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EXISTENCE THEOREMS FOR ATTRACTIVE POINTS OF SEMIGROUPS OF BREGMAN GENERALIZED NONSPREADING MAPPINGS IN BANACH SPACES

BASHIR ALI,[1] MURTALA HARUNA HARBAU,[2*] and LAWAL HARUNA YUSUF[3]

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Abstract. In this paper, we establish new attractive point theorems for semigroups of generalized Bregman nonspreading mappings in reflexive Banach spaces. Our theorems improve and extend many results announced recently in the literature.

1. Introduction

Throughout this paper, the set of natural numbers is denoted by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$ and the set $(-\infty, +\infty]$ by $\bar{\mathbb{R}}$, the extended real line.The concept of attractive points for nonlinear mappings in a Hilbert space was first introduced by Takahashi and Takeuchi [11]. Let C be a nonempty subset of a real Hilbert space H, and $T : C \to C$ be a mapping. A point $u \in H$ is called an attractive point of T if

$$\|Tv - u\| \leq \|v - u\| \qquad \forall v \in C.$$

Denote by $A(T)$, the set of all attractive points of T i.e

$$A(T) = \{u \in H : \|Tv - u\| \leq \|v - u\|, \forall v \in C\}.$$

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*Corresponding author.
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Remark 1.1. Observe that every fixed point of quasi-nonexpansive mapping is an attractive point.

A mapping $T : C \to H$ is called (α, β)-generalized hybrid see Takahashi et al. [7] if $\exists\ \alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tu - Tv\|^2 + (1-\alpha)\|u - Tv\|^2 \leq \beta\|Tu - v\|^2 + (1-\beta)\|u - v\|^2 \qquad \forall u, v \in C.$$

Putting $\alpha = 1, \beta = 0$ we obtain $(1, 0)$-generalized hybrid mapping which is called nonexpansive i.e

$$\|Tu - Tv\| \leq \|u - v\| \qquad \forall u, v \in C.$$

Also, putting $\alpha = \frac{3}{2}, \beta = \frac{1}{2}$ we obtain $(\frac{3}{2}, \frac{1}{2})$-generalized hybrid mappings which are called hybrid mapping see [8] i.e

$$3\|Tu - Tv\|^2 \leq \|u - Tv\|^2 + \|Tu - v\|^2 + \|u - v\|^2 \qquad \forall u, v \in C.$$

In the case of $\alpha = 2, \beta = 1$, we obtain $(2, 1)$-generalized hybrid mapping which is called a nonspreading mapping, see [8] i.e

$$2\|Tu - Tv\|^2 \leq \|u - Tv\|^2 + \|Tu - v\|^2 \qquad \forall u, v \in C.$$

$T : C \to H$ is called $(\alpha, \beta, \gamma, \delta)$- normally generalized hybrid see Takahashi et al. [12] if $\exists \alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

(i) $\alpha + \beta + \gamma + \delta \geq 0$
(ii) $\alpha + \beta > 0$ or $\gamma + \delta > 0$; and
(iii) $\alpha\|Tu - Tv\|^2 + \beta\|u - Tv\|^2 + \gamma\|Tu - v\|^2 + \delta\|u - v\|^2 \leq 0\ \forall u, v \in C.$

Let E be a real smooth Banach space and E^* be the dual space of E. Let C be a nonempty closed convex subset of E. The normalised duality map $J : E \to 2^{E^*}$ is defined by

$$Ju = \left\{u^* \in E^*\ : \langle u, u^*\rangle = \|u\|^2 = \|u^*\|^2\right\} \qquad \forall u \in E.$$

Let $\phi : E \times E \to \mathbb{R}$ be a Lyapanov functional defined by

$$\phi(u, v) = \|u\|^2 - 2\langle u, Jv\rangle + \|v\|^2, \quad \forall u, v \in E.$$

A mapping T from C into itself is called generalized nonspreading mapping if $\exists \alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha\phi(Tx, Ty) \quad &+ \quad (1-\alpha)\phi(x, Ty) + \gamma\{\phi(Ty, Tx) - \phi(Ty, x)\} \\ &\leq \quad \beta\phi(Tx, y) + (1-\beta)\phi(x, y) + \delta\{\phi(y, Tx) - \phi(y, x)\}, \forall x, y \in C. \end{aligned}$$

If such a mapping T is called $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping then A (1,1,1,0)-generalized nonspreading mapping is called a nonspreading mapping [8], i.e

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x) \forall x, y \in C.$$

Also, A (1,0,0,0)-generalized nonspreading mapping is called a ϕ-nonenpansive mapping i.e

$$\phi(Tx, Ty) \leq \phi(x, y) \forall x, y \in C.$$

Let $f : E \to \bar{\mathbb{R}}$ be a convex and Gateaux differentiable function. The function $D_f : \text{dom} f \times \text{ intdom} f \to \bar{\mathbb{R}}$, defined as follows:

$$D_f(u, v) := f(u) - f(v) - \langle \bigtriangledown f(v), u - v\rangle, \tag{1.1}$$

is called the Bregman distance with respect to f.

