

## EQUIVALENT CONDITIONS OF A HARDY-TYPE INTEGRAL INEQUALITY RELATED TO THE EXTENDED RIEMANN ZETA FUNCTION

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ABSTRACT. By the use of techniques of real analysis and weight functions, we obtain two lemmas and build a few equivalent conditions of a Hardy-type integral inequality with a non-homogeneous kernel, related to a parameter where the constant factor is expressed in terms of the extended Riemann zeta function. Meanwhile, a few equivalent conditions for two kinds of Hardy-type integral inequalities with the homogeneous kernel are deduced. We also consider the operator expressions.

### 1. INTRODUCTION

If

$$0 < \int_0^\infty f^2(x)dx < \infty \text{ and } 0 < \int_0^\infty g^2(y)dy < \infty ,$$

then we have the following Hilbert integral inequality (cf. [14]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \pi \left( \int_0^\infty f^2(x)dx \int_0^\infty g^2(y)dy \right)^{\frac{1}{2}} , \quad (1.1)$$

where, the constant factor  $\pi$  is the best possible.

In 1925, by introducing one pair of conjugate exponents  $(p, q)$ , Hardy [2] gave an

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extension of (1.1) as follows:

For  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $f(x), g(y) \geq 0$ ,

$$0 < \int_0^\infty f^p(x)dx < \infty \quad \text{and} \quad 0 < \int_0^\infty g^q(y)dy < \infty,$$

we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} \left( \int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left( \int_0^\infty g^q(y)dy \right)^{\frac{1}{q}}, \quad (1.2)$$

where, the constant factor  $\frac{\pi}{\sin(\pi/p)}$  is the best possible. Inequalities (1.1) and (1.2) are important in Mathematical Analysis and its applications (cf. [3, 11, 19, 12, 13]).

In 1934, Hardy et al. presented the following extension of (1.2):  
If  $k_1(x, y)$  is a non-negative homogeneous function of degree  $-1$ ,

$$k_p = \int_0^\infty k_1(u, 1)u^{-\frac{1}{p}} du \in \mathbf{R}_+ = (0, \infty),$$

then we have the following Hardy–Hilbert-type integral inequality

$$\int_0^\infty \int_0^\infty k_1(x, y)f(x)g(y)dx dy < k_p \left( \int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left( \int_0^\infty g^q(y)dy \right)^{\frac{1}{q}}, \quad (1.3)$$

where, the constant factor  $k_p$  is the best possible (cf. [3], Theorem 319). Additionally, the following Hilbert-type integral inequality with the non-homogeneous kernel is proved:

If  $h(u) > 0$ ,

$$\phi(\sigma) = \int_0^\infty h(u)u^{\sigma-1}du \in \mathbf{R}_+,$$

then

$$\begin{aligned} & \int_0^\infty \int_0^\infty h(xy)f(x)g(y)dx dy \\ & < \phi\left(\frac{1}{p}\right) \left( \int_0^\infty x^{p-2}f^p(x)dx \right)^{\frac{1}{p}} \left( \int_0^\infty g^q(y)dy \right)^{\frac{1}{q}}, \end{aligned} \quad (1.4)$$

where, the constant factor  $\phi\left(\frac{1}{p}\right)$  is the best possible (cf. [3], Theorem 350).

In 1998, by introducing an independent parameter  $\lambda > 0$ , Yang provided an extension of (1.1) with the kernel  $\frac{1}{(x+y)^\lambda}$  (cf. [20]). In 2004, Yang [21] introduced another pair conjugate exponents  $(r, s)$ , and gave the following extension of (1.2):  
If  $\lambda > 0$ ,  $r > 1$ ,  $\frac{1}{r} + \frac{1}{s} = 1$ ,  $f(x), g(y) \geq 0$ ,

$$0 < \int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)dx < \infty \quad \text{and} \quad 0 < \int_0^\infty y^{q(1-\frac{\lambda}{s})-1}g^q(y)dy < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dx dy < \frac{\pi}{\lambda \sin(\pi/r)} \left[ \int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{q(1-\frac{\lambda}{s})-1} g^q(y) dy \right]^{\frac{1}{q}}, \tag{1.5}$$

where, the constant factor  $\frac{\pi}{\lambda \sin(\pi/r)}$  is the best possible.

For  $\lambda = 1, r = q, s = p$ , (1.5) reduces to (1.2); For  $\lambda = 1, r = p, s = q$ , (1.5) reduces to the dual form of (1.2) as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x + y} dx dy < \frac{\pi}{\sin(\pi/p)} \left( \int_0^\infty x^{p-2} f^p(x) dx \right)^{\frac{1}{p}} \left( \int_0^\infty y^{q-2} g^q(y) dy \right)^{\frac{1}{q}}. \tag{1.6}$$

For  $p = q = 2$ , both (1.2) and (1.6) reduce to (1.1).

In [22], inequalities (1.2) and (1.5) are also extended with the kernel  $\frac{1}{(x+y)^\lambda}$ . Moreover, Krnić et al. [6]-[16] proved some extensions and particular cases of (1.2), (1.3) and (1.4) with parameters.

In 2009, Yang (cf. [17, 19]) gave an extension of (1.3), (1.5). Namely he showed that:

If  $\lambda_1 + \lambda_2 = \lambda \in \mathbf{R} = (-\infty, \infty)$ ,  $k_\lambda(x, y)$  is a non-negative homogeneous function of degree  $-\lambda$ , satisfying

$$k_\lambda(ux, uy) = u^{-\lambda} k_\lambda(x, y) \quad (u, x, y > 0),$$

with

$$k(\lambda_1) = \int_0^\infty k_\lambda(u, 1) u^{\lambda_1-1} du \in \mathbf{R}_+ = (0, \infty),$$

then we have

$$\int_0^\infty \int_0^\infty k_\lambda(x, y) f(x)g(y) dx dy < k(\lambda_1) \left( \int_0^\infty x^{p(1-\lambda_1)-1} f^p(x) dx \right)^{\frac{1}{p}} \left( \int_0^\infty y^{q(1-\lambda_2)-1} g^q(y) dy \right)^{\frac{1}{q}}, \tag{1.7}$$

where, the constant factor  $k(\lambda_1)$  is the best possible.

For  $\lambda = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$ , (1.7) reduces to (1.3) and for  $k_\lambda(x, y) = \frac{1}{x^\lambda + y^\lambda}$  ( $\lambda > 0$ ), (1.7) reduces to (1.5). Furthermore, the following extension of (1.4) was proved:

$$\int_0^\infty \int_0^\infty h(xy) f(x)g(y) dx dy < \phi(\sigma) \left( \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right)^{\frac{1}{p}} \left( \int_0^\infty y^{q(1-\sigma)-1} g^q(y) dy \right)^{\frac{1}{q}}, \tag{1.8}$$

where, the constant factor  $\phi(\sigma)$  is the best possible (cf. [18]).

