamma(x-\delta)\delta^\alpha}{\alpha} + \frac{1}{\alpha}\int_{(x-\delta,x]}(x-t)^\alpha d\phi_\gamma(t)\right) \\
=\ & \frac{\epsilon}{\Psi_\gamma(x)}\int_{(x-\delta,x]}(1-\frac{t}{x})^{\alpha-1}\phi_\gamma(t)dt \le \epsilon.
\end{aligned}$$

For $\alpha > 1$ we have $(x-y)^{\alpha-1} \approx \delta^{\alpha-1} + (x-\delta-y)^{\alpha-1}$, so

$$\begin{aligned}
&\limsup_{\gamma\to\infty} \frac{x^{1-\alpha}}{\Psi_\gamma(x)} \int_0^{x-\delta} (x-y)^{\alpha-1}\phi_\gamma(y)\,|f(y)-f(x)|\,dy \\
\approx\;& \limsup_{\gamma\to\infty} \frac{x^{1-\alpha}\delta^{\alpha-1}}{\Psi_\gamma(x)} \int_0^{x-\delta} \phi_\gamma(y)\,|f(y)-f(x)|\,dy \\
+\;& \limsup_{\gamma\to\infty} \frac{x^{1-\alpha}}{\Psi_\gamma(x)} \int_0^{x-\delta} (x-\delta-y)^{\alpha-1}\phi_\gamma(y)\,|f(y)-f(x)|\,dy \\
:=\;& J_1 + J_2.
\end{aligned}$$

Since $\phi_\gamma(x)$ is nondecreasing function,

$$\begin{aligned}
J_1 &\leq \limsup_{\gamma\to\infty} \frac{x^{1-\alpha}\delta^{\alpha-1}\phi_\gamma(x-\delta)}{\Psi_\gamma(x)} \int_0^{x-\delta} |f(y)-f(x)|\,dy, \\
J_2 &\leq \limsup_{\gamma\to\infty} \frac{x^{1-\alpha}(x-\delta)^{\alpha-1}\phi_\gamma(x-\delta)}{\Psi_\gamma(x)} \int_0^{x-\delta} |f(y)-f(x)|\,dy,
\end{aligned}$$

applying (2.1), then

$$\limsup_{\gamma\to\infty} \frac{1}{\Psi_\gamma(x)} \int_0^{x-\delta} (1-\frac{y}{x})^{\alpha-1}\phi_\gamma(y)\,|f(y)-f(x)|\,dy = 0.$$

Since ϵ>0 was arbitrary,

$$\lim_{\gamma\to\infty} \frac{1}{\Psi_\gamma(x)} \int_0^{x} (1-\frac{y}{x})^{\alpha-1}\phi_\gamma(y)\,|f(y)-f(x)|\,dy = 0.$$

□

2.3. **Convergence of $\mathcal{G}_{\phi_\gamma}$ in $L^p_{x^\alpha}(0,a)$.** In approximation theory and also in the theory of partial differential equations, the spaces with weights are of interest. Let ω be a weight, that is, a measurable almost everywhere positive function on a measurable set $\Omega \subseteq \mathbb{R}_+$. Let $0 < p < \infty$. Then $L^p(\Omega,\omega)$ denotes the set of all measurable functions f defined almost everywhere on Ω and such that

$$\|f\|_{L^p_\omega(\Omega)} := \left(\int_\Omega \big(\omega(x)\,|f(x)|\,\big)^p dx \right)^{\frac{1}{p}} < \infty.$$

The space $L^p(\Omega,\omega)$ is complete and also separable for $0 < p < \infty$.

Theorem 2.2. *Assume that*
1. $a > 0$*,*
2. there exists $\Phi_\gamma(u)$ *such that for all* $u \in (0,1)$ *the inequality* $\frac{x\phi_\gamma(ux)}{\Psi_\gamma(x)} \leq \Phi_\gamma(u)$ *holds for all* $x \in (0,a)$*,*
3. $\limsup_{\gamma\to\infty} \|\Phi_\gamma\|_{L^1(0,1)} = C < \infty$*,*
4. for all $\beta > 0$ *and* $0 < \zeta < 1$*,* $\lim_{\gamma\to\infty} \|u^{-\beta}\Phi_\gamma(u)\|_{L^1(0,\zeta)} = 0$*.*
Then for $f \in L^p_{x^\alpha}(0,a), 1 \leq p < \infty, \alpha > 1$*,*

$$\lim_{\gamma\to\infty} \|\frac{1}{\Psi_\gamma(x)} \int_0^{x} (1-\frac{y}{x})^{\alpha-1}\phi_\gamma(y)f(y)dy - f(x)\|_{L^p_{x^\alpha}(0,a)} = 0.$$