Remark 1.2. If E is smooth and strictly convex Banach space and $f(u) = \|u\|^2$ for all $u \in E$, then we have $\bigtriangledown f(u) = 2Ju$ for all $u \in E$ and hence $D_f(u, v) = \phi(u, v)$.

Observe that from (1.1), we have

$$D_f(u, w) := D_f(u, v) + D_f(v, w) + \langle \bigtriangledown f(v) - \bigtriangledown f(w), u - v\rangle. \tag{1.2}$$

Nonspreading and hybrid mappings are generally not continuous, see for example [6]. Takahashi and Takeuchi [11] proved the following attractive point and mean convergence theorems without convexity in Hilbert spaces:

Theorem 1.3. *(Takahashi and Takeuchi)* [11] *Let H be a Hilbert space and C be a nonempty subset of H. Let $T : C \to C$ be a generalized hybrid mapping . Then T has an attractive point if and only if $\exists z \in C$ such that $\{T^n z : n = 0, 1, ...\}$ is bounded.*

Theorem 1.4. *(Takahashi and Takeuchi)* [11] *Let H be a real Hilbert space and let C be a nonempty subset of H. Let $T : C \to C$ be a generalized hybrid mapping. Let $\{v_n\}$ and $\{b_n\}$ be sequences defined by*

$$v_1 \in C, \qquad v_{n+1} = Tv_n, \qquad b_n = \frac{1}{n}\sum_{k=1}^{n} v_k \qquad \forall n \in \mathbb{N}.$$

If $\{v_n\}$ is bounded then the following hold:

(i) *$A(T)$ is nonempty, closed and convex;*
(ii) *$\{b_n\}$ converges weakly to $u_0 \in A(T)$ where $u_0 = \lim_{n\to\infty} P_{A(T)} v_n$ and $P_{A(T)}$ is the metric projection of H onto $A(T)$.*

In 1975, Riech [15](see also Goebel and Reich [5]) proved that, for any nonexpansive mapping $T : C \to C$, if a specfied given sequence of iterate is bounded, then T has a fixed point

For commutative semigroups of nonexpansive mappings, Atsushita and Takahashi [2] proved some attractive point theorem in a real Hilbert space. Takahashi et al. [13] proved an attractive point and mean convergence theorems for semigroup of mappings without continuity in Hilbert spaces which unified the results of [2] and [11].

More recently, Takahashi et al. [14] extended these results to some Banach space setting much more general than Hilbert spaces. In fact they proved the following theorems:

Theorem 1.5. *Takahashi, Wong and Yao* [14] *Let E be a smooth and reflexive Banach space with the duality mappping J and C be a nonempty subset of E. Let S be a commutative semitopological semigroup with identity. Let $S = \{T_s :$*

$s \in S\}$ be a continuous representation of S as mapping of C into itself such that $\{T_s x : s \in S\}$ is bounded for some $x \in C$. Let μ be a mean on $C(S)$. Suppose that $\mu_s \phi(T_s x, T_t y) \leq \mu_s \phi(T_s x, y), \forall y \in C$ and $t \in S$ then $A(S) = \cap\{A(T_s) : s \in S\} \neq \emptyset$. In particular, if E is strictly convex and C is closed and convex, then $F(S) = \cap\{F(T_t) : t \in S\} \neq \emptyset$.

Theorem 1.6. *Takahashi, Wong and Yao.* [14] *Let E be a smooth and reflexive Banach space and C be a nonempty subset of E. Let $T : C \to C$ be a generalized nonspreading mapping, then the following are equivalent:*

(i) $A(T) \neq \emptyset$
(ii) $\{T^n v_0\}$ *is bounded for some* $v_0 \in C$.

Additionally if E is strictly convex and C is closed and convex, then the following are equivalent:

(i) $F(T) \neq \emptyset$
(ii) $\{T^n v_0\}$ *is bounded for some* $v_0 \in C$.