For  $\sigma = \frac{1}{p}$ , (1.8) reduces to (1.4).

Some inequalities equivalent to (1.7) and (1.8) are considered in [19]. In 2013, Yang [18] also studied the equivalency between (1.7) and (1.8) by the addition of

a particular condition. In 2017, Hong [5] studied an equivalent condition between (1.7) with a few parameters.

*Remark 1.1.* (cf. [18]) If  $h(xy) = 0$ , for  $xy > 1$ , then

$$\phi(\sigma) = \int_0^1 h(u)u^{\sigma-1}du = \phi_1(\sigma) \in \mathbf{R}_+,$$

and (1.8) reduces to the following Hardy-type integral inequality with the non-homogeneous kernel:

$$\begin{aligned} & \int_0^\infty g(y) \left( \int_0^{\frac{1}{y}} h(xy)f(x)dx \right) dy \\ & < \phi_1(\sigma) \left( \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx \right)^{\frac{1}{p}} \left( \int_0^\infty y^{q(1-\sigma)-1} g^q(y)dy \right)^{\frac{1}{q}}; \quad (1.9) \end{aligned}$$

if  $h(xy) = 0$ , for  $xy < 1$ , then

$$\phi(\sigma) = \int_1^\infty h(u)u^{\sigma-1}du = \phi_2(\sigma) \in \mathbf{R}_+,$$

and (1.8) reduces to another kind of Hardy-type integral inequality with the non-homogeneous kernel, namely:

$$\begin{aligned} & \int_0^\infty g(y) \left( \int_{\frac{1}{y}}^\infty h(xy)f(x)dx \right) dy \\ & < \phi_2(\sigma) \left( \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx \right)^{\frac{1}{p}} \left( \int_0^\infty y^{q(1-\sigma)-1} g^q(y)dy \right)^{\frac{1}{q}}. \quad (1.10) \end{aligned}$$

In this paper, by the use of techniques of real analysis and weight functions, we obtain two lemmas and build a few equivalent conditions of a Hardy-type integral inequalities with the non-homogeneous kernel

$$\frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} \quad (\beta, \lambda > 0),$$

related to a parameter where the constant factor is expressed in terms of the extended Riemann zeta function. Meanwhile, a few equivalent conditions of two kinds of Hardy-type integral inequalities with the homogeneous kernel are deduced. We also consider the operator expressions.

## 2. TWO LEMMAS

For  $\beta, \lambda > 0$ , set

$$h(u) := \frac{|\ln u|^\beta}{|u^\lambda - 1|} \quad (u > 0).$$

For  $\sigma > 0$ , by the Lebesgue term by term integration theorem (cf. [7]), we obtain

$$\begin{aligned} k_1(\sigma) & : = \int_0^1 \frac{|\ln u|^\beta}{|u^\lambda - 1|} u^{\sigma-1} du = \int_0^1 \frac{(-\ln u)^\beta}{1 - u^\lambda} u^{\sigma-1} du \\ & = \int_0^1 (-\ln u)^\beta \sum_{k=0}^\infty u^{k\lambda + \sigma - 1} du = \sum_{k=0}^\infty \int_0^1 (-\ln u)^\beta u^{k\lambda + \sigma - 1} du. \end{aligned}$$

Setting  $v = (k\lambda + \sigma)(-\ln u)$  in the above integral, we derive that

$$\begin{aligned} k_1(\sigma) & = \sum_{k=0}^\infty \frac{1}{(k\lambda + \sigma)^{\beta+1}} \int_0^\infty v^\beta e^{-v} dv \\ & = \frac{\Gamma(\beta + 1)}{\lambda^{\beta+1}} \zeta(\beta + 1, \frac{\sigma}{\lambda}) \in \mathbf{R}_+, \end{aligned} \tag{2.1}$$

where,

$$\Gamma(\eta) := \int_0^\infty v^{\eta-1} e^{-v} dv \ (\eta > 0)$$

is the gamma function and

$$\zeta(s, a) := \sum_{k=0}^\infty \frac{1}{(k + a)^s} \ (Re(s) > 1, a > 0)$$

is the extended Riemann zeta function ( $\zeta(s, 1) = \zeta(s)$  is the Riemann zeta function) (cf. [15]). In particular, for  $\sigma = \lambda$ , we have

$$k_1(\sigma) = \frac{\Gamma(\beta + 1)}{\lambda^{\beta+1}} \zeta(\beta + 1).$$

For  $0 < \sigma < \lambda$ ,  $\mu = \lambda - \sigma > 0$ , setting  $v = \frac{1}{u}$ , by (2.1), we deduce that

$$\begin{aligned} k_2(\sigma) & : = \int_1^\infty \frac{|\ln u|^\beta}{|u^\lambda - 1|} u^{\sigma-1} du \\ & = \int_1^\infty \frac{(\ln u)^\beta}{u^\lambda - 1} u^{\sigma-1} du = \int_0^1 \frac{(-\ln v)^\beta}{1 - v^\lambda} v^{\mu-1} dv \\ & = \frac{\Gamma(\beta + 1)}{\lambda^{\beta+1}} \zeta(\beta + 1, \frac{\mu}{\lambda}) = k_1(\mu) \in \mathbf{R}_+. \end{aligned} \tag{2.2}$$

In the sequel, we will always assume that

$$p > 1, \frac{1}{p} + \frac{1}{q} = 1, \text{ and } \sigma_1, \mu_1 \in \mathbf{R}.$$

**Lemma 2.1.** *If  $\beta, \sigma, \lambda > 0$ , there exists a constant  $M_1$  such that for any non-negative measurable functions  $f(x), g(y)$  with  $x, y \in (0, \infty)$ , the inequality*

$$\begin{aligned} & \int_0^\infty g(y) \left[ \int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x) dx \right] dy \\ & \leq M_1 \left[ \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}} \end{aligned} \tag{2.3}$$

holds. Then  $\sigma_1 = \sigma$ , and  $M_1 \geq k_1(\sigma)$ .

