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BOUNDEDNESS OF MULTILINEAR INTEGRAL OPERATORS AND THEIR COMMUTATORS ON GENERALIZED MORREY SPACES

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Abstract. In this paper, we obtain some boundedness of multilinear Calderón-Zygmund Operators, multilinear fractional integral operators and their commutators on generalized Morrey Spaces.

1. Introduction and preliminaries

Let T be a multilinear operator initially defined on the m-fold product of Schwartz spaces and taking values into the space of tempered distributions, i.e.

$$T : (\mathbb{R}^n) \times \cdots \times (\mathbb{R}^n) \to (\mathbb{R}^n).$$

In [5], it is said that a function K belongs to the class $m - CZK(A, \varepsilon)$ if

(1) $|K(y_0, y_1, \ldots, y_m)| \leqslant \frac{A}{(\sum_{k,l=0}^{m} |y_k - y_l|)^{mn}}$,

(2) if $|y_j - y_j'| \leqslant \frac{1}{2} \max_{0 \leqslant k \leqslant m} |y_j - y_k|$,

$$|K(y_0, \ldots, y_j, \ldots, y_m) - K(y_0, \ldots, y_j', \ldots, y_m)| \leqslant \frac{A|y_j - y_j'|^{\varepsilon}}{(\sum_{k,l=0}^{m} |y_k - y_l|)^{mn+\varepsilon}}$$

for some $\varepsilon > 0$ and $j = 0, 1, 2, \ldots, m$. In [9], the operator T is said to be an m-linear Calderón-Zygmund operator if there exists a function $K \in m - CZK(A, \varepsilon)$

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defined away from the diagonal $x = y_1 = y_2 \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$ such that

$$T(f_1, \ldots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) f_1(y_1) \ldots f_m(y_m) dy_1 \ldots y_m$$

for $x \notin \cap_{j=1}^m \operatorname{supp} f_j$ and that T extends to a bounded multilinear operator from $L^{q_1} \times \cdots \times L^{q_m}$ to L^q for some $1 \leqslant q_j < \infty$ with $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$.

It was shown in [5] that if $\frac{1}{r_1} + \cdots + \frac{1}{r_m} = \frac{1}{r}$, then an m-linear Calderón-Zygmund operator satisfies

$$T : L^{r_1} \times \cdots \times L^{r_m} \to L^r$$

when $1 < r_j < \infty$ for $j = 1, \ldots, m$ and

$$T : L^{r_1} \times \cdots \times L^{r_m} \to L^{r,\infty}$$

when $1 \le r_j < \infty$ for $j = 1, \ldots, m$ and at least one $r_j = 1$. In particular,

$$T : L^1 \times \cdots \times L^1 \to L^{1/m,\infty}.$$

The theory of multiple weight associated with m-linear Calderón-Zygmund operators was developed by Lerner, Ombrosi, Pérez, Torres and Trujillo-González in [9]. Let $1 < p_j < \infty$ for $j = 1, \ldots, m$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $\vec{p} = (p_1 \ldots, p_m)$, we say $\vec{\omega} = (\omega_1, ..., \omega_m) \in A_{\vec{p}}$ if

$$\sup_B \left(\frac{1}{|B|} \int_B v_{\vec{\omega}} \right)^{1/p} \prod_{j=1}^m \left(\frac{1}{|B|} \int_B \omega_j^{1-p'_j} \right)^{1/p'_j} < \infty,$$

where B is the ball in $\mathbb{R}^n$ and $v_{\vec{\omega}} = \prod_{j=1}^m \omega_j^{p/p_j}$. When $p_j = 1$, denote $p'_j = \infty$, $(\frac{1}{|B|} \int_B \omega_j^{1-p'_j})^{1/p'_j}$ is understood as $(\inf_B \omega_j)^{-1}$. They showed that if $\vec{\omega} \in A_{\vec{p}}$ then

$$\|T(\vec{f})\|_{L^p(v_{\vec{\omega}})} \leqslant C \prod_{j=1}^m \|f_j\|_{L^{p_j}(\omega_j)} . \tag{1.1}$$

If $1 \leqslant p_j < \infty$ for $j = 1, \ldots, m$ and at least one of the $p_j = 1$, they also proved

$$\|T(\vec{f})\|_{L^{p,\infty}(v_{\vec{\omega}})} \leqslant C \prod_{j=1}^m \|f_j\|_{L^{p_j}(\omega_j)} . \tag{1.2}$$

Let $\vec{b} = (b_1, \ldots, b_m)$ be a vector-valued locally integrable function. If $\vec{b} = (b_1, \ldots, b_m)$ in $(BMO)^m$, we denote $\|\vec{b}\|_{(BMO)^m} = \sup_{j=1,\ldots,m} \|b_j\|_{BMO}$ (see [9]), the definition of $\|\cdot\|_{BMO}$ see Section 2. The commutator generated by an m-linear Calderón-Zygmund operator T and a $(BMO)^m$ function $\vec{b}$ is defined by

$$T_{\vec{b}}(f_1, \ldots, f_m) = \sum_{j=1}^m T_{\vec{b}}^j(\vec{f}),$$

where each term is the commutator of b_j and T in the jth entry of T, that is,

$$T_{\vec{b}}^j(\vec{f}) = b_j T(f_1, \ldots, f_j, \ldots, f_m) - T(f_1, \ldots, b_j f_j, \ldots, f_m).$$

Pérez and Torres [12] proved that if $\vec{b} \in (BMO)^m$ then

$$T_{\vec{b}} : L^{p_1} \times \cdots \times L^{p_m} \to L^p$$

for $1 < p_j < \infty$ for $j = 1, \ldots, m$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $1 < p < \infty$. In [9], the authors proved that if $\vec{\omega} \in A_{\vec{p}}$ and $\vec{b} \in (BMO)^m$ then

$$\|T_{\vec{b}}(\vec{f})\|_{L^p(v_{\vec{\omega}})} \leqslant C\|\vec{b}\|_{(BMO)^m} \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\omega_j)} \tag{1.3}$$

for $1 < p_j < \infty$ for $j = 1, \ldots, m$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$.

Feuto [2] introduced the generalized weighted Morrey space $(L^q(\omega), L^p)^\alpha$. Let $1 \leqslant q \leqslant \alpha \leqslant p \leqslant \infty$, ω be a weight and $\omega(B) = \int_B \omega(x)dx$. The space $(L^q(\omega), L^p)^\alpha$ is defined to be the set of all measurable functions f satisfying $\|f\|_{(L^q(\omega),L^p)^\alpha} < \infty$, where

$$\|f\|_{(L^q(\omega),L^p)^\alpha} = \sup_{r>0} {}_r\|f\|_{(L^q(\omega),L^p)^\alpha}$$

with

$${}_r\|f\|_{(L^q(\omega),L^p)^\alpha} := \left[\int_{\mathbb{R}^n} \left(\omega(B(y,r))^{1/\alpha-1/q-1/p}\|f\chi_{B(y,r)}\|_{L^q(\omega)}\right)^p dy\right]^{1/p}.$$

When $\omega \equiv 1$, the space $(L^q, L^p)^\alpha$ was introduced in [3]. If $q < \alpha$ and $p = \infty$, the space $(L^q(\omega), L^\infty)^\alpha$ is just the weighted Morrey space $L^{q,\kappa}(\omega)$ with $\kappa = 1 - q/\alpha$ defined by Komori and Shirai [8].