Motivated by the above results, in this paper, we established Bregman attractive point theorems using Bregman distance in a reflexive Banach spaces. Our results improved and generalized recently announced results of Takahashi et al. [14], Lin et al. [9] and many others.

2. Preliminaries

Definition 2.1. A normed space E is called smooth if for every $u \in E$, $\|u\| = 1$, there exists a unique $u^* \in E^*$ such that $\|u^*\| = 1$ and $\langle u, u^* \rangle = \|u\|$.

Definition 2.2. Let E be a normed linear space and $\varphi : E \to E^{**}$ be the canonical embedding. If φ is onto, then E is reflexive.

Let E be a real reflexive Banach space with norm $\|\cdot\|$ and E^* the the dual space of E. Let $f : E \to \bar{\mathbb{R}}$ be a proper, lower semi-continuous and convex function. The Fenchel conjugate of f is the convex function $f^* : E^* \to \bar{\mathbb{R}}$ defined by

$$f^*(u^*) = \sup\{\langle u^*, u \rangle - f(u) : u \in E\}.$$

Let $u \in \text{int dom} f$; the subdifferential of f at u is the convex set defined by

$$\partial f(u) = \{u^* \in E^* : f(u) + \langle u^*, v - u \rangle \leq f(v), \ \forall v \in E\}.$$

For any $u \in \text{intdom} f$ and $v \in E$, the right-hand derivative of f at u in the direction v is defined by

$$f^\circ(u, v) := \lim_{t \to 0^+} \frac{f(u + tv) - f(u)}{t}.$$

The function f is said to be Gateaux differentiable at u if $\lim_{t \to 0^+} \frac{f(u+tv)-f(u)}{t}$ exists for any v. In this case, $f^\circ(u, v)$ coincides with $\bigtriangledown f(u)$, the value of the gradient $\bigtriangledown f$ of f at u. The function f is said to be Gateaux differentiable if it is Gateaux differentiable for any $u \in \text{intdom} f$. The function f is said to be Fréchet differentiable at u if this limit is attained uniformly in $\|v\| = 1$. Finally, f is said to be Gateaux differentiable on a subset C of E if the limit is attained uniformly for $u \in C$ and $\|v\| = 1$. it is well known that if f is Gateaux differentiable (resp. Frechet differentiable) on intdom$f$, then f is continuous and its Gateaux differentiable $\bigtriangledown f$ is norm-to-weak* continuous (resp. continuous) on intdom$f$ (see also [1]) .

Definition 2.3. The function f is said to be:
(i)Essentially smooth, if ∂f is both locally bounded and single-valued on its domain;
(ii)Essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every subset of dom$f$;
(iii)Legendre, if it is both essentially smooth and essentially strictly convex.

Remark 2.4. Let E be reflexive Banach space. Then we have:
(i) f is essentially smooth if and only if f^* is essentially strictly convex;
(ii) $(\partial f)^{-1} = \partial f^*$
(iii) f is Legendre if and only if f^* is Legendre
(iv)If f is Legendre, then $\bigtriangledown f$ is a bijection satifying $\bigtriangledown f = (\bigtriangledown f^*)^{-1}$, ran$\bigtriangledown f$ = dom$\bigtriangledown f^*$ = intdomf^* and ran $\bigtriangledown f^*$ = dom$f$ = intdom$f$.

Examples of Legendre functions were given in [3]. One important and interesting Legendre function is $\frac{1}{p}\|\cdot\|^p$ $(1 < p < \infty)$ when E is a smooth and strictly convex Banach space. In this case the gradient $\bigtriangledown f$ of f is coincident with the generalized duality mapping of E, i.e, $\bigtriangledown f = J_p$ $(1 < p < \infty)$. In particular, $\bigtriangledown f = I$ the identity mapping in Hilbert spaces. In the rest of this paper, we always assume that $f : E \to \bar{\mathbb{R}}$ is a Legendre function.

Definition 2.5. (Conjugate Operator)[17] Let E be a reflexive Banach space and C be a subset of E. Let $f : E \to \bar{\mathbb{R}}$ be Legendre and $T : C \subset intdomf \to intdomf$. The conjugate operator $T^*_f : \bigtriangledown f(C) \to intdomf^*$ associated with T denoted by T^* is defined by

$$T^* = \bigtriangledown f o T o \bigtriangledown f^*. \tag{2.1}$$

Lemma 2.6. *(see [4], Theorem 7.3 (vi), (vii)) Suppose $u \in E$ and $v \in domf$. Then*
(i)If f is essentially strictly convex, then $D_f(u, v) = 0 \Leftrightarrow u = v$
(ii)If f is differentiable on intdomf and essentially strictly convex, then $D_f(u, v) = D_{f^*}(\bigtriangledown f(v), \bigtriangledown f(u))$.