*Proof.* If  $\sigma_1 > \sigma$ , then for  $n \geq \frac{1}{\sigma_1 - \sigma}$  ( $n \in \mathbf{N}$ ), we set:

$$f_n(x) := \begin{cases} x^{\sigma + \frac{1}{pn} - 1}, & 0 < x \leq 1 \\ 0, & x > 1 \end{cases}, \quad g_n(y) := \begin{cases} 0, & 0 < y < 1 \\ y^{\sigma_1 - \frac{1}{qn} - 1}, & y \geq 1 \end{cases},$$

and obtain

$$\begin{aligned} J_1 & : = \left[ \int_0^\infty x^{p(1-\sigma)-1} f_n^p(x) dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{q(1-\sigma_1)-1} g_n^q(y) dy \right]^{\frac{1}{q}} \\ & = \left( \int_0^1 x^{\frac{1}{n}-1} dx \right)^{\frac{1}{p}} \left( \int_1^\infty y^{-\frac{1}{n}-1} dy \right)^{\frac{1}{q}} = n. \end{aligned}$$

For  $u = xy$ , we derive that

$$\begin{aligned} I_1 & : = \int_0^\infty g_n(y) \left( \int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f_n(x) dx \right) dy \\ & = \int_1^\infty \left( \int_0^{\frac{1}{y}} \frac{(-\ln xy)^\beta}{1 - (xy)^\lambda} x^{\sigma + \frac{1}{pn} - 1} dx \right) y^{\sigma_1 - \frac{1}{qn} - 1} dy \\ & = \int_1^\infty y^{(\sigma_1 - \sigma) - \frac{1}{n} - 1} dy \int_0^1 \frac{(-\ln u)^\beta}{1 - u^\lambda} u^{\sigma + \frac{1}{pn} - 1} du, \end{aligned}$$

and thus by (2.3), we have

$$\begin{aligned} & \int_1^\infty y^{(\sigma_1 - \sigma) - \frac{1}{n} - 1} dy \int_0^1 \frac{(-\ln u)^\beta}{1 - u^\lambda} u^{\sigma + \frac{1}{pn} - 1} du \\ & = I_1 \leq M_1 J_1 = M_1 n < \infty. \end{aligned} \tag{2.4}$$

Since

$$(\sigma_1 - \sigma) - \frac{1}{n} \geq 0,$$

it follows that

$$\int_1^\infty y^{(\sigma_1 - \sigma) - \frac{1}{n} - 1} dy = \infty.$$

By (2.4), in view of

$$\int_0^1 \frac{(-\ln u)^\beta}{1 - u^\lambda} u^{\sigma + \frac{1}{pn} - 1} du > 0,$$

we get that  $\infty < \infty$ , which is a contradiction.

If  $\sigma_1 < \sigma$ , then for  $n \geq \frac{1}{\sigma - \sigma_1}$  ( $n \in \mathbf{N}$ ), we set:

$$\tilde{f}_n(x) := \begin{cases} 0, & 0 < x < 1 \\ x^{\sigma - \frac{1}{pn} - 1}, & x \geq 1 \end{cases}, \quad \tilde{g}_n(y) := \begin{cases} y^{\sigma_1 + \frac{1}{qn} - 1}, & 0 < y \leq 1 \\ 0, & y > 1 \end{cases},$$

and find

$$\begin{aligned} \tilde{J}_1 & : = \left[ \int_0^\infty x^{p(1-\sigma)-1} \tilde{f}_n^p(x) dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{q(1-\sigma_1)-1} \tilde{g}_n^q(y) dy \right]^{\frac{1}{q}} \\ & = \left( \int_1^\infty x^{-\frac{1}{n}-1} dx \right)^{\frac{1}{p}} \left( \int_0^1 y^{\frac{1}{n}-1} dy \right)^{\frac{1}{q}} = n. \end{aligned}$$

For  $u = xy$ , we obtain

$$\begin{aligned} \tilde{I}_1 & : = \int_0^\infty \tilde{f}_n(x) \left[ \int_0^{\frac{1}{x}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} \tilde{g}_n(y) dy \right] dx \\ & = \int_1^\infty \left[ \int_0^{\frac{1}{x}} \frac{(-\ln xy)^\beta}{1 - (xy)^\lambda} y^{\sigma_1 + \frac{1}{qn} - 1} dy \right] x^{\sigma - \frac{1}{pn} - 1} dx \\ & = \int_1^\infty x^{(\sigma - \sigma_1) - \frac{1}{n} - 1} dx \int_0^1 \frac{(-\ln u)^\beta}{1 - u^\lambda} u^{\sigma_1 + \frac{1}{qn} - 1} du, \end{aligned}$$

and thus by Fubini's theorem (cf. [7]) and (2.3), we have

$$\begin{aligned} & \int_1^\infty x^{(\sigma - \sigma_1) - \frac{1}{n} - 1} dx \int_0^1 \frac{(-\ln u)^\beta}{1 - u^\lambda} u^{\sigma_1 + \frac{1}{qn} - 1} du \\ & = \tilde{I}_1 = \int_0^\infty \tilde{g}_n(y) \left[ \int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta \tilde{f}_n(x)}{|(xy)^\lambda - 1|} dx \right] dy \leq M_1 \tilde{J}_1 = M_1 n. \end{aligned} \tag{2.5}$$

Since

$$(\sigma - \sigma_1) - \frac{1}{n} \geq 0,$$

it follows that

$$\int_1^\infty x^{(\sigma - \sigma_1) - \frac{1}{n} - 1} dx = \infty.$$

By (2.5), in view of

$$\int_0^1 \frac{(-\ln u)^\beta}{1 - u^\lambda} u^{\sigma_1 + \frac{1}{qn} - 1} du > 0,$$

we deduce that  $\infty < \infty$ , which is a contradiction.

Hence, we conclude that  $\sigma_1 = \sigma$ .

For  $\sigma_1 = \sigma$ , inequality (2.5) is reduced to the following

$$M_1 \geq \int_0^1 \frac{(-\ln u)^\beta}{1 - u^\lambda} u^{\sigma + \frac{1}{qn} - 1} du. \tag{2.6}$$

Since

$$\left\{ \frac{(-\ln u)^\beta}{1 - u^\lambda} u^{\sigma + \frac{1}{qn} - 1} \right\}_{n=1}^\infty$$

is nonnegative and increasing in  $(0, 1]$ , by Levi's theorem (cf. [7]), we get

$$\begin{aligned} M_1 & \geq \lim_{n \rightarrow \infty} \int_0^1 \frac{(-\ln u)^\beta}{1 - u^\lambda} u^{\sigma + \frac{1}{qn} - 1} du \\ & = \int_0^1 \lim_{n \rightarrow \infty} \frac{(-\ln u)^\beta}{1 - u^\lambda} u^{\sigma + \frac{1}{qn} - 1} du = k_1(\sigma). \end{aligned}$$

The lemma is proved. □

**Lemma 2.2.** *If  $\beta > 0, 0 < \sigma < \lambda$ , there exists a constant  $M_2$ , such that for any non-negative measurable functions  $f(x), g(y)$  with  $x, y \in (0, \infty)$ , the inequality*

$$\begin{aligned} & \int_0^\infty g(y) \left[ \int_{\frac{1}{y}}^\infty \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x) dx \right] dy \\ & \leq M_2 \left[ \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}} \end{aligned} \quad (2.7)$$

holds. Then  $\sigma_1 = \sigma$ , and  $M_2 \geq k_2(\sigma)$ .