Similarly, the weak space $(L^{q,\infty}(\omega), L^p)^\alpha$ is defined with

$${}_r\|f\|_{(L^{q,\infty}(\omega),L^p)^\alpha} := \left[\int_{\mathbb{R}^n} \left(\omega(B(y,r))^{1/\alpha-1/q-1/p}\|f\chi_{B(y,r)}\|_{L^{q,\infty}(\omega)}\right)^p dy\right]^{1/p}.$$

When $q = 1$, the space $(L^{1,\infty}(\omega), L^p)^\alpha$ was introduced in [2].

Feuto has proved in [2] that Calderón-Zygmund singular integral operators, Marcinkiewicz operators, the maximal operators associated to Bochner-Riesz operators and their commutators are bounded on $(L^q(\omega), L^p)^\alpha$.

A nature question is whether the m-linear Calderón-Zygmund operator T and its commutator $T_{\vec{b}}$ have the similar properties. Now, we first introduce the following space.

Definition 1.1. Let $1 \leqslant p \leqslant \alpha \leqslant q \leqslant \infty$. The space $\left(L^{\vec{p}}(u, \vec{\omega}), L^q\right)^\alpha$ is defined as the set of vector-valued measurable functions $\vec{f} = (f_1, \ldots, f_m)$ satisfying $\|\vec{f}\|_{\left(L^{\vec{p}}(u,\vec{\omega}),L^q\right)^\alpha} := \sup_{r>0} {}_r\|\vec{f}\|_{\left(L^{\vec{p}}(u,\vec{\omega}),L^q\right)^\alpha} < \infty$ with

$${}_r\|\vec{f}\|_{\left(L^{\vec{p}}(u,\vec{\omega}),L^q\right)^\alpha} := \left[\int_{\mathbb{R}^n} \left(u(B(y,r))^{1/\alpha-1/p-1/q} \prod_{i=1}^{m} \|f_i\chi_{B(y,r)}\|_{L^{p_i}(\omega_i)}\right)^q dy\right]^{1/q}$$

for $r > 0$, where u is a weight, $\vec{\omega} = (\omega_1, \ldots, \omega_m)$ is vector weight, $\vec{p} = (p_1, \ldots, p_m)$ and $p_i \geq 1$ for $i = 1, \ldots, m$.

When $m = 1$ and $u = \omega$, it is just the space $(L^q(\omega), L^p)^\alpha$ in [2]. When $m = 1$, $q = \infty$ and $p < \alpha$, the space $(L^p(u,\omega), L^\infty)^\alpha$ are the weighted Morrey space $L^{p,\kappa}(u,\omega)$ with $\kappa = 1 - p/\alpha$ in [8]. By the works above, we state our main results as follows.

Theorem 1.2. *Let T be an m-linear Calderón-Zygmund operator, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $\vec{\omega} \in A_{\vec{p}}$.*
(1) If $1 < p_j < \infty$, $j = 1, \dots, m$ and $p \leqslant \alpha < q \leqslant \infty$, then T is bounded from $\left(L^{\vec{p}}(v_{\vec{\omega}}, \vec{\omega}), L^q\right)^\alpha$ to $(L^p(v_{\vec{\omega}}), L^q)^\alpha$;
(2) if $1 \leqslant p_j < \infty$, $j = 1, \dots, m$ and at least one of the $p_j = 1$, $p \leqslant \alpha < q \leqslant \infty$, then T is bounded from $\left(L^{\vec{p}}(v_{\vec{\omega}}, \vec{\omega}), L^q\right)^\alpha$ to $(L^{p,\infty}(v_{\vec{\omega}}), L^q)^\alpha$.

Theorem 1.3. *Let $T_{\vec{b}}$ be a multilinear commutator, $\vec{b} \in (BMO)^m$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $1 < p_j < \infty$ and $\vec{\omega} \in A_{\vec{p}}$. If $p \leqslant \alpha < q \leqslant \infty$, then $T_{\vec{b}}$ is bounded from $\left(L^{\vec{p}}(v_{\vec{\omega}}, \vec{\omega}), L^q\right)^\alpha$ to $(L^p(v_{\vec{\omega}}), L^q)^\alpha$.*

Remark 1.4. When $m = 1$, Theorem 1.2 is just the Theorem **2.1** in [2] and Theorem 1.3 is just the Theorem **2.5** in [2].

Another purpose of this paper is to establish the boundedness of multilinear fractional integral operators and their commutators on the generalized Morrey spaces. Let us introduce the following definition.

Definition 1.5. For $1 \leqslant q \leqslant \beta \leqslant \gamma \leqslant \infty$, we denote the space $\left(L^{q,\vec{p}}(u,\vec{v}), L^\gamma\right)^\beta$ as the space of all vector-valued measurable functions $\vec{f} = (f_1, \dots, f_m)$ satisfying $\|\vec{f}\|_{\left(L^{q,\vec{p}}(u,\vec{v}), L^\gamma\right)^\beta} = \sup_{r>0} {}_r\|\vec{f}\|_{\left(L^{q,\vec{p}}(u,\vec{v}), L^\gamma\right)^\beta} < \infty$ with

$$ {}_r\|\vec{f}\|_{\left(L^{q,\vec{p}}(u,\vec{v}), L^\gamma\right)^\beta} := \left[\int_{\mathbb{R}^n} \left(u(B(y,r))^{1/\beta - 1/q - 1/\gamma} \prod_{i=1}^m \|f_i \chi_{B(y,r)}\|_{L^{p_i}(v_i)} \right)^\gamma dy \right]^{1/\gamma} $$

for $r > 0$, where u is a weight, $\vec{v} = (v_1 \dots, v_m)$ is vector weight and $\vec{p} = (p_1, \dots, p_m)$ with $p_i \geq 1 (i = 1, \dots, m)$.

When $m = 1$, $\gamma = \infty$, $q < \beta$ the space $(L^{q,p}(u,v), L^\infty)^\beta$ is the weighted Morrey space $L^{q,\kappa}(u,v)$ with $\kappa = p/q - p/\beta$ in [8].