Lemma 2.7. [16] *If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then $\bigtriangledown f$ is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

3. Semitopological Semigroups and Invariant Means

A semitopological semigroup S is a semigroup with a Hausdorff topology such that for each $x \in S$, the mappings $s \to x.s$ and $s \to s.x$ from S into itself are continuous. If S is commutative then st is denoted by $s+t$. Let the Banach space of all bounded real-valued functions on S with supremum norm be $B(S)$ and let the subspace of such Banach space $B(S)$ of all bounded real-valued continuous functions on S be $C(S)$. Let μ be an element of the dual space of $C(S)$. The value of μ at $f \in C(S)$ is denoted by $\mu(f)$. It is sometimes denoted by $\mu_t(f(t))$. For each $s \in S$ and $f \in C(S)$, defining the two $(r_s f)$ and $(l_s f)$ functions are defined as follows:

$$(r_s f)(t) = f(ts) \text{ } and \text{ } (l_s f)(t) = f(st) \ \forall t \in S.$$

An element μ of $C(S)^*$ is called a mean on $C(S)$ if

$$\mu(e) = \|\mu\| = 1,$$

where $e(s) = 1 \ \forall s \in S$. It is known that $\mu \in C(S)^*$ is a mean on $C(S)$ if and only if

$$\inf_{s \in S} f(s) \leq \mu(s) \leq \sup_{s \in S} f(s) \qquad \forall f \in C(S).$$

A right invariant mean is a mean μ on $C(S)$ for which $\mu(r_s f) = \mu(f) \ \forall f \in C(S)$ and $s \in S$. Similarly, a left invariant mean is a mean μ on $C(S)$ for which $\mu(l_s f) = \mu(f) \ \forall f \in C(S)$ and $s \in S$. A right and left invariant mean on $C(S)$ is called an invariant mean on $C(S)$. if $S = \mathbb{N}$, an invariant mean on $C(S) = B(S)$ is called a Banach limit on l^∞.

Theorem 3.1. *Takahashi,* [10]. *Let S be a commutative semitopological semigroup. Then $\exists$ an invariant mean on $C(S)$; i.e $\exists$ an element μ on $C(S)^*$ such that $\mu(e) = \|\mu\| = 1$ and $\mu(r_s f) = \mu(f) \forall f \in C(S)$ and $s \in S$.*

Let C be a nonempty subset of a Banach space and E. Let S be a semitopological semigroup and let $S^| = \{T_s : s \in S\}$ be a family of mappings of C into itself. Then $S^| = \{T_s : s \in S\}$ is called a continuous representation of S as mappings on C if $T_{st} = T_s T_t$ for all $s, t \in S$ and $s \to T_s x$ is continuous for each $x \in C$. Denoting the set of common fixed point of T_s for $s \in S$ by $F(S^|)$ i.e

$$F(S^|) = \cap\{F(T_s) : s \in S\}.$$

The following definition is very essential in the studies of nonlinear ergodic theory of abstract semigroups:

Definition 3.2. Let E be a reflexive space and let E^* be the dual space of E. Let $u : S \to E$ be a continuous function such that $\{u(s) : s \in S\}$ is bounded and let μ be a mean on $C(S)$. Then there exists a unique point $z_0 \in \bar{co}\{u(s) : s \in S\}$ such that

$$\mu_s \langle u(s), y^* \rangle = \langle z_0, y^* \rangle \qquad \forall y^* \in E^*.$$

Such $z_0 \in \bar{co}\{u(s) : s \in S\}$ is called the mean vector of u for μ. see [14] for details.