*Proof.* If  $\sigma_1 < \sigma$ , then for  $n \geq \frac{1}{\sigma - \sigma_1}$  ( $n \in \mathbf{N}$ ), we consider two functions  $\tilde{f}_n(x)$  and  $\tilde{g}_n(y)$  as in Lemma 1, and obtain that

$$\tilde{J}_1 = \left[ \int_0^\infty x^{p(1-\sigma)-1} \tilde{f}_n^p(x) dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{q(1-\sigma_1)-1} \tilde{g}_n^q(y) dy \right]^{\frac{1}{q}} = n.$$

Setting  $u = xy$ , we obtain

$$\begin{aligned} \tilde{I}_2 & : = \int_0^\infty \tilde{g}_n(y) \left[ \int_{\frac{1}{y}}^\infty \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} \tilde{f}_n(x) dx \right] dy \\ & = \int_0^1 \left[ \int_{\frac{1}{y}}^\infty \frac{(\ln xy)^\beta}{(xy)^\lambda - 1} x^{\sigma - \frac{1}{pn} - 1} dx \right] y^{\sigma_1 + \frac{1}{qn} - 1} dy \\ & = \int_0^1 y^{(\sigma_1 - \sigma) + \frac{1}{n} - 1} dy \int_1^\infty \frac{(\ln u)^\beta}{u^\lambda - 1} u^{\sigma - \frac{1}{pn} - 1} du, \end{aligned}$$

and then by (2.7), we obtain

$$\begin{aligned} & \int_0^1 y^{(\sigma_1 - \sigma) + \frac{1}{n} - 1} dy \int_1^\infty \frac{(\ln u)^\beta}{u^\lambda - 1} u^{\sigma - \frac{1}{pn} - 1} du \\ & = \tilde{I}_2 \leq M_2 \tilde{J}_1 = M_2 n < \infty. \end{aligned} \quad (2.8)$$

Since

$$(\sigma_1 - \sigma) + \frac{1}{n} \leq 0,$$

it follows that

$$\int_0^1 y^{(\sigma_1 - \sigma) + \frac{1}{n} - 1} dy = \infty.$$

By (2.8), in view of

$$\int_1^\infty \frac{(\ln u)^\beta}{u^\lambda - 1} u^{\sigma - \frac{1}{pn} - 1} du > 0,$$

we deduce that  $\infty < \infty$ , which is a contradiction.

If  $\sigma_1 > \sigma$ , then for  $n \geq \frac{1}{\sigma_1 - \sigma}$  ( $n \in \mathbf{N}$ ), we consider two functions  $f_n(x)$  and  $g_n(y)$  as in Lemma 1, and find

$$J_1 = \left[ \int_0^\infty x^{p(1-\sigma)-1} f_n^p(x) dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{q(1-\sigma_1)-1} g_n^q(y) dy \right]^{\frac{1}{q}} = n.$$



Setting  $u = xy$ , we obtain

$$\begin{aligned} I_2 & : = \int_0^\infty f_n(x) \left[ \int_{\frac{1}{x}}^\infty \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} g_n(y) dy \right] dx \\ & = \int_0^1 \left[ \int_{\frac{1}{x}}^\infty \frac{(\ln xy)^\beta}{(xy)^\lambda - 1} y^{\sigma_1 - \frac{1}{qn} - 1} dy \right] x^{\sigma + \frac{1}{pn} - 1} dx \\ & = \int_0^1 x^{(\sigma - \sigma_1) + \frac{1}{n} - 1} dx \int_1^\infty \frac{(\ln u)^\beta}{u^\lambda - 1} u^{\sigma_1 - \frac{1}{qn} - 1} du, \end{aligned}$$

and then by Fubini's theorem (cf. [7]) and (2.7), we obtain

$$\begin{aligned} & \int_0^1 x^{(\sigma - \sigma_1) + \frac{1}{n} - 1} dx \int_1^\infty \frac{(\ln u)^\beta}{u^\lambda - 1} u^{\sigma_1 - \frac{1}{qn} - 1} du \\ & = I_2 = \int_0^\infty g_n(y) \left[ \int_{\frac{1}{y}}^\infty \frac{|\ln xy|^\beta f_n(x)}{|(xy)^\lambda - 1|} dx \right] dy \leq M_2 J_1 = M_2 n. \end{aligned} \tag{2.9}$$

Since

$$(\sigma - \sigma_1) + \frac{1}{n} \leq 0,$$

it follows that

$$\int_0^1 x^{(\sigma - \sigma_1) + \frac{1}{n} - 1} dx = \infty.$$

By (2.9), in view of

$$\int_1^\infty \frac{(\ln u)^\beta}{u^\lambda - 1} u^{\sigma_1 - \frac{1}{qn} - 1} du > 0,$$

we deduce that  $\infty < \infty$ , which is a contradiction.

Hence, we conclude that  $\sigma_1 = \sigma$ .

For  $\sigma_1 = \sigma$ , inequality (2.9) reduces to the following

$$M_2 \geq \int_1^\infty \frac{(\ln u)^\beta}{u^\lambda - 1} u^{\sigma - \frac{1}{qn} - 1} du. \tag{2.10}$$

Since

$$\left\{ \frac{(\ln u)^\beta}{u^\lambda - 1} u^{\sigma - \frac{1}{qn} - 1} \right\}_{n=1}^\infty$$

is nonnegative and increasing in  $[1, \infty)$ , again by the application of Levi's theorem (cf. [7]), we obtain that

$$\begin{aligned} M_2 & \geq \lim_{n \rightarrow \infty} \int_1^\infty \frac{(\ln u)^\beta}{u^\lambda - 1} u^{\sigma - \frac{1}{qn} - 1} du \\ & = \int_1^\infty \lim_{n \rightarrow \infty} \frac{(\ln u)^\beta}{u^\lambda - 1} u^{\sigma - \frac{1}{qn} - 1} du = k_2(\sigma). \end{aligned}$$

This completes the proof of the lemma. □

## 3. MAIN RESULTS AND COROLLARIES

**Theorem 3.1.** *If  $\beta, \sigma, \lambda > 0$ , then the following conditions are equivalent:*

(i) *There exists a constant  $M_1$ , such that for any  $f(x) \geq 0$  ( $x \in (0, \infty)$ ), satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

*we have the following Hardy-type integral inequality of the first kind with the non-homogeneous kernel:*

$$\begin{aligned} J & : = \left\{ \int_0^\infty y^{p\sigma_1-1} \left[ \int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} \\ & < M_1 \left[ \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}; \end{aligned} \quad (3.1)$$

(ii) *There exists a constant  $M_1$ , such that for any  $f(x), g(y) \geq 0$  ( $x, y \in (0, \infty)$ ), satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

*and*

$$0 < \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy < \infty,$$

*we have the following inequality:*

$$\begin{aligned} I & : = \int_0^\infty g(y) \left[ \int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x) dx \right] dy \\ & < M_1 \left[ \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}}; \end{aligned} \quad (3.2)$$

(iii)  $\sigma_1 = \sigma$ .