Kenig and Stein [7], Grafakos [4], Grafakos and Kalton [6] studied the multilinear fractional integral operators. Their works originated from the bilinear fractional integral operator

$$ \mathcal{B}_\alpha(f,g) = \int_{\mathbb{R}^n} \frac{f(x+t)g(x-t)}{|t|^{n-\alpha}} dt. $$

They showed that $\mathcal{B}_\alpha$ is bounded from $L^{p_1} \times L^{p_2}$ to L^q, where $1/q = 1/p_1 + 1/p_2 - \alpha/n$. A further multilinear extension of ordinary fractional integration is

$$ \mathcal{I}_\alpha \vec{f}(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1) f_2(y_2) \cdots f_m(y_m)}{\left(\sum_{i=1}^m |x - y_i|\right)^{mn-\alpha}} d\vec{y}, $$

where $0 < \alpha < mn$. Moen [10] showed that

$$\left(\int_{\mathbb{R}^n}\left(|\mathcal{I}_\alpha\vec{f}(x)|\Big(\prod_{i=1}^m\omega_i(x)\Big)\right)^q dx\right)^{1/q} \leqslant C\prod_{i=1}^m\left(\int_{\mathbb{R}^n}(|f_i(x)|\omega_i(x))^{p_i}\,dx\right)^{1/p_i}$$

if and only if $\vec{\omega}$ satisfies $A_{(\vec{p},q)}$ condition:

$$\sup_B\left(\frac{1}{|B|}\int_B\prod_{i=1}^m\omega_i^q(x)dx\right)^{1/q}\prod_{i=1}^m\left(\frac{1}{|B|}\int_B\omega_i^{-p_i'}(x)dx\right)^{1/p'_i} < \infty.$$

The corresponding multilinear fractional integral with homogeneous kernels is defined by

$$\mathcal{I}_{\Omega,\alpha}\vec{f}(x) = \int_{(\mathbb{R}^n)^m}\frac{\prod_{i=1}^m\Omega_i(x-y_i)f_i(y_i)}{\big(\sum_{i=1}^m|x-y_i|\big)^{mn-\alpha}}\mathrm{d}\vec{y},$$

where each $\Omega_i \in L^s(S^{n-1})$ $(i = 1,\dots,m)$ for some $s > 1$ is a homogeneous function of degree zero on $\mathbb{R}^n$, i.e. $\Omega_i(\lambda x) = \Omega_i(x)$ for any $\lambda > 0$, $x \in \mathbb{R}^n$ and S^{n-1} denotes the unit sphere in $\mathbb{R}^n$ $(n \geq 2)$. Let $\vec{b} = (b_1,\dots,b_m)$ be a vector-valued locally integrable function. The multilinear commutator of $\mathcal{I}_\alpha$ is defined as

$$\mathcal{I}_{\vec{b},\alpha}(f_1,\dots,f_m) = \sum_j^m \mathcal{I}^j_{\vec{b},\alpha}(\vec{f}),$$

where each term is the commutator of b_j and $\mathcal{I}_\alpha$ in the jth entry of $\mathcal{I}_\alpha$, that is,

$$\mathcal{I}^j_{\vec{b},\alpha}(\vec{f}) = b_j\mathcal{I}_\alpha(f_1,\dots,f_j,\dots,f_m) - \mathcal{I}_\alpha(f_1,\dots,b_jf_j,\dots,f_m).$$

Chen and Xue [1] proved the weighted estimates of $\mathcal{I}_{\Omega,\alpha}$ and $\mathcal{I}_{\vec{b},\alpha}$. For $1 \leqslant s' < p_1,\dots,p_m < \infty$, if $\vec{\omega}^{s'} = (\omega_1^{s'},\dots,\omega_m^{s'}) \in A_{(\vec{p}/s',q/s')} \cap A_{(\vec{p}/s',q_\varepsilon/s')} \cap A_{(\vec{p}/s',q_{-\varepsilon}/s')}$, $1/q_\varepsilon = 1/p - (\alpha+\varepsilon)/n$ and $1/q_{-\varepsilon} = 1/p - (\alpha-\varepsilon)/n$, $0 < \varepsilon < \min\{\alpha, mn-\alpha\}$, they showed that

$$\|\mathcal{I}_{\Omega,\alpha}(\vec{f})\|_{L^q((\prod_{i=1}^m\omega_i)^q)} \leqslant C\prod_{i=1}^m\|f_i\|_{L^{p_i}(\omega_i^{p_i})} \tag{1.4}$$

and for $s > 1$, with $0 < s\alpha < mn$, if $\vec{\omega}^s \in A_{(\vec{p}/s,q/s)}$ and $\prod_i^m\omega_i^q \in A_\infty$, they got

$$\|\mathcal{I}_{\vec{b},\alpha}(\vec{f})\|_{L^q((\prod_{i=1}^m\omega_i)^q)} \leqslant C\|\vec{b}\|_{(BMO)^m}\prod_{i=1}^m\|f_i\|_{L^{p_i}(\omega_i^{p_i})}\,. \tag{1.5}$$

Our main results for $\mathcal{I}_{\Omega,\alpha}$ and $\mathcal{I}_{\vec{b},\alpha}$ are as follows.

Theorem 1.6. *Let $0 < \alpha < nm$, $1 \leqslant s' < p_1,\dots,p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$ and $1/q = 1/p - \alpha/n$. Denote $\vec{\omega}^{s'} = (\omega_1^{s'},\dots,\omega_m^{s'})$, and $\vec{p}/s' = (p_1/s',\dots,p_m/s')$. Assume $\vec{\omega}^{s'} \in A_{(\vec{p}/s',q/s')} \cap A_{(\vec{p}/s',q_\varepsilon/s')} \cap A_{(\vec{p}/s',q_{-\varepsilon}/s')}$, where $0 < \varepsilon < \min\{\alpha, mn-\alpha\}$, $1/q_\varepsilon = 1/p - (\alpha+\varepsilon)/n$ and $1/q_{-\varepsilon} = 1/p - (\alpha-\varepsilon)/n$. If $q \leqslant \beta < \gamma \leqslant \infty$, then $\mathcal{I}_{\Omega,\alpha}$ is bounded from $(L^{q,\vec{p}}(\prod_{i=1}^m\omega_i^q,\vec{\omega}^{\vec{p}}), L^\gamma)^\beta$ to $(L^q(\prod_i^m\omega_i^q), L^\gamma)^\beta$, where $\vec{\omega}^{\vec{p}} = (\omega_1^{p_1},\dots,\omega_m^{p_m})$.*

Remark 1.7. Since $A_{\vec{p}}$ is not monotone increasing with the natural partial order, we have to assume that $\vec{\omega}^{s'} \in A_{(\vec{p}/s', q_{\varepsilon}/s')} \cap A_{(\vec{p}/s', q_{-\varepsilon}/s')}$ *(see* [9] *and* [1]*).*

Theorem 1.8. *Let* $0 < \alpha < nm$, $1 < p_i < \infty$ *for* $i = 1, \ldots, m$, $1/p = 1/p_1 + \cdots + 1/p_m$ *and* $1/q = 1/p - \alpha/n$*. For* $s > 1$ *with* $0 < s\alpha < mn$*, assume* $\vec{\omega}^s \in A_{(\vec{p}/s, q/s)}$ *and* $\prod_{i=1}^m \omega_i^q \in A_\infty$*, if* $q \leqslant \beta < \gamma \leqslant \infty$*, then* $\mathcal{I}_{\vec{b},\alpha}$ *is bounded from* $(L^{q,\vec{p}}(\prod_{i=1}^m \omega_i^q, \vec{\omega}^{\vec{p}}), L^\gamma)^\beta$ *to* $(L^q(\prod_i^m \omega_i^q), L^\gamma)^\beta$*, where* $\vec{\omega}^{\vec{p}} = (\omega_1^{p_1}, \ldots, \omega_m^{p_m})$*.*