4. Main Results

Let E be a reflexive Banach space and $f : E \to \bar{\mathbb{R}}$ be convex and Gâteaux differentiable function. Let T be a mapping from C into itself where C is a nonempty subset of E. Denoting the set of Bregman attractive point of T by $A^B(T)$ i.e

$$A^B(T) = \{v \in E : D_f(v, Tz) \leq D_f(v, z), \forall z \in C\}.$$

Now, let S be a semitopological semigroup with identity. Let $S^| = \{T_s : s \in S\}$ be a continous representation of S as mapping of C into itself. Denoting the set of common Bregman attractive points of $S^|$ by $A^B(S^|)$ i.e

$$A^B(S^|) = \{A^B(T_t) : t \in S\}.$$

We now prove the following Bregman attractive point theorems:

Theorem 4.1. *Let E be a reflexive Banach space and $f : E \to \bar{\mathbb{R}}$ be a convex and Gâteaux differentiable function and C be a nonempty subset of E. Let S be a semitopological semigroup with identity. Let $S^| = \{T_s : s \in S\}$ be a continuous representation of S as mapping of C into itself such that $S^| = \{T_s x : s \in S\}$ is bounded for some $x \in C$. Let μ be a mean on $C(S)$. Suppose that $\mu_s D_f(T_s x, T_s y) \leq \mu_s D_f(T_s x, y), \forall y \in C$ and $t \in S$ then $A^B(S^|) = \cap\{A^B(T_s) : s \in S\} \neq \emptyset$.*

Proof. Using a mean μ on a bounded set $\{T_s x : s \in S\}$ and define a function $g : E^* \to \mathbb{R}$ by

$$g(x^*) = \mu_s \langle T_s x, x^* \rangle \ \forall x^* \in E^*.$$

Since E is reflexive as shown by (Takahashi [14] in section 3) $\exists!\ z \in E^*$ such that

$$g(x^*) = \mu_s \langle T_s x, x^* \rangle = \langle z, x^* \rangle \qquad \forall x^* \in E^*.$$

and such $z \in \bar{co}\{T_s : s \in S\}$. Now, for s,t $\in S$, we have

$$\begin{aligned} D_f(T_s x, y) &= D_f(T_s x, T_t y) + D_f(T_t y, y) + \langle \triangledown f(T_t y) - \triangledown f(y), T_s x - T_t y \rangle \\ &= D_f(T_s x, T_t y) + D_f(T_t y, y) + \langle \triangledown f(T_t y), T_s x - T_t y \rangle - \langle \triangledown f(y), T_s x - T_t y \rangle. \end{aligned}$$

Applying mean μ_s on both side of the above expression we get,

$$\begin{aligned} \mu_s D_f(T_s x, y) &= \mu_s D_f(T_s x, T_t y) + \mu_s D_f(T_t y, y) + \mu_s \langle \triangledown f(T_t y), T_s x - T_t y \rangle \\ &- \mu_s \langle \triangledown f(y), T_s x - T_t y \rangle \\ &= \mu_s D_f(T_s x, T_t y) + D_f(T_t y, y) + \langle \triangledown f(T_t y), z - T_t y \rangle - \langle \triangledown f(y), z - T_t y \rangle \\ &\leq \mu_s D_f(T_s x, y) + D_f(T_t y, y) + \langle \triangledown f(T_t y), z - T_t y \rangle - \langle \triangledown f(y), z - T_t y \rangle. \end{aligned}$$

This implies,

$$\begin{aligned} 0 &\leq D_f(T_t y, y) + \langle \triangledown f(T_t y), z - T_t y \rangle - \langle \triangledown f(y), z - T_t y \rangle \\ &= D_f(T_t y, y) + D_f(z, T_t y) - D_f(z, T_t y) + \langle \triangledown f(T_t y) - \triangledown f(y), z - T_t y \rangle \\ &= D_f(z, T_t y) + D_f(T_t y, y) + \langle \triangledown f(T_t y) - \triangledown f(y), z - T_t y \rangle - D_f(z, T_t y) \\ &= D_f(z, y) - D_f(z, T_t y). \end{aligned}$$

Hence

$$D_f(z, T_t y) \leq D_f(z, y).$$

This implies $z \in A^B(T_t)$ for each t and so $z \in A^B(S^|)$. Hence $A^B(S^|) \neq \emptyset$. This completes the proof.

□

Definition 4.2. Let E be a reflexive Banach space and C be a nonempty subset of E. A mapping T from C into itself is called generalized Bregman nonspreading mapping if $\exists \alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha D_f(Tx, Ty) + (1-\alpha) D_f(x, Ty) + \gamma\{D_f(Ty, Tx) - D_f(Ty, x)\} \\ &\quad \leq \ \beta D_f(Tx, y) + (1-\beta) D_f(x, y) + \delta\{D_f(y, Tx) - D_f(y, x)\},\ \forall x, y \in C. \end{aligned}$$