*If Condition (iii) holds true, then  $M_1 \geq k_1(\sigma)$  and the constant factor*

$$M_1 = k_1(\sigma) = \frac{\Gamma(\beta + 1)}{\lambda^{\beta+1}} \zeta\left(\beta + 1, \frac{\sigma}{\lambda}\right)$$

*in (3.1) and (3.2) is the best possible.*

*Proof.*

“(i)  $\Rightarrow$  (ii)”. By Hölder’s inequality (cf. [8]), we have

$$\begin{aligned} I & = \int_0^\infty \left[ y^{\sigma_1 - \frac{1}{p}} \int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x) dx \right] \left( y^{\frac{1}{p} - \sigma_1} g(y) \right) dy \\ & \leq J \left[ \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}}. \end{aligned} \quad (3.3)$$

Thus by (3.1), we deduce (3.2).

“(ii)  $\Rightarrow$  (iii)”. By Lemma 1, we have  $\sigma_1 = \sigma$ .

“(iii) ⇒ (i)”. Setting  $u = xy$ , we obtain the following weight function:

$$\begin{aligned} \omega_1(\sigma, y) & : = y^\sigma \int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} x^{\sigma-1} dx \\ & = \int_0^1 \frac{(-\ln u)^\beta}{1 - u^\lambda} u^{\sigma-1} du = k_1(\sigma) \quad (y > 0). \end{aligned} \tag{3.4}$$

By the weighted Hölder inequality and (3.4), for  $y \in (0, \infty)$ , we have

$$\begin{aligned} & \left[ \int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x) dx \right]^p \\ & = \left\{ \int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} \left[ \frac{y^{(\sigma-1)/p}}{x^{(\sigma-1)/q}} f(x) \right] \left[ \frac{x^{(\sigma-1)/q}}{y^{(\sigma-1)/p}} \right] dx \right\}^p \\ & \leq \int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} \frac{y^{\sigma-1} f^p(x)}{x^{(\sigma-1)p/q}} dx \left[ \int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} \frac{x^{\sigma-1} dx}{y^{(\sigma-1)q/p}} \right]^{p-1} \\ & = [\omega_1(\sigma, y) y^{q(1-\sigma)-1}]^{p-1} \int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx \\ & = (k_1(\sigma))^{p-1} y^{-p\sigma+1} \int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx. \end{aligned} \tag{3.5}$$

If (3.5) obtains the form of equality for a  $y \in (0, \infty)$ , then (cf. [8]) there exist constants  $A$  and  $B$ , such that they are not all zero and

$$A \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) = B \frac{x^{\sigma-1}}{y^{(\sigma-1)q/p}} \quad \text{a.e. in } \mathbf{R}_+.$$

Let  $A \neq 0$  (otherwise  $B = A = 0$ ). It follows that

$$x^{p(1-\sigma)-1} f^p(x) = y^{q(1-\sigma)} \frac{B}{Ax} \quad \text{a.e. in } \mathbf{R}_+,$$

which contradicts the fact that

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty.$$

Hence, (3.5) assumes the form of strict inequality. Then for  $\sigma_1 = \sigma$ , by (3.5) and Fubini's theorem (cf. [7]), we obtain

$$\begin{aligned} J &< (k_1(\sigma))^{\frac{1}{q}} \left\{ \int_0^\infty \left[ \int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx \right] dy \right\}^{\frac{1}{p}} \\ &= (k_1(\sigma))^{\frac{1}{q}} \left\{ \int_0^\infty \left[ \int_0^{\frac{1}{x}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} \frac{y^{\sigma-1}}{x^{(\sigma-1)(p-1)}} dy \right] f^p(x) dx \right\}^{\frac{1}{p}} \\ &= (k_1(\sigma))^{\frac{1}{q}} \left[ \int_0^\infty \omega_1(\sigma, x) x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \\ &= k_1(\sigma) \left[ \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}. \end{aligned}$$

Setting  $M_1 \geq k_1(\sigma)$ , (3.1) follows.

Therefore, Condition (i), Condition (ii) and Condition (iii) are equivalent.

When Condition (iii) is satisfied, if there exists a constant factor  $M_1 \leq k_1(\sigma)$ , such that (3.2) is valid, then by Lemma 1, we have  $M_1 \geq k_1(\sigma)$ . Hence, the constant factor  $M_1 = k_1(\sigma)$  in (3.2) is the best possible. The constant factor  $M_1 = k_1(\sigma)$  in (3.1) is still the best possible. Otherwise, by (3.3) (for  $\sigma_1 = \sigma$ ), we would conclude that the constant factor  $M_1 = k_1(\sigma)$  in (3.2) is not the best possible.  $\square$

Setting

$$y = \frac{1}{Y}, \quad G(Y) = Y^{\lambda-2} g\left(\frac{1}{Y}\right), \quad \mu_1 = \lambda - \sigma_1$$

in Theorem 1, then replacing  $Y$  ( $G(Y)$ ) by  $y$  ( $g(y)$ ), since  $\mu = \lambda - \sigma$ , we have

**Corollary 3.2.** *If  $\beta, \sigma, \lambda > 0$ , then the following conditions are equivalent:*

(i) *There exists a constant  $M_1$ , such that for any  $f(x) \geq 0$  ( $x \in (0, \infty)$ ), for which*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

*the following Hardy-type inequality of the first kind with the homogeneous kernel is satisfied:*

$$\begin{aligned} &\left\{ \int_0^\infty y^{p\mu_1-1} \left[ \int_0^y \frac{|\ln(x/y)|^\beta}{|x^\lambda - y^\lambda|} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} \\ &< M_1 \left[ \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}; \end{aligned} \tag{3.6}$$

(ii) *There exists a constant  $M_1$ , such that for any  $f(x), g(y) \geq 0$  ( $x, y \in (0, \infty)$ ), satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty \quad \text{and} \quad 0 < \int_0^\infty y^{q(1-\mu_1)-1} g^q(y) dy < \infty,$$

we have the following inequality:

$$\begin{aligned} & \int_0^\infty g(y) \left[ \int_0^y \frac{|\ln(x/y)|^\beta}{|x^\lambda - y^\lambda|} f(x) dx \right] dy \\ & < M_1 \left[ \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{q(1-\mu_1)-1} g^q(y) dy \right]^{\frac{1}{q}}; \end{aligned} \tag{3.7}$$

(iii)  $\mu_1 = \mu$ .