Proof. For γ sufficiently large and for $r, x \in (0,a)$ and applying Hölder's inequality, we obtain

$$\int_0^r (1-\frac{y}{x})^{\alpha-1}\phi_\gamma(y)|f(y)|dy$$

$$\leq \Big(\int_0^r \Big((1-\frac{y}{x})^{\alpha-1}y^\alpha|f(y)|\Big)^p dy\Big)^{\frac{1}{p}}\Big(\int_0^r (y^{-\alpha}\phi_\gamma(y))^{p'}dy\Big)^{\frac{1}{p'}}$$

$$\leq \|f\|_{L^p_{y^\alpha}(0,a)}\Big(\phi_\gamma(r)\Big)^{\frac{1}{p}}\Big(\int_0^r y^{-\alpha p'}\phi_\gamma(y)dy\Big)^{\frac{1}{p'}}$$

and

$$\Big(\int_0^r y^{-\alpha p'}\phi_\gamma(y)dy\Big)^{\frac{1}{p'}}$$

$$= r^{-(\alpha+\frac{\alpha}{p'})}\Big(\Psi_\gamma(r)\Big)^{\frac{1}{p'}}\Big(\int_0^1 u^{-\alpha p'}\frac{r^{\alpha+1}\phi_\gamma(ur)}{\Psi_\gamma(r)}du\Big)^{\frac{1}{p'}}$$

$$\leq r^{-(\alpha+\frac{\alpha}{p'})}\Big(\Psi_\gamma(r)\Big)^{\frac{1}{p'}}\Big(\int_0^1 u^{-\alpha p'}\Phi_\gamma(u)du\Big)^{\frac{1}{p'}} := A_*.$$

Let $\alpha p' = \beta$, for $0<\zeta<1$,

$$\begin{aligned}\int_0^1 u^{-\alpha p'}\Phi_\gamma(u)du &= \int_0^\zeta u^{-\beta}\Phi_\gamma(u)du + \int_\zeta^1 u^{-\beta}\Phi_\gamma(u)du \\ &\leq C + \zeta^{-\beta}\int_\zeta^1 \Phi_\gamma(u)du < \infty,\end{aligned}$$

so $A_* < \infty$. Thus, $(1-\frac{y}{x})^{\alpha-1}\phi_\gamma(y)f(y) \in L^1(0,r)$ for all $r\in(0,a)$. For $0<\zeta<1$,

$$\|\frac{1}{\Psi_\gamma(x)}\int_0^x (1-\frac{y}{x})^{\alpha-1}\phi_\gamma(y)f(y)dy - f(x)\|_{L^p_{x^\alpha}}$$

$$\leq \Big(\int_0^a \Big(\frac{x^\alpha}{\Psi_\gamma(x)}\int_0^x (1-\frac{y}{x})^{\alpha-1}\phi_\gamma(y)|f(y)-f(x)|dy\Big)^p dx\Big)^{\frac{1}{p}}$$

$$\leq \Big(\int_0^a \Big(\frac{x^\alpha}{\Psi_\gamma(x)}\int_0^x \phi_\gamma(y)|f(y)-f(x)|dy\Big)^p dx\Big)^{\frac{1}{p}}$$

$$= \Big(\int_0^a \Big(\frac{x^{\alpha+1}}{\Psi_\gamma(x)}\int_0^1 \phi_\gamma(ux)|f(ux)-f(x)|du\Big)^p dx\Big)^{\frac{1}{p}}$$

$$\leq \int_0^1 \Big(\int_0^a \Big(\frac{x^{\alpha+1}}{\Psi_\gamma(x)}\phi_\gamma(ux)|f(ux)-f(x)|\Big)^p dx\Big)^{\frac{1}{p}}du$$

$$\begin{aligned} &= \int_0^\zeta \Big(\int_0^a \Big(\frac{x\phi_\gamma(ux)}{\Psi_\gamma(x)}x^\alpha|f(ux)-f(x)|\Big)^p dx\Big)^{\frac{1}{p}}du \\ &+ \int_\zeta^1 \Big(\int_0^a \Big(\frac{x\phi_\gamma(ux)}{\Psi_\gamma(x)}x^\alpha|f(ux)-f(x)|\Big)^p dx\Big)^{\frac{1}{p}}du := G_0 + G_1.\end{aligned}$$