2. Notations and Preliminaries

We first recall the definition of A_p weight. A nonnegative locally integrable function ω belongs to A_p $(p > 1)$ if

$$\sup_B \left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{1-p'}dx\right)^{p-1} < \infty,$$

where p' is the conjugate index of p i.e. $1/p + 1/p' = 1$. We say that $\omega \in A_1$ if there is a constant $C > 0$ such that

$$\frac{1}{|B|}\int_B \omega(x)\mathrm{d}x \leqslant C \inf_{x\in B} \omega(x).$$

If $\omega \in A_p$, then there exists $\delta > 0$ such that

$$\frac{\omega(E)}{\omega(B)} \lesssim \left(\frac{|E|}{|B|}\right)^\delta \tag{2.1}$$

for any measurable subset E of a ball B, where $\omega(B) = \int_B \omega(x)\mathrm{d}x$. Since the A_p classes are increasing with respect to p, we use the following notation by $A_\infty = \cup_{p>1} A_p$. $A \lesssim B$ means $A \leq CB$, where C is a positive constant independent of the main parameters. For $\lambda > 0$ and a ball $B \subset \mathbb{R}^n$, we write λB for the ball with same center as B and radius λ times radius of B.

Obviously, if $m = 1$, $A_{\vec{p}}$ is the classical A_p class. $A_{\vec{p}}$ has the following characterization.

Lemma 2.1. [9] *Let* $\vec{\omega} = (\omega_1, \ldots, \omega_m)$*. Then* $\vec{\omega} \in A_{\vec{p}}$ *if and only if*

$$\omega_j^{1-p'_j} \in A_{mp'_j} \text{ and } v_{\vec{\omega}} \in A_{mp},$$

where the condition $\omega_j^{1-p'_j} \in A_{mp_j'}$ *is understood as* $\omega_j^{1/m} \in A_1$ *in the case* $p_j = 1$*.*

Lemma 2.2. [9] *Assume that* $\vec{\omega} = (\omega_1, \ldots, \omega_m)$ *satisfies* $A_{\vec{p}}$ *condition. Then there exists a finite constant* $r > 1$ *such that* $\vec{\omega} \in A_{\vec{p}/r}$*.*

The class $A(p, q)$ was also first introduced by Muckenhout and Wheeden in [11]. A weight function ω belongs to $A(p, q)$ for $1 < p < q < \infty$ if there exists a constant C such that

$$\sup_B \left(\frac{1}{|B|}\int_B \omega(x)^q dx\right)^{1/q}\left(\frac{1}{|B|}\int_B \omega(x)^{-p'}dx\right)^{1/p'} < \infty$$

for every ball $B \subset \mathbb{R}^n$.

The multiple weight class $A_{(\vec{p},q)}$ is defined as follows.

Definition 2.3. [10] Let $1 < p_i < \infty$ for $i = 1, \ldots, m$ and q be a number $\frac{1}{m} < p \leqslant q < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$, $\vec{p} = (p_1, \ldots, p_m)$. We say that a vector of weights $\overrightarrow{\omega} = (\omega_1, \cdots, \omega_m)$ is in the class $A_{(\vec{p},q)}$, if

$$\sup_B \left(\frac{1}{|B|} \int_B \prod_{i=1}^m \omega_i^q(x)dx \right)^{1/q} \prod_{i=1}^m \left(\frac{1}{|B|} \int_B \omega_i^{-p'_i}(x)dx \right)^{1/p'_i} < \infty.$$

When $m = 1$, the $A_{(\vec{p},q)}$ is the classical $A(p, q)$ weight. Moen [10] got the following property of $A_{(\vec{p},q)}$.

Lemma 2.4. [10] *Suppose* $1 < p_i < \infty$ $(i = 1, \ldots, m)$ *and* $\vec{\omega} \in A_{(\vec{p},q)}$, *then*

$$\omega_i^{-p'_i} \in A_{mp'_i} \text{ and } \left(\prod_{i=1}^m \omega_i \right)^q \in A_{mq}.$$

A locally integrable function b belongs to in BMO if

$$\|b\|_{BMO} = \sup_B \frac{1}{|B|} \int\limits_B |b(x) - b_B|dx < \infty,$$

where $b_B = \frac{1}{|B|} \int_B b(x)dx$ and the supremum is taken over all ball in $\mathbb{R}^n$. In order to prove the results for commutators, we need the following properties of BMO. For $b \in BMO$, $1 < p < \infty$ and $\omega \in A_\infty$ we get

$$\|b\|_{BMO} \sim \sup_B \left(\frac{1}{|B|} \int_B |b(x) - b_B|^p dx \right)^{1/p}$$

and for all balls B

$$\left(\frac{1}{\omega(B)} \int_B |b(x) - b_B|^p \omega(x)dx \right)^{1/p} \leq C\|b\|_{BMO}. \tag{2.2}$$

For all nonnegative integers k, we obtain

$$|b_{2^{k+1}B} - b_B| \leq C(k + 1)\|b\|_{BMO}, \tag{2.3}$$

where $\omega(B) = \int_B \omega(x)\mathrm{d}x$, $b_B = \frac{1}{|B|} \int_B b(x)\mathrm{d}x$ (see [2]).

3. Proof of the Main Results

Proof of Theorem 1.2. (1) Let $B = B(y, r)$ be a ball of $\mathbb{R}^n$, $f_i = f_i\chi_{2B} + f_i\chi_{(2B)^c}$ and denote $f_i\chi_{2B}$ by f_i^0 and $f_i\chi_{(2B)^c}$ by f_i^∞ $(i = 1, \ldots, m)$, χ_E denotes the characteristic function of set E. For $x \in B(y, r)$, we have

$$\begin{aligned} |T\vec{f}(x)| &\leqslant |T(f_1^0, \ldots, f_m^0)(x)| + \sum_{\alpha_1,\ldots,\alpha_m \in \{0,\infty\}} |T(f_1^{\alpha_1}, \ldots, f_m^{\alpha_m})(x)| \\ &\quad + |T(f_1^\infty, \ldots, f_m^\infty)(x)| \\ &= I + II + III, \end{aligned}$$

where $\alpha_1, \dots, \alpha_m$ are not all equal to 0 or ∞ at the same time. We first estimate III. We have

$$\begin{aligned} III &= \left| \int_{(\mathbb{R}^n \setminus 2B)^m} \frac{f_1(y_1)\cdots f_m(y_m)}{(\sum_{i=1}^m |x-y_i|)^{mn}} d\vec{y} \right| \\ &\leqslant \int_{(\mathbb{R}^n \setminus 2B)^m} \frac{|f_1(y_1)\cdots f_m(y_m)|}{(\sum_{i=1}^m |x-y_i|)^{mn}} d\vec{y} \\ &= \sum_{k=1}^{\infty} \int_{(2^{k+1}B \setminus 2^k B)^m} \frac{|f_1(y_1)\cdots f_m(y_m)|}{(\sum_{i=1}^m |x-y_i|)^{mn}} d\vec{y} \\ &\lesssim \sum_{k=1}^{\infty} \prod_{i=1}^m \int_{2^{k+1}B \setminus 2^k B} \frac{|f_i(y_i)|}{|x-y_i|^n} dy_i \\ &\lesssim \sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|^m} \prod_{i=1}^m \int_{2^{k+1}B} |f_i(y_i)| dy_i, \end{aligned}$$