If Condition (iii) holds true, then we have  $M_1 \geq k_1(\sigma)$ , and the constant  $M_1 = k_1(\sigma)$  in (3.6) and (3.7) is the best possible.

Remark 3.3. On the other hand, setting

$$y = \frac{1}{Y}, \quad G(Y) = Y^{\lambda-2} g\left(\frac{1}{Y}\right), \quad \mu_1 = \lambda - \sigma_1$$

in Corollary 1, and replacing  $Y$  ( $G(Y)$ ) by  $y$  ( $g(y)$ ), we have Theorem 1. Hence, we conclude that Theorem 1 and Corollary 1 are equivalent.

Similarly, for  $0 < \sigma = \lambda - \mu < \lambda$ , we obtain the following weight function:

$$\begin{aligned} \omega_2(\sigma, y) & : = y^\sigma \int_{\frac{1}{y}}^\infty \frac{|\ln xy|^\beta x^{\sigma-1}}{|(xy)^\lambda - 1|} dx \\ & = \int_1^\infty \frac{\ln^\beta u}{u^\lambda - 1} u^{\sigma-1} du = k_2(\sigma) \quad (y > 0), \end{aligned}$$

and thus in view of Lemma 2 and similarly to the way we showed Theorem 1, we deduce the following theorem.

**Theorem 3.4.** *If  $\beta > 0$ ,  $0 < \sigma = \lambda - \mu < \lambda$ , then the following conditions are equivalent:*

(i) *There exists a constant  $M_2$ , such that for any  $f(x) \geq 0$  ( $x \in (0, \infty)$ ), satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

*we have the following Hardy-type inequality of the second kind with the non-homogeneous kernel:*

$$\begin{aligned} & \left\{ \int_0^\infty y^{p\sigma_1-1} \left[ \int_{\frac{1}{y}}^\infty \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} \\ & < M_2 \left[ \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}; \end{aligned} \tag{3.8}$$

(ii) *There exists a constant  $M_2$ , such that for any  $f(x), g(y) \geq 0$  ( $x, y \in (0, \infty)$ ), satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty \quad \text{and} \quad 0 < \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy < \infty,$$

we have the following inequality:

$$\begin{aligned} & \int_0^\infty g(y) \left[ \int_{\frac{1}{y}}^\infty \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x) dx \right] dy \\ & < M_2 \left[ \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}}; \end{aligned} \tag{3.9}$$

(iii)  $\sigma_1 = \sigma$ .

If Condition (iii) holds true, then we have  $M_2 \geq k_2(\sigma)$ , and the constant factor

$$M_2 = k_2(\sigma) = \frac{\Gamma(\beta + 1)}{\lambda^{\beta+1}} \zeta(\beta + 1, \frac{\mu}{\lambda}) = k_1(\mu)$$

in (3.8) and (3.9) is the best possible.

Setting

$$y = \frac{1}{Y}, \quad G(Y) = Y^{\lambda-2} g\left(\frac{1}{Y}\right), \quad \mu_1 = \lambda - \sigma_1$$

in Theorem 2, and replacing  $Y$  ( $G(Y)$ ) by  $y$  ( $g(y)$ ), we get

**Corollary 3.5.** *If  $\beta > 0$ ,  $0 < \sigma = \lambda - \mu < \lambda$ , then the following conditions are equivalent:*

(i) *There exists a constant  $M_2$ , such that for any  $f(x) \geq 0$  ( $x \in (0, \infty)$ ), satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

*we obtain the following Hardy-type inequality of the second kind with the homogeneous kernel:*

$$\begin{aligned} & \left\{ \int_0^\infty y^{p\mu_1-1} \left[ \int_y^\infty \frac{|\ln(x/y)|^\beta}{|x^\lambda - y^\lambda|} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} \\ & < M_2 \left[ \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}; \end{aligned} \tag{3.10}$$

(ii) *There exists a constant  $M_2$ , such that for any  $f(x), g(y) \geq 0$  ( $x, y \in (0, \infty)$ ), satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty \quad \text{and} \quad 0 < \int_0^\infty y^{q(1-\mu_1)-1} g^q(y) dy < \infty,$$

*we derive the following inequality:*

$$\begin{aligned} & \int_0^\infty g(y) \left[ \int_y^\infty \frac{|\ln(x/y)|^\beta}{|x^\lambda - y^\lambda|} f(x) dx \right] dy \\ & < M_2 \left[ \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{q(1-\mu_1)-1} g^q(y) dy \right]^{\frac{1}{q}}; \end{aligned} \tag{3.11}$$

(iii)  $\mu_1 = \mu$ .

If Condition (iii) holds true, then we have  $M_2 \geq k_2(\sigma)$ , and the constant  $M_2 = k_2(\sigma) = k_1(\mu)$  in (3.10) and (3.11) is the best possible.

*Remark 3.6.* Similarly, Theorem 2 and Corollary 2 are equivalent.

#### 4. OPERATOR EXPRESSIONS

For  $\sigma, \lambda > 0, \mu = \lambda - \sigma$ , we set the following functions:

$$\varphi(x) := x^{p(1-\sigma)-1}, \quad \psi(y) := y^{q(1-\sigma)-1}, \quad \phi(y) := y^{q(1-\mu)-1},$$

wherefrom,

$$\psi^{1-p}(y) = y^{p\sigma-1}, \quad \phi^{1-p}(y) = y^{p\mu-1} \quad (x, y \in \mathbf{R}_+).$$

Define the following real normed linear spaces:

$$L_{p,\varphi}(\mathbf{R}_+) := \left\{ f : \|f\|_{p,\varphi} := \left( \int_0^\infty \varphi(x)|f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\},$$

wherefrom,

$$L_{q,\psi}(\mathbf{R}_+) = \left\{ g : \|g\|_{q,\psi} := \left( \int_0^\infty \psi(y)|g(y)|^q dy \right)^{\frac{1}{q}} < \infty \right\},$$

$$L_{q,\phi}(\mathbf{R}_+) = \left\{ g : \|g\|_{q,\phi} := \left( \int_0^\infty \phi(y)|g(y)|^q dy \right)^{\frac{1}{q}} < \infty \right\},$$

$$L_{p,\psi^{1-p}}(\mathbf{R}_+) = \left\{ h : \|h\|_{p,\psi^{1-p}} = \left( \int_0^\infty \psi^{1-p}(y)|h(y)|^p dy \right)^{\frac{1}{p}} < \infty \right\},$$

$$L_{p,\phi^{1-p}}(\mathbf{R}_+) = \left\{ h : \|h\|_{p,\phi^{1-p}} = \left( \int_0^\infty \phi^{1-p}(y)|h(y)|^p dy \right)^{\frac{1}{p}} < \infty \right\}.$$