Since for $0 < p < \infty$, $\alpha > 0$, continuous functions with compact support are dense in $L^p_{x^\gamma}$ then $\lim_{t\to 1} \|f(tx) - f(x)\|_{L^p_{x^\alpha}} = 0$, i.e. for every $\epsilon > 0$, there exists $0 < \zeta_\epsilon < 1$ such that for $\zeta_\epsilon < u < 1$,

$$\left(\int_0^a \left(x^\alpha |f(ux) - f(x)|\right)^p dx\right)^{\frac{1}{p}} < \frac{\epsilon}{C}.$$

So

$$\begin{aligned} G_1 &\leq \int_{\zeta_\epsilon}^1 \Phi_\gamma(u)\bigg(\int_0^a \left(x^\alpha |f(ux) - f(x)|\right)^p dx\bigg)^{\frac{1}{p}} du \\ &< \frac{\epsilon}{C}\int_{\zeta_\epsilon}^1 \Phi_\gamma(u)du \leq \frac{\epsilon}{C}\int_0^1 \Phi_\gamma(u)du, \end{aligned}$$

and

$$\limsup_{\gamma\to\infty} \int_{\zeta_\epsilon}^1 \left(\int_0^a \left(\frac{x\phi_\gamma(ux)}{\Psi_\gamma(x)} x^\alpha |f(ux) - f(x)|\right)^p dx\right)^{\frac{1}{p}} du \leq \epsilon.$$

Also

$$\begin{aligned} &\limsup_{\gamma\to\infty} \int_0^{\zeta_\epsilon} \left(\int_0^a \left(\frac{x\phi_\gamma(ux)}{\Psi_\gamma(x)} x^\alpha |f(ux) - f(x)|\right)^p dx\right)^{\frac{1}{p}} du \\ &\leq \limsup_{\gamma\to\infty} \int_0^{\zeta_\epsilon} \Phi_\gamma(u)\bigg(\int_0^a \left(x^\alpha |f(ux) - f(x)|\right)^p dx\bigg)^{\frac{1}{p}} du \\ &\leq \limsup_{\gamma\to\infty} \int_0^{\zeta_\epsilon} \Phi_\gamma(u)\bigg(\bigg(\int_0^a \left(x^\alpha |f(ux)|\right)^p dx\bigg)^{\frac{1}{p}} + \bigg(\int_0^a \left(x^\alpha |f(x)|\right)^p dx\bigg)^{\frac{1}{p}}\bigg) du \\ &\leq \|f\|_{L^p_{x^\alpha}} \limsup_{\gamma\to\infty} \int_0^{\zeta} \Phi_\gamma(u)\bigg(\frac{1}{u^{\alpha+\frac{1}{p}}} + 1\bigg) du = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} &\limsup_{\gamma\to\infty} \|\frac{1}{\Psi_\gamma(x)}\int_0^x (1 - \frac{x}{y})^{\alpha-1}\phi_\gamma(y)f(y)dy - f(x)\|_{L^p_{x^\alpha}} \\ &\leq \limsup_{\gamma\to\infty} \int_0^{\zeta_\epsilon} \left(\int_0^a \left(\frac{x\phi_\gamma(ux)}{\Psi_\gamma(x)} x^\alpha |f(ux) - f(x)|\right)^p dx\right)^{\frac{1}{p}} du \\ &+ \limsup_{\gamma\to\infty} \int_{\zeta_\epsilon}^1 \left(\int_0^a \left(\frac{x\phi_\gamma(ux)}{\Psi_\gamma(x)} x^\alpha |f(ux) - f(x)|\right)^p dx\right)^{\frac{1}{p}} du \leq \epsilon, \end{aligned}$$

and so $\lim_{\gamma\to\infty} \|\frac{1}{\Psi_\gamma(x)}\int_0^x (1 - \frac{x}{y})^{\alpha-1}\phi_\gamma(y)f(y)dy - f(x)\|_{L^p_{x^\alpha}(0,a)} = 0$. ☐

In the following Theorem 2.3 we will study the infinity behavior of $\mathcal{G}_{\phi_\gamma}$ for uniformly continuous functions.