If such a mapping T is called $(\alpha, \beta, \gamma, \delta)$-generalized Bregman nonspreading mapping, then a (1,1,1,0)-generalized Bregman nonspreading mapping is called a Bregman nonspreading mapping. i.e

$$D_f(Tx, Ty) + D_f(Ty, Tx) \leq D_f(Tx, y) + D_f(Ty, x) \forall x, y \in C.$$

Putting $\alpha = 1$ and $\beta = \gamma = \delta = 0$, we obtain Bregman nonexpansive mapping

$$D_f(Tx, Ty) \leq D_f(x, y) \qquad \forall x, y \in C.$$

Theorem 4.3. *Let E be a reflexive Banach space and $f : E \to \bar{\mathbb{R}}$ be a convex and Gâteaux differentiable function and C be a nonempty subset of E. Let $T : C \to C$ be a generalized Bregman nonspreading mapping, then $A^B(T) \neq \emptyset$ if and only if $\{T^n v_0\}$ is bounded for some $v_0 \in C$*

Proof. By Theorem 4.1, following the line of proof of (Theorem 4.3 in [14]) and considering the generalized Bregman nonspreading mapping in Definition 4.2, the result follows:

□

Theorem 4.4. *Let E be a reflexive Banach space and $f : E \to \bar{\mathbb{R}}$ be essentially strictly convex and Gâteaux differentiable function. Let D a nonempty, closed and convex subset of E. Let S be a semitopological semigroup with identity and let $C(S)$ be a Banach space of all bounded real-valued continuous functions on S with supremum norm.Let $u : S \to E$ be a continuous function such that $\{u(s) : s \in S\} \subset D$ is bounded and let μ be a mean on $C(S)$. if $g : D \to \mathbb{R}$ is defined by*

$$g(z) = \mu_s D_f(u(s), z) \forall z \in D,$$

then the mean vector z_0 of $\{u(s) : s \in S\}$ for μ_s is a unique minimizer in D such that

$$g(z_0) = min\{g(z) : z \in D\}.$$

Proof. For a bounded net $\{u(s)\} \subset D$ and a mean on $C(S)$, we see that a function $g : D \to \mathbb{R}$ defined by

$$g(z) = \mu_s D_f(u(s), z) \forall z \in D$$

is well defined as $u(s) \in E, z \in D \subseteq E$ and $D_f(u(s), z) = f(u(s)) - f(z) - \langle \bigtriangledown f(z), u(s) - z \rangle$. Since $f : E \to (-\infty, +\infty]$, it is clear that f can see both $u(s)$

and z. Also, from (section 3 of [14]) $\exists$ a mean vector z_0 of $\{u(s)\}$ for μ that is $\exists$ $z_0 \in \bar{co}\{u(s) : s \in S\}$ such that

$$\mu_s\langle u(s), y^*\rangle = \langle z_0, y^*\rangle \ \forall y^* \in E^*.$$

Since D is closed and convex and $\{u(s)\} \subset D$, we have $z_0 \in D$.
Using the three point identity of Bregman distance, we have

$$\begin{aligned}
g(z) - g(z_0) &= \mu_s D_f(u(s), z) - \mu_s D_f(u(s), z_0) = \mu_s\big(D_f(u(s), z) - D_f(u(s), z_0)\big)\\
&= \mu_s\big(D_f(u(s), z) - D_f(u(s), z) - D_f(z, z_0) - \langle \bigtriangledown f(z) - \bigtriangledown f(z_0), u(s) - z\rangle\big)\\
&= -\mu_s\big(D_f(z, z_0) + \langle \bigtriangledown f(z) - \bigtriangledown f(z_0), u(s) - z\rangle\big)\\
&= -D_f(z, z_0) - \mu_s\langle \bigtriangledown f(z) - \bigtriangledown f(z_0), u(s) - z\rangle\\
&= -D_f(z, z_0) - \mu_s\langle \bigtriangledown f(z), u(s) - z\rangle + \mu_s\langle \bigtriangledown f(z_0), u(s) - z\rangle\\
&= -D_f(z, z_0) - \langle \bigtriangledown f(z), z_0 - z\rangle + \langle \bigtriangledown f(z_0), z_0 - z\rangle\\
&= -f(z) + f(z_0) + \langle \bigtriangledown f(z_0), z - z_0\rangle - \langle \bigtriangledown f(z), z_0 - z\rangle + \langle \bigtriangledown f(z_0), z_0 - z\rangle\\
&= f(z_0) - f(z) - \langle \bigtriangledown f(z), z_0 - z\rangle\\
&= D_f(z_0, z).
\end{aligned}$$

Thus,

$$g(z) = D_f(z_0, z) + g(z_0) \ \forall z \in D.$$

This implies $z_0 \in D$ is a minimizer , that is,

$$g(z_0) = min\{g(z) : z \in D\}.$$

Now, suppose $u \in D$ satisfies $g(u) = g(z_0)$, then we get $g(u) = D_f(z_0, u) + g(z_0)$ and $D_f(z_0, u) = 0 \Leftrightarrow z_0 = u$ as f is an essential strictly convex. Hence $z_0 \in D$ is a unique minimizer. This completes the proof.