(a) In view of Theorem 1 (for  $\sigma_1 = \sigma$ ), where  $f \in L_{p,\varphi}(\mathbf{R}_+)$ , setting

$$h_1(y) := \int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x) dx \quad (y \in \mathbf{R}_+),$$

by (3.1), we obtain that

$$\|h_1\|_{p,\psi^{1-p}} = \left[ \int_0^\infty \psi^{1-p}(y)h_1^p(y)dy \right]^{\frac{1}{p}} < M_1 \|f\|_{p,\varphi} < \infty. \tag{4.1}$$

**Definition 4.1.** We define a Hardy-type integral operator of the first kind with the non-homogeneous kernel

$$T_1^{(1)} : L_{p,\varphi}(\mathbf{R}_+) \rightarrow L_{p,\psi^{1-p}}(\mathbf{R}_+)$$

as follows:

For any  $f \in L_{p,\varphi}(\mathbf{R}_+)$ , there exists a unique representation  $T_1^{(1)}f = h_1 \in L_{p,\psi^{1-p}}(\mathbf{R}_+)$ , satisfying

$$T_1^{(1)}f(y) = h_1(y),$$

for any  $y \in \mathbf{R}_+$ .

In view of (4.1), it follows that

$$\|T_1^{(1)}f\|_{p,\psi^{1-p}} = \|h_1\|_{p,\psi^{1-p}} \leq M_1\|f\|_{p,\varphi},$$

and thus the operator  $T_1^{(1)}$  is bounded satisfying

$$\|T_1^{(1)}\| = \sup_{f(\neq\theta)\in L_{p,\varphi}(\mathbf{R}_+)} \frac{\|T_1^{(1)}f\|_{p,\psi^{1-p}}}{\|f\|_{p,\varphi}} \leq M_1.$$

If we define the formal inner product of  $T_1^{(1)}f$  and  $g$  as follows:

$$(T_1^{(1)}f, g) := \int_0^\infty \left[ \int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x)dx \right] g(y)dy,$$

then we can rewrite Theorem 1 in the following manner:

**Theorem 4.2.** *For  $\beta, \sigma, \lambda > 0$ , the following conditions are equivalent:*

(i) *There exists a constant  $M_1$ , such that for any  $f(x) \geq 0, f \in L_{p,\varphi}(\mathbf{R}_+)$ ,  $\|f\|_{p,\varphi} > 0$ , the following inequality holds true:*

$$\|T_1^{(1)}f\|_{p,\psi^{1-p}} < M_1\|f\|_{p,\varphi}; \tag{4.2}$$

(ii) *There exists a constant  $M_1$ , such that for any*

$$f(x), g(y) \geq 0, f \in L_{p,\varphi}(\mathbf{R}_+), g \in L_{q,\psi}(\mathbf{R}_+), \|f\|_{p,\varphi}, \|g\|_{q,\psi} > 0,$$

*the following inequality holds true*

$$(T_1^{(1)}f, g) < M_1\|f\|_{p,\varphi}\|g\|_{q,\psi}. \tag{4.3}$$

*We still have  $\|T_1^{(1)}\| = k_1(\sigma) \leq M_1$ .*

(b) In view of Corollary 1 (for  $\mu_1 = \mu$ ), where  $f \in L_{p,\varphi}(\mathbf{R}_+)$ , setting

$$h_2(y) := \int_0^y \frac{|\ln(x/y)|^\beta}{|x^\lambda - y^\lambda|} f(x)dx \quad (y \in \mathbf{R}_+),$$

by (3.6), we deduce that

$$\|h_2\|_{p,\phi^{1-p}} = \left[ \int_0^\infty \phi^{1-p}(y)h_2^p(y)dy \right]^{\frac{1}{p}} < M_1\|f\|_{p,\varphi} < \infty. \tag{4.4}$$

**Definition 4.3.** We define a Hardy-type integral operator of the first kind with the homogeneous kernel

$$T_1^{(2)} : L_{p,\varphi}(\mathbf{R}_+) \rightarrow L_{p,\phi^{1-p}}(\mathbf{R}_+)$$

as follows:

For any  $f \in L_{p,\varphi}(\mathbf{R})$ , there exists a unique representation

$$T_1^{(2)}f = h_2 \in L_{p,\phi^{1-p}}(\mathbf{R}_+),$$

satisfying  $T_1^{(2)}f(y) = h_2(y)$ , for any  $y \in \mathbf{R}_+$ .



In view of (4.4), it follows that

$$\|T_1^{(2)}f\|_{p,\phi^{1-p}} = \|h_2\|_{p,\phi^{1-p}} \leq M_1\|f\|_{p,\varphi},$$

and thus the operator  $T_1^{(2)}$  is bounded satisfying

$$\|T_1^{(2)}\| = \sup_{f(\neq\theta)\in L_{p,\varphi}(\mathbf{R}_+)} \frac{\|T_1^{(2)}f\|_{p,\phi^{1-p}}}{\|f\|_{p,\varphi}} \leq M_1.$$

If we define the formal inner product of  $T_1^{(2)}f$  and  $g$  as

$$(T_1^{(2)}f, g) := \int_0^\infty \left[ \int_0^y \frac{|\ln(x/y)|^\beta}{|x^\lambda - y^\lambda|} f(x)dx \right] g(y)dy,$$

then Corollary 1 assumes the following form:

**Corollary 4.4.** *For  $\beta, \sigma, \lambda > 0$ , the following conditions are equivalent:*

(i) *There exists a constant  $M_1$ , such that for any  $f(x) \geq 0, f \in L_{p,\varphi}(\mathbf{R}_+)$ ,  $\|f\|_{p,\varphi} > 0$ , the following inequality holds true:*

$$\|T_1^{(2)}f\|_{p,\phi^{1-p}} < M_1\|f\|_{p,\varphi}; \tag{4.5}$$

(ii) *There exists a constant  $M_1$ , such that for any  $f(x), g(y) \geq 0, f \in L_{p,\varphi}(\mathbf{R}_+)$ ,  $g \in L_{q,\phi}(\mathbf{R}_+)$ ,  $\|f\|_{p,\varphi}, \|g\|_{q,\phi} > 0$ , the following inequality holds true:*

$$(T_1^{(2)}f, g) < M_1\|f\|_{p,\varphi}\|g\|_{q,\phi}. \tag{4.6}$$

We still have  $\|T_1^{(2)}\| = k_1(\sigma) \leq M_1$ .

*Remark 4.5.* Similarly, Theorem 3 and Corollary 3 are equivalent.