Theorem 2.3. *For all $x \in (0, a), 0 < a < \infty$ and $u \in (0, 1)$, assume that*
1. there exists $\Phi_\gamma(u)$, so that $\frac{x\phi_\gamma(ux)}{\Psi_\gamma(x)} \leq \Phi_\gamma(u)$,

2. $\lim_{\gamma\to\infty}\Phi_\gamma(u)=0.$
Then for any uniformly continuous function f on $(0,a)$,

$$\lim_{\gamma\to\infty}\sup_{0<x<a}\left|\frac{1}{\Psi_\gamma(x)}\int_0^x(1-\frac{y}{x})^{\alpha-1}\phi_\gamma(y)f(y)dy-f(x)\right|=0.$$

Proof. Let $\epsilon>0$, Since $\lim_{\gamma\to\infty}\Phi_\gamma(u)=0$, so there exists γ_0 such that for $\gamma\ge\gamma_0$,

$$\Phi_\gamma\left(\frac{a-\delta}{a}\right)<\frac{\epsilon}{2\sup_{0<t<a}|f(t)|}.$$

The function f is uniformly continuous on $(0,a)$, it follows that there exists $0<\delta<a$, such that $|f(u)-f(v)|<\epsilon$ for $u,v\in(0,a)$, and $|u-v|<\delta$. For $0<x\le\delta$,

$$\begin{aligned}
&\left|\frac{1}{\Psi_\gamma(x)}\int_0^x(1-\frac{y}{x})^{\alpha-1}\phi_\gamma(y)f(y)dy-f(x)\right|\\
\le\ &\frac{1}{\Psi_\gamma(x)}\int_0^x(1-\frac{y}{x})^{\alpha-1}\phi_\gamma(y)\,|f(y)-f(x)|\,dy\\
\le\ &\frac{\epsilon}{\Psi_\gamma(x)}\int_0^x(1-\frac{y}{x})^{\alpha-1}\phi_\gamma(y)dy=\epsilon.
\end{aligned}$$

For $\delta<x<a$,

$$\begin{aligned}
&\frac{1}{\Psi_\gamma(x)}\int_0^x(1-\frac{y}{x})^{\alpha-1}\phi_\gamma(y)\,|f(y)-f(x)|\,dy\\
=\ &\frac{1}{\Psi_\gamma(x)}\int_0^{x-\delta}(1-\frac{y}{x})^{\alpha-1}\phi_\gamma(y)\,|f(y)-f(x)|\,dy\\
+\ &\frac{1}{\Psi_\gamma(x)}\int_{x-\delta}^x(1-\frac{y}{x})^{\alpha-1}\phi_\gamma(y)\,|f(y)-f(x)|\,dy\\
\le\ &\frac{1}{\Psi_\gamma(x)}\big(2\sup_{0<t<a}|f(t)|\big)(x-\delta)\phi_\gamma(x-\delta)\\
+\ &\frac{\epsilon}{\Psi_\gamma(x)}\int_{x-\delta}^x(1-\frac{y}{x})^{\alpha-1}\phi_\gamma(y)dy\\
\le\ &\frac{x\phi_\gamma(x-\delta)}{\Psi_\gamma(x)}\big(2\sup_{0<t<a}|f(t)|\big)+\epsilon\\
\le\ &\big(2\sup_{0<t<a}|f(t)|\big)\Phi_\gamma\left(\frac{a-\delta}{a}\right)+\epsilon.
\end{aligned}$$

Hence

$$\limsup_{\gamma\to\infty}\sup_{0<x<a}\left|\frac{1}{\Psi_\gamma(x)}\int_0^x(1-\frac{y}{x})^{\alpha-1}\phi_\gamma(y)f(y)dy-f(x)\right|\le 2\epsilon,$$

and so,

$$\lim_{\gamma\to\infty}\sup_{0<x<a}\left|\frac{1}{\Psi_\gamma(x)}\int_0^x(1-\frac{y}{x})^{\alpha-1}\phi_\gamma(y)f(y)dy-f(x)\right|=0.$$

□

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