□

Let E be a reflexive Banach space and C be a nonempty subset of E. Let T be a mapping from C into E. Denoting the set of skew-Bregman attractive point of T by $B^B(T)$ i.e

$$B^B(T) = \{z \in E : D_f(Tx, z) \leq D_f(x, z), \forall x \in C\}.$$

Lemma 4.5. *Let E be a reflexive Banach space and $f : E \to \bar{\mathbb{R}}$ be Legendre. Let $T : C \subset intdomf \to intdomf$ be an operator and T^* be the conjugate operator associated with T . Then the following hold:*

(i) $B^B(T) = \bigtriangledown f^*\big(A^B(T^*)\big)$,

(ii) $A^B(T) = \bigtriangledown f^*\big(B^B(T^*)\big)$.

Proof.

$$\begin{aligned}
z \in B^B(T) \quad &\Leftrightarrow \quad D_f(Tx,z) \le D_f(x,z) \qquad \forall x \in C \\
&\Leftrightarrow \quad D_{f^*}\big(\bigtriangledown f(z), \bigtriangledown f(Tx)\big) \\
&\qquad \le D_{f^*}\big(\bigtriangledown f(z), \bigtriangledown fx\big)\ \forall x \in C \\
&\Leftrightarrow \quad D_{f^*}\big(\bigtriangledown f(z), \bigtriangledown fT \bigtriangledown f^* \bigtriangledown fx\big) \\
&\qquad \le D_{f^*}\big(\bigtriangledown f(z), \bigtriangledown fx\big)\ \forall x \in C \\
&\Leftrightarrow \quad D_{f^*}\big(\bigtriangledown f(z), T^*x^*\big) \le D_{f^*}\big(\bigtriangledown f(z), x^*\big)
\end{aligned}$$

where $x^* = \bigtriangledown fx \in intdomf^*$

$$\begin{aligned}
&\Leftrightarrow \quad \bigtriangledown f(z) \in A(T^*) \\
&\Leftrightarrow \quad z \in \bigtriangledown f^*\big(A^B(T^*)\big).
\end{aligned}$$

Hence $B^B(T) = \bigtriangledown f^*\big(A^B(T^*)\big)$. This completes the proof.

□

Let E be a reflexive Banach space and $f : E \to \bar{\mathbb{R}}$ be convex and Gâteaux differentiable function. Let $S^| = \{T_s : s \in S\}$ be a family of mappings of C into itself where C is a nonempty subset of E. Denoting the set of Bregman skew-attractive point of T by $B^B(S^|)$ i.e

$$B^B(S^|) = \cap\{B^B(T_t) : t \in S\}.$$

We now obtain the following skew-attractive point theorem for semigroup of mappings without continuity in a Banach space.

Theorem 4.6. *Let E be a reflexive Banach space and $f : E \to (-\infty, +\infty]$ be Legendre. Let S be a commutative semitopological semigroup with identity. Let $S = \{T_s : s \in S\}$ be a continuous representation of S as mapping of C into itself such that $S = \{T_s x : s \in S\}$ is bounded for some $x \in C$. Let μ be a mean on $C(S)$. Suppose that $\mu_s D_f(T_t y, T_s x) \le D_f(y, T_s x)$ $\forall y \in C$ and $t \in S$. Then $B^B(S^|) = \cap\{B(T_t) : t \in S\} \neq \emptyset$.*

Proof. Since $S = \{T_s x : s \in S\}$ is bounded subset of C for some $x \in C$. Put $x^* = \bigtriangledown fx$ and $y^* = \bigtriangledown fy$. Then by Definition 2.5 we have

$$\begin{aligned}
T_s^* T_t^* \quad &= \quad \bigtriangledown f T_s \bigtriangledown f^* \bigtriangledown f T_t \bigtriangledown f^* \\
&= \quad \bigtriangledown f T_s T_t \bigtriangledown f^* \\
&= \quad \bigtriangledown f T_{s+t} \bigtriangledown f^* \\
&= \quad T_{s+t}^* \qquad \forall s \in S.
\end{aligned}$$

Since $\bigtriangledown f$ is continuous then for any $y^* \in \bigtriangledown f(C)$ we have,

$\|T_s^* y^* - T_t^* y^*\| = \|\bigtriangledown f T_s \bigtriangledown f^* \bigtriangledown f(y) - \bigtriangledown f T_t \bigtriangledown f^* \bigtriangledown f(y)\| = |\bigtriangledown f T_s y - \bigtriangledown f T_t y\| \to 0$ as $s \to t$.