(c) In view of Theorem 2 (for  $\sigma_1 = \sigma$ ), where  $f \in L_{p,\varphi}(\mathbf{R}_+)$ , setting

$$H_1(y) := \int_{\frac{1}{y}}^\infty \frac{|\ln xy|^\beta}{|(xy)^\lambda - 1|} f(x)dx \quad (y \in \mathbf{R}_+),$$

by (3.8), we derive that

$$\|H_1\|_{p,\psi^{1-p}} = \left[ \int_0^\infty \psi^{1-p}(y) H_1^p(y) dy \right]^{\frac{1}{p}} < M_2\|f\|_{p,\varphi} < \infty. \tag{4.7}$$

**Definition 4.6.** We define a Hardy-type integral operator of the second kind with the non-homogeneous kernel

$$T_2^{(1)} : L_{p,\varphi}(\mathbf{R}_+) \rightarrow L_{p,\psi^{1-p}}(\mathbf{R}_+)$$

as follows:

For any  $f \in L_{p,\varphi}(\mathbf{R}_+)$ , there exists a unique representation  $T_2^{(1)}f = H_1 \in L_{p,\psi^{1-p}}(\mathbf{R}_+)$ , satisfying  $T_2^{(1)}f(y) = H_1(y)$ , for any  $y \in \mathbf{R}_+$ .

In view of (4.7), it follows that

$$\|T_2^{(1)}f\|_{p,\psi^{1-p}} = \|H_1\|_{p,\psi^{1-p}} \leq M_2\|f\|_{p,\varphi},$$

and thus the operator  $T_2^{(1)}$  is bounded satisfying

$$\|T_2^{(1)}\| = \sup_{f(\neq\theta)\in L_{p,\varphi}(\mathbf{R}_+)} \frac{\|T_2^{(1)}f\|_{p,\psi^{1-p}}}{\|f\|_{p,\varphi}} \leq M_2.$$

If we define the formal inner product of  $T_2^{(1)}f$  and  $g$  as follows:

$$(T_2^{(1)}f, g) := \int_0^\infty \left[ \int_{\frac{1}{y}}^\infty \frac{(\ln xy)^\beta}{(xy)^\lambda - 1} f(x) dx \right] g(y) dy,$$

then Theorem 2 obtains the following form:

**Theorem 4.7.** *For  $\beta > 0, 0 < \sigma = \lambda - \mu < \lambda$ , the following conditions are equivalent:*

(i) *There exists a constant  $M_2$ , such that for any  $f(x) \geq 0, f \in L_{p,\varphi}(\mathbf{R}_+)$ ,  $\|f\|_{p,\varphi} > 0$ , the following inequality holds true:*

$$\|T_2^{(1)}f\|_{p,\psi^{1-p}} < M_2\|f\|_{p,\varphi}; \tag{4.8}$$

(ii) *There exists a constant  $M_2$ , such that for any  $f(x), g(y) \geq 0, f \in L_{p,\varphi}(\mathbf{R}_+)$ ,  $g \in L_{q,\psi}(\mathbf{R}_+)$ ,  $\|f\|_{p,\varphi}, \|g\|_{q,\psi} > 0$ , we have the following inequality:*

$$(T_2^{(1)}f, g) < M_2\|f\|_{p,\varphi}\|g\|_{q,\psi}. \tag{4.9}$$

We still have  $\|T_2^{(1)}\| = k_2(\sigma) = k_1(\mu) \leq M_2$ .

(d) In view of Corollary 2 (for  $\mu_1 = \mu$ ), where  $f \in L_{p,\varphi}(\mathbf{R}_+)$ , setting

$$H_2(y) := \int_y^\infty \frac{|\ln(x/y)|^\beta}{|x^\lambda - y^\lambda|} f(x) dx \quad (y \in \mathbf{R}_+),$$

by (3.10), we derive that

$$\|H_2\|_{p,\phi^{1-p}} = \left[ \int_0^\infty \phi^{1-p}(y) H_2^p(y) dy \right]^{\frac{1}{p}} < M_2\|f\|_{p,\varphi} < \infty. \tag{4.10}$$

**Definition 4.8.** We define a Hardy-type integral operator of the second kind with the homogeneous kernel

$$T_2^{(2)} : L_{p,\varphi}(\mathbf{R}_+) \rightarrow L_{p,\phi^{1-p}}(\mathbf{R}_+)$$

as follows:

For any  $f \in L_{p,\varphi}(\mathbf{R}_+)$ , there exists a unique representation  $T_2^{(2)}f = H_2 \in L_{p,\phi^{1-p}}(\mathbf{R}_+)$ , satisfying  $T_2^{(2)}f(y) = H_2(y)$ , for any  $y \in \mathbf{R}_+$ .

In view of (4.10), it follows that

$$\|T_2^{(2)}f\|_{p,\phi^{1-p}} = \|H_2\|_{p,\phi^{1-p}} \leq M_2\|f\|_{p,\varphi},$$

and thus the operator  $T_2^{(2)}$  is bounded satisfying

$$\|T_2^{(2)}\| = \sup_{f(\neq\theta)\in L_{p,\varphi}(\mathbf{R}_+)} \frac{\|T_2^{(2)}f\|_{p,\phi^{1-p}}}{\|f\|_{p,\varphi}} \leq M_2.$$

If we define the formal inner product of  $T_1^{(2)}f$  and  $g$  as follows:

$$(T_2^{(2)}f, g) := \int_0^\infty \left[ \int_y^\infty \frac{[\ln(x/y)]^\beta}{x^\lambda - y^\lambda} f(x) dx \right] g(y) dy,$$

then Corollary 2 obtains the following form:

**Corollary 4.9.** *For  $\beta > 0, 0 < \sigma = \lambda - \mu < \lambda$ , the following conditions are equivalent:*

(i) *There exists a constant  $M_2$ , such that for any  $f(x) \geq 0, f \in L_{p,\varphi}(\mathbf{R}_+)$ ,  $\|f\|_{p,\varphi} > 0$ , the following inequality holds true:*

$$\|T_2^{(2)}f\|_{p,\phi^{1-p}} < M_2 \|f\|_{p,\varphi}; \quad (4.11)$$

(ii) *There exists a constant  $M_2$ , such that for any  $f(x), g(y) \geq 0, f \in L_{p,\varphi}(\mathbf{R}_+)$ ,  $g \in L_{q,\phi}(\mathbf{R}_+)$ ,  $\|f\|_{p,\varphi}, \|g\|_{q,\phi} > 0$ , we have the following inequality:*

$$(T_2^{(2)}f, g) < M_2 \|f\|_{p,\varphi} \|g\|_{q,\phi}. \quad (4.12)$$

We still have  $\|T_2^{(2)}\| = k_2(\sigma) = k_1(\mu) \leq M_2$ .

*Remark 4.10.* Similarly, Theorem 4 and Corollary 4 are equivalent.

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