Since $2^{k-1}r \leqslant |x-y_i| \leqslant 2^{k+2}r$. The Hölder inequality gives us that

$$\begin{aligned} &\int_{2^{k+1}B} |f_i(y_i)| dy_i \\ &\quad = \int_{2^{k+1}B} |f_i(y_i)| \omega(y_i)^{1/p_i} \omega(y_i)^{-1/p_i} dy_i \\ &\quad \leqslant \left(\int_{2^{k+1}B} |f(y_i)|^{p_i} \omega_i(y_i) dy_i \right)^{1/p_i} \left(\int_{2^{k+1}B} \omega(y_i)^{-p'_i/p_i} dy_i \right)^{1/p'_i}. \end{aligned} \tag{3.1}$$

By $m = 1/p + 1/p'_1 + \cdots + 1/p'_m$ and the definition of $A_{\vec{p}}$ condition, we obtain

$$III \leq \sum_{k=1}^{\infty} \frac{1}{(\int_{2^{k+1}B} v_{\vec{\omega}})^{1/p}} \prod_{i=1}^m \|f_i \chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)}.$$

For II, we just consider this case: $\alpha_i = \infty$ for $i = 1, \dots, l$ and $\alpha_j = 0$ for $j = l+1, \dots, m$.

$$\begin{aligned} &|T(f_1^\infty, \dots, f_l^\infty, f_{l+1}^0, \dots, f_m^0)(x)| \\ &\quad = \left| \int_{(\mathbb{R}^n \setminus 2B)^l} \int_{(2B)^{m-l}} \frac{f_1(y_1)\dots f_m(y_m)}{(\sum_{i=1}^m |x-y_i|)^{mn}} d\vec{y} \right| \\ &\quad \leqslant \int_{(\mathbb{R}^n \setminus 2B)^l} \int_{(2B)^{m-l}} \frac{|f_1(y_1)\dots f_m(y_m)|}{(\sum_{i=1}^m |x-y_i|)^{mn}} d\vec{y} \\ &\quad \lesssim \prod_{i=l+1}^m \int_{2B} |f_i(y_i)| dy_i \sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|^m} \prod_{i=1}^l \int_{2^{k+1}B \setminus 2^k B} |f_i(y_i)| dy_i \\ &\quad \lesssim \sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|^m} \prod_{i=1}^m \int_{2^{k+1}B} |f_i(y_i)| dy_i, \end{aligned}$$

In view of (3.1) and the definition of $A_{\vec{p}}$ condition, we have

$$II \lesssim \sum_{k=1}^{\infty} \frac{1}{(\int_{2^{k+1}B} v_{\vec{\omega}})^{1/p}} \prod_{i=1}^{m} \|f_i \chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)}.$$

Combining all the cases together, we obtain

$$\begin{aligned} |T\vec{f}(x)| &\lesssim \left| \int_{(2B)^m} K(x, y_1, \ldots, y_m) f_1(y_1) \ldots f_m(y_m) d\vec{y} \right| \\ &\quad + \sum_{k=1}^{\infty} \frac{1}{(\int_{2^{k+1}B} v_{\vec{\omega}})^{1/p}} \prod_{i=1}^{m} \|f\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)}. \end{aligned} \tag{3.2}$$

Taking $L^p(v_{\vec{\omega}})$ norm on the ball $B(y, r)$ in both sides of (3.2), by (1.1), we get

$$\begin{aligned} \|T\vec{f}\chi_{B(y,r)}\|_{L^p(v_{\vec{\omega}})} &\lesssim \prod_{i=1}^{m} \|f_i \chi_{B(y,2r)}\|_{L^{p_i}(\omega_i)} \\ &\quad + \sum_{k=1}^{\infty} \frac{(\int_B v_{\vec{\omega}})^{1/p}}{(\int_{2^{k+1}B} v_{\vec{\omega}})^{1/p}} \prod_{i=1}^{m} \|f_i \chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)}. \end{aligned} \tag{3.3}$$

Multiplying both sides of (3.3) by $v_{\vec{\omega}}(B)^{1/\alpha - 1/q - 1/p}$, by Lemma 2.1 and (2.1), we obtain

$$\begin{aligned} & v_{\vec{\omega}}(B)^{1/\alpha - 1/q - 1/p} \|T\vec{f}\chi_{B(y,r)}\|_{L^p(v_{\vec{\omega}})} \\ &\lesssim \sum_{k=0}^{\infty} \frac{v_{\vec{\omega}}(2^{k+1}B)^{1/\alpha - 1/q - 1/p}}{2^{nk\delta(1/\alpha - 1/q)}} \prod_{i=1}^{m} \|f_i \chi_{B(y,2^{k+1}r)}\|_{L^{p_i}(\omega_i)}. \end{aligned} \tag{3.4}$$

Since $\sum_{k=0}^{\infty} \frac{1}{2^{nk\delta(1/\alpha - 1/q)}} < \infty$, we obtain the expected result by (3.4).

(2) For $\lambda > 0$, by (3.2) and (1.2), we get

$$\begin{aligned} \lambda v_{\vec{\omega}}(x \in B(y, r) : |T\vec{f}(x)| > \lambda)^{1/p} &\lesssim \prod_{i=1}^{m} \|f_i \chi_{B(y,2r)}\|_{L^{p_i}(\omega_i)} \\ &\quad + \sum_{k=1}^{\infty} \frac{(\int_B v_{\vec{\omega}})^{1/p}}{(\int_{2^{k+1}B} v_{\vec{\omega}})^{1/p}} \prod_{i=1}^{m} \|f_i \chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)}. \end{aligned}$$

That is,

$$\begin{aligned} \|Tf\chi_{B(y,r)}\|_{L^{p,\infty}(v_{\vec{\omega}})} &\lesssim \prod_{i=1}^{m} \|f_i \chi_{B(y,2r)}\|_{L^{p_i}(\omega_i)} \\ &\quad + \sum_{k=1}^{\infty} \frac{(\int_B v_{\vec{\omega}})^{1/p}}{(\int_{2^{k+1}B} v_{\vec{\omega}})^{1/p}} \prod_{i=1}^{m} \|f_i \chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)}. \end{aligned} \tag{3.5}$$

Multiplying both sides of (3.5) by $v_{\vec{\omega}}(B)^{1/\alpha - 1/q - 1/p}$, we conclude as in the case (1).