Therefore, $S^{|*} = \{T_s^* : s \in S\}$ is a continuous representation of S as mappings of $\bigtriangledown f(C)$ into $intdomf^*$. Further more Since $S = \{T_s : s \in S\}$ is bounded and

$\mu_s D_f(Tx,z) \leq \mu_s D_f(x,z)$ $\forall y \in C$ and $t \in S$, we have

$$\begin{aligned}\mu_s D_{f^*}(T_s^* x^*, T_t^* y^*) &= \mu_s D_{f^*}(\triangledown f T_s \triangledown f^* \triangledown fx, \triangledown f T_t \triangledown f^* \triangledown fy) \\ &= \mu_s D_{f^*}(\triangledown f T_s x, \triangledown f T_t y) \\ &= \mu_s D_f(T_t y, T_s x) \leq \mu_s D_f(y, T_s x) = \mu_s D_{f^*}(\triangledown f T_s x, \triangledown f y) \\ &= \mu_s D_{f^*}(\triangledown f T_s \triangledown f^* \triangledown fx, \triangledown f y) = \mu_s D_{f^*}(T_s^* x^*, y^*).\end{aligned}$$

Therefore $\mu_s D_{f^*}(T_s^* x^*, T_t^* y^*) \leq \mu_s D_{f^*}(T_s^* x^*, y^*)$ $\forall y^* \in \triangledown f(C)$ and $t \in S$.

Using Theorem 4.1 we see that

$$A^B(S^{|*}) = \cap\{A^B(T_t^*) : t \in S\} \neq \emptyset.$$

Since $\triangledown f : intdomf \to intdomf^*$ is bijection and using Lemma 4.5 we have

$$\begin{aligned}B^B(S^{|}) &= \cap\{B^B(T_t) : t \in S\} \\ &= \cap\{\triangledown f^* A^B(T_t^*) : t \in S\} \\ &= \triangledown f^*\{\cap(A^B(T_t^*)) : t \in S\} \\ &= \triangledown f^*(A^B(S^{|*})).\end{aligned}$$

Since $A^B(S^{|*})$ is nonempty $\Rightarrow B^B(S^{|}) \neq \emptyset$. This completes the proof.

□

Definition 4.7. Let E be reflexive Banach space and C be a nonempty subset of E. A mapping T from C into itself is called skew-generalized Bregman nonspreading mapping if $\exists \alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned}&\alpha D_f(Ty,Tx) + (1-\alpha) D_f(Ty,x) + \gamma\{D_f(Tx,Ty) - D_f(x,Ty)\} \\ \leq\ & \beta D_f(y,Tx) + (1-\beta) D_f(y,x) + \delta\{D_f(Tx,y) - D_f(x,y)\}, \forall x,y \in C.\end{aligned}$$

Theorem 4.8. *Let E be a reflexive Banach space and $f : E \to \bar{\mathbb{R}}$ be a convex and Gâteaux differentiable function and C be a nonempty subset of E. Let $T : C \to C$ be a skew-generalized Bregman nonspreading mapping, then $B^B(T) \neq \emptyset$ if and only if $\{T^n v_0\}$ bounded for some $v_0 \in C$.*

Proof. By Theorem 4.6 above, following the line of proof of (Theorem 5.4 in [14]) and considering the skew-generalized Bregman nonspreading mapping in Definition 4.7 above, the result follows. □

Remark 4.9. Observe that using Remark 1.2, we see that the results in this paper extend and improve all the results in Takahashi et al. [14] and many others.

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[1]Department of Mathematical Sciences, Bayero University, Kano, Nigeria.
E-mail address: bashiralik@yahoo.com

[2]Department of Science and Technology Education, Bayero University, Kano, Nigeria.
E-mail address: murtalaharbau@yahoo.com

[3]Department of Mathematics, Ahmadu Bello University, Zaria, Kaduna, Nigeria.
E-mail address: yulah121@gmail.com