Proof of Theorem 1.3. It suffices to prove the theorem for $T_{\vec{b}}^{j}$. For $B = B(y, r)$, $x \in B$

$$\begin{aligned} T_{\vec{b}}^{j}(\vec{f})(x) &= T_{\vec{b}}^{j}(\vec{f}\chi_{2B})(x) + \sum_{\alpha_1,\dots,\alpha_m \in \{0,\infty\}} \big((b_j(x)T(f_1^{\alpha_1},\dots,f_j^{\alpha_j},\dots,f_m^{\alpha_m}) \\ &\quad -T(f_1^{\alpha_1},\dots,b_jf_j^{\alpha_j},\dots,f_m^{\alpha_m})(x)\big) \\ &\quad +b_j(x)T(f_1^{\infty},\dots,f_j^{\infty},\dots,f_m^{\infty}) - T(f_1^{\infty},\dots,b_jf_j^{\infty},\dots,f_m^{\infty})(x) \\ &= I^{'} + II^{'} + III^{'}, \end{aligned}$$

where $\alpha_1, \dots, \alpha_m$ are not all equal to 0 or ∞ at the same time. We first deal with $III^{'}$. By estimate of III in Theorem 1.2 and $|b_j(y_j) - b_B| \leqslant |b_j(y_j) - b_{2^{k+1}B}| + |b_{2^{k+1}B} - b_B|$,

$$\begin{aligned} |III^{'}| &\leq |(b_j(x) - b_B)T(f_1^{\infty},\dots,f_j^{\infty},\dots,f_m^{\infty})| \\ &\quad +|T(f_1^{\infty},\dots,(b_j - b_B)f_j^{\infty},\dots,f_m^{\infty})(x)| \\ &\leq |(b_j(x) - b_B)| \sum_{k=1}^{\infty} \frac{1}{(\int_{2^{k+1}B} v_{\vec{\omega}})^{1/p}} \prod_{i=1}^{m} \|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)} \\ &\quad + \sum_{k=1}^{\infty} \frac{|b_{2^{k+1}B} - b_B|}{(\int_{2^{k+1}B} v_{\vec{\omega}})^{1/p}} \prod_{i=1}^{m} \|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)} \\ &\quad + \sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|^m} \int_{(2^{k+1}B)^m} \prod_{\substack{j=1,\\ j\neq i}}^{m} |f_i(y_i)f_j(y_j)\big(b_j(y_j) - b_{2^{k+1}B}\big)|d\vec{y}. \end{aligned}$$

There exists an $s > 1$ such that $\vec{\omega} \in A_{\vec{p}/s}$ by Lemma 2.2. Then characterization $A_{\vec{p}/s}$ and (2.2) yield

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|^m} \int_{(2^{k+1}B)^m} \prod_{\substack{j=1,\\ j\neq i}}^{m} |f_i(y_i)f_j(y_j)\big(b_j(y_j) - b_{2^{k+1}B}\big)|d\vec{y} \\ &\quad \leq \sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|^{m/s}} \left(\prod_{\substack{j=1,\\ j\neq i}}^{m} \int_{2^{k+1}B} |f_i(y_i)|^s dy_i \right)^{1/s} \\ &\qquad \times \left(\int_{2^{k+1}B} |(b_j(y_j) - b_{2^{k+1}B})f_j(y_j)|^s dy_j \right)^{1/s} \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|^{m/s}} \prod_{\substack{j=1,\\ j\neq i}}^{m} \left(\int_{2^{k+1}B} |f_i(y_i)|^{p_i}\omega_i(y_i)dy_i\right)^{1/p_i} \\
&\quad \left(\int_{2^{k+1}B} \omega_i(y_i)^{-s/(p_i-s)})\,\mathrm{d}y_i\right)^{(p_i-s)/p_i s} \\
&\quad \times\left(\int_{2^{k+1}B} |b_j(y_j)-b_{2^{k+1}B}|^{p_j s/(p_j-s)}\omega_i(y_j)^{-s/(p_j-s)}dy_j\right)^{1/s} \\
&\quad \left(\int_{2^{k+1}B} |f_j(y_j)|^{p_j}\omega_j(y_j)dy_j\right)^{1/p_j} \\
&\lesssim \|b_j\|_{BMO} \sum_{k=1}^{\infty} \frac{1}{(\int_{2^{k+1}B} v_{\vec{\omega}})^{1/p}} \prod_{i=1}^{m} \|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)}. \qquad (3.6)
\end{aligned}$$

So we have

$$\begin{aligned}
|III^{'}| \;\leq\; & |(b_j(x)-b_B)| \sum_{k=1}^{\infty} \frac{1}{(\int_{2^{k+1}B} v_{\vec{\omega}})^{1/p}} \prod_{i=1}^{m} \|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)} \\
&+\sum_{k=1}^{\infty} \frac{|b_{2^{k+1}B}-b_B|}{(\int_{2^{k+1}B} v_{\vec{\omega}})^{1/p}} \prod_{i=1}^{m} \|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)} \\
&+\|b_j\|_{BMO} \sum_{k=1}^{\infty} \frac{1}{(\int_{2^{k+1}B} v_{\vec{\omega}})^{1/p}} \prod_{i=1}^{m} \|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)}\,.
\end{aligned}$$

For $II^{'}$, we just consider this case: $\alpha_i = \infty$ for $i = 1,\ldots,l$ and $\alpha_j = 0$ for $j = l+1,\ldots,m$. There are two cases:

$b_j(x)T(f_1^{\infty},\ldots,f_j^{\infty},\ldots,f_l^{\infty},f_{l+1}^0,\ldots,f_m^0) - T(f_1^{\infty},\ldots,b_jf_j^{\infty},\ldots,f_l^{\infty},f_{l+1}^0,\ldots,f_m^0)(x)$

or

$b_j(x)T(f_1^{\infty},\ldots,f_l^{\infty},f_{l+1}^0,\ldots,f_j^0,\ldots,f_m) - T(f_1^{\infty},\ldots,f_l^{\infty},f_{l+1}^0,\ldots,b_jf_j^0,\ldots,f_m^0)(x).$

We just consider the following case, the other case completely analogous.

$$\begin{aligned}
&|b_j(x)T(f_1^{\infty},\ldots,f_j^{\infty},\ldots,f_l^{\infty},f_{l+1}^0,\ldots,f_m^0) - \\
&\quad T(f_1^{\infty},\ldots,b_jf_j^{\infty},\ldots,f_l^{\infty},f_{l+1}^0,\ldots,f_m^0)(x)| \\
&\leq |(b_j(x)-b_B)T(f_1^{\infty},\ldots,f_j^{\infty},\ldots,f_l^{\infty},f_{l+1}^0,\ldots,f_m^0)| \\
&\quad +|T(f_1^{\infty},\ldots,(b_j-b_B)f_j^{\infty},\ldots,f_l^{\infty},f_{l+1}^0,\ldots,f_m^0)(x)| \\
&\leq |(b_j(x)-b_B)| \sum_{k=1}^{\infty} \frac{1}{(\int_{2^{k+1}B} v_{\vec{\omega}})^{1/p}} \prod_{i=1}^{m} \|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)} \\
&\quad +\sum_{k=1}^{\infty} \frac{|b_{2^{k+1}B}-b_B|}{(\int_{2^{k+1}B} v_{\vec{\omega}})^{1/p}} \prod_{i=1}^{m} \|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)} \\
&\quad +\sum_{k=1}^{\infty} \frac{\prod_{i=l+1}^{m}\int_{2B}|f_i(y_i)|dy_i}{|2^{k+1}B|^m} \int_{(2^{k+1}B)^m} \prod_{\substack{j=1,\\ j\neq i}}^{l} |f_i(y_i)f_j(y_j)\big(b_j(y_j)-b_{2^{k+1}B}\big)|d\vec{y}.
\end{aligned}$$

The estimate for

$$\sum_{k=1}^{\infty}\frac{\prod_{i=l+1}^{m}\int_{2B}|f_i(y_i)|dy_i}{|2^{k+1}B|^m}\int_{(2^{k+1}B)^m}\prod_{\substack{j=1,\\ j\neq i}}^{l}|f_i(y_i)f_j(y_j)\big(b_j(y_j)-b_{2^{k+1}B}\big)|d\vec{y}$$

is similar to (3.6). We get

$$\begin{aligned}
&\sum_{k=1}^{\infty}\frac{\prod_{i=l+1}^{m}\int_{2B}|f_i(y_i)|dy_i}{|2^{k+1}B|^m}\int_{(2^{k+1}B)^m}\prod_{\substack{j=1,\\ j\neq i}}^{l}|f_i(y_i)f_j(y_j)\big(b_j(y_j)-b_{2^{k+1}B}\big)|d\vec{y}\\
&\quad\leq \sum_{k=1}^{\infty}\frac{1}{|2^{k+1}B|^m}\prod_{\substack{j=1,\\ j\neq i}}^{m}\int_{2^{k+1}B}|f_i(y_i)|dy_i\times\int_{2^{k+1}B}|b_j(y_j)-b_{2^{k+1}B}|f(y_j)dy_j\\
&\quad\lesssim \|b_j\|_{BMO}\sum_{k=1}^{\infty}\frac{1}{(\int_{2^{k+1}B}v_{\vec{\omega}})^{1/p}}\prod_{i=1}^{m}\|f\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)},
\end{aligned}\tag{3.7}$$

so we have

$$\begin{aligned}
|T_{\vec{b}}^{j}(\vec{f})(x)| \leq\ & |T_{\vec{b}}^{j}(\vec{f}\chi_{2B})(x)|\\
&+|(b_j(x)-b_B)|\sum_{k=1}^{\infty}\frac{1}{(\int_{2^{k+1}B}v_{\vec{\omega}})^{1/p}}\prod_{i=1}^{m}\|f\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)}\\
&+\sum_{k=1}^{\infty}\frac{|b_{2^{k+1}B}-b_B|}{(\int_{2^{k+1}B}v_{\vec{\omega}})^{1/p}}\prod_{i=1}^{m}\|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)}\\
&+\|b_j\|_{BMO}\sum_{k=1}^{\infty}\frac{1}{(\int_{2^{k+1}B}v_{\vec{\omega}})^{1/p}}\prod_{i=1}^{m}\|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)}.
\end{aligned}\tag{3.8}$$

Take $L^p(v_{\vec{\omega}})$ norm on the ball $B(y,r)$ in both sides of (3.8). By (1.3), (2.2), (2.3), we have

$$\begin{aligned}
&\|T_{\vec{b}}^{j}(\vec{f})\chi_{B(y,r)}\|_{L^p(v_{\vec{\omega}})}\\
&\quad\lesssim \|b_j\|_{BMO}\prod_{i=1}^{m}\|f_i\chi_{B(y,2r)}\|_{L^{p_i}(\omega_i)}\\
&\qquad+\|b_j\|_{BMO}\sum_{k=1}^{\infty}\frac{(k+1)(\int_B v_{\vec{\omega}})^{1/p}}{(\int_{2^{k+1}B}v_{\vec{\omega}})^{1/p}}\prod_{i=1}^{m}\|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)}.
\end{aligned}\tag{3.9}$$

Multiplying both sides of (3.9) by $v_{\vec{\omega}}(B)^{1/\alpha-1/q-1/p}$, by Lemma 2.1 and (2.1), we obtain

$$\begin{aligned}
&v_{\vec{\omega}}(B)^{1/\alpha-1/q-1/p}\|T_{\vec{b}}^{j}(\vec{f})\chi_{B(y,r)}\|_{L^p(v_{\vec{\omega}})}\\
&\quad\lesssim \sum_{k=0}^{\infty}\frac{(k+1)\|b_j\|_{BMO}}{2^{nk\delta(1/\alpha-1/q)}}v_{\vec{\omega}}(2^{k+1}B)^{1/\alpha-1/q-1/p}\prod_{i=1}^{m}\|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i)},
\end{aligned}$$

the conclusion following easily as in the proof of Theorem 1.2.

Proof of Theorem 1.6. Let $B = B(y,r)$ be a ball of $\mathbb{R}^n$. For $x \in B(y,r)$, we get

$$\begin{aligned}|\mathcal{I}_{\Omega,\alpha}\vec{f}(x)| &\leqslant |\mathcal{I}_{\Omega,\alpha}(f_1^0,\dots,f_m^0)(x)| + \sum_{\alpha_1,\dots,\alpha_m\in\{0,\infty\}} |\mathcal{I}_{\Omega,\alpha}(f_1^{\alpha_1},\dots,f_i^{\alpha_m})(x)| \\ &\quad + |\mathcal{I}_{\Omega,\alpha}(f_1^\infty,\dots,f_m^\infty)(x)| \\ &= U + V + W,\end{aligned}$$

where $\alpha_1,\dots,\alpha_m$ are not all equal to 0 or ∞ at the same time. We first estimate W. An application of the Hölder inequality gives us that

$$\begin{aligned}W &\leqslant \sum_{k=1}^\infty \frac{1}{|2^{k+1}|^{m-\alpha/n}} \int_{(2^{k+1}B)^m} |\prod_{i=1}^m \Omega_i(x-y_i) f_i(y_i)| d\vec{y} \\ &\leqslant \sum_{k=1}^\infty \frac{1}{|2^{k+1}B|^{m/s'-\alpha/n}} \left(\prod_{i=1}^m \|\Omega_i\|_{L^s(S^{n-1})}\right) \\ &\quad \times \left(\int_{(2^{k+1}B)^m} \prod_{i=1}^m |f(y_i)|^{s'} d\vec{y}\right)^{1/s'}. \end{aligned} \tag{3.10}$$

Let $\vec{v} = \vec{\omega}^{s'}$. By the Hölder inequality and the definition of $A_{(\vec{p}/s',q/s')}$, we obtain

$$\begin{aligned}&\prod_{i=1}^m \left(\int_{2^{k+1}B} |f(y_i)|^{s'} d\vec{y}\right)^{1/s'} \\ &\quad = \prod_{i=1}^m \left(\int_{2^{k+1}B} |f(y_i)|^{s'} v_i(y_i) v_i^{-1}(y_i) d\vec{y}\right)^{1/s'} \\ &\quad \leqslant \prod_{i=1}^m \left(\int_{2^{k+1}B} |f_i(y_i)|^{t_i s'} v_i^{t_i'}(y_i) dy_i\right)^{1/(t_i s')} \left(\int_{2^{k+1}B} v_i^{-t_i'}(y_i) dy_i\right)^{1/(t_i' s')} \\ &\quad \lesssim \prod_{i=1}^m \left(\int_{2^{k+1}B} |f_i(y_i)\omega_i(y_i)|^{p_i} dy_i\right)^{1/p_i} \frac{|2^{k+1}B|^{1/q+m/s'-1/p}}{\left(\int_{2^{k+1}B}(\prod_{i=1}^m \omega_i)^q\right)^{1/q}}, \end{aligned} \tag{3.11}$$

where $t_i = p_i/s'$. So we have

$$W \lesssim \sum_{k=1}^\infty \frac{\prod_{i=1}^m \left(\int_{2^{k+1}B} |f_i(y_i)\omega_i(y_i)|^{p_i} dy_i\right)^{1/p_i}}{\left(\int_{2^{k+1}B}(\prod_{i=1}^m \omega_i)^q\right)^{1/q}}.$$

For V, we also just consider the case: $\alpha_i = \infty$ for $i = 1, \dots, l$ and $\alpha_j = 0$ for $j = l+1, \dots, m$.

$$
\begin{aligned}
&|\mathcal{I}_{\Omega,\alpha}(f_1^\infty, \dots, f_l^\infty, f_{l+1}^\infty, \dots, f_m^\infty)(x)| \\
&= \left| \int_{(\mathbb{R}^n)^l \setminus (2B)^l} \int_{(2B)^{m-l}} \frac{\Omega_1(y_i)\dots\Omega_m(y_m) f_1(y_1)\dots f_m(y_m)}{(\sum_{i=1}^m |x-y_i|)^{mn-\alpha}} d\vec{y} \right| \\
&\leqslant \int_{(\mathbb{R}^n)^l \setminus (2B)^l} \int_{(2B)^{m-l}} \frac{|\Omega_1(y_1)\dots\Omega_m(y_m) f_1(y_1)\dots f_m(y_m)|}{(\sum_{i=1}^m |x-y_i|)^{mn-\alpha}} d\vec{y} \\
&\lesssim \prod_{i=l+1}^m \int_{2B} |\Omega(y_i) f_i(y_i)| dy_i \times \sum_{k=1}^\infty \frac{1}{|2^{k+1}B|^{m-\alpha/n}} \prod_{i=1}^l \int_{2^{k+1}B \setminus 2^k B} |\Omega(y_i) f_i(y_i)| dy_i \\
&\lesssim \sum_{k=1}^\infty \frac{1}{|2^{k+1}B|^{m-\alpha/n}} \prod_{i=1}^m \int_{2^{k+1}B} |\Omega_i(y_i) f_i(y_i)| dy_i.
\end{aligned}
$$

By (3.10) and (3.11) we get

$$
\begin{aligned}
&|\mathcal{I}_{\Omega,\alpha}(f_1^\infty, \dots, f_l^\infty, f_{l+1}^\infty, \dots, f_m^\infty)(x)| \\
&\quad \lesssim \sum_{k=1}^\infty \frac{\prod_{i=1}^m \left(\int_{2^{k+1}B} |f_i(y_i)\omega_i(y_i)|^{p_i} dy_i \right)^{1/p_i}}{\left(\int_{2^{k+1}B} (\prod_{i=1}^m \omega_i)^q \right)^{1/q}}.
\end{aligned}
$$

Combining all the cases together, we have

$$
\begin{aligned}
|\mathcal{I}_{\Omega,\alpha}\vec{f}(x)| \lesssim & \left| \int_{(2B)^m} \frac{\prod_{i=1}^m \Omega_i(x-y_i) f_i(y_i)}{(\sum_{i=1}^m |x-y_i|)^{mn-\alpha}} d\vec{y} \right| \\
& + \sum_{k=1}^\infty \frac{\prod_{i=1}^m \left(\int_{2^{k+1}B} |f_i(y_i)\omega_i(y_i)|^{p_i} dy_i \right)^{1/p_i}}{\left(\int_{2^{k+1}B} (\prod_{i=1}^m \omega_i)^q \right)^{1/q}}. \qquad (3.12)
\end{aligned}
$$

Taking $L^q \left(\prod_{i=1}^m \omega_i^q \right)$ norm on the ball $B(y,r)$ in both sides of (3.12), by (1.4) we get

$$
\begin{aligned}
&\|\mathcal{I}_{\Omega,\alpha}\vec{f}\chi_{By,r}\|_{L^q(\prod_{i=1}^m \omega_i^q)} \\
&\quad \lesssim \prod_{i=1}^m \|f_i \chi_{B(y,2r)}\|_{L^{p_i}(\omega_i^{p_i})} \\
&\quad + \sum_{k=1}^\infty \frac{\prod_{i=1}^m \left(\int_{2^{k+1}B} |f_i(y_i)\omega_i(y_i)|^{p_i} dy_i \right)^{1/p_i} \left(\int_B (\prod_{i=1}^m \omega_i)^q \right)^{1/q}}{\left(\int_{2^{k+1}B} (\prod_{i=1}^m \omega_i)^q \right)^{1/q}}. \qquad (3.13)
\end{aligned}
$$

Multiplying both sides of (3.13) by $\left(\prod_{i=1}^{m}\omega_i^q\right)(B)^{1/\beta-1/q-1/\gamma}$, by Lemma 2.4 and (2.1) we obtain

$$\begin{aligned}
&\left(\prod_{i=1}^{m}\omega_i^q\right)(B)^{1/\beta-1/q-1/\gamma}\|\mathcal{I}_{\Omega,\alpha}\vec{f}\chi_{B(y,r)}\|_{L^q(\prod_{i=1}^{m}\omega_i^q)} \\
&\quad\lesssim \sum_{k=1}^{\infty}\frac{1}{2^{nk\delta(1/\beta-1/\gamma)}}\left(\prod_{i=1}^{m}\omega_i^q\right)(2^{k+1}B)^{1/\beta-1/q-1/\gamma}\|f_i\chi_{B(y,2^{k+1}r)}\|_{L^{p_i}(\omega_i^{p_i})},
\end{aligned}$$

the conclusion following easily as in proof of Theorem 1.2.

Proof of Theorem 1.8. Following analogous reasoning as in previous proofs, it suffices to estimate $\mathcal{I}^j_{\vec{b},\alpha}$. For $B=B(y,r)$ and $x\in B$

$$\begin{aligned}
\mathcal{I}^j_{\vec{b},\alpha}(\vec{f})(x) &= \mathcal{I}^j_{\vec{b},\alpha}(\vec{f}\chi_{2B})(x) \\
&\quad+\sum_{\alpha_1,\dots,\alpha_m\in\{0,\infty\}}\Big((b_j(x)\mathcal{I}_\alpha(f_1^{\alpha_1},\dots,f_j^{\alpha_j},\dots,f_m^{\alpha_m}) \\
&\quad-\mathcal{I}_\alpha(f_1^{\alpha_1},\dots,b_jf_j^{\alpha_j},\dots,f_m^{\alpha_m})(x)\Big) \\
&\quad+b_j(x)T(f_1^{\infty},\dots,f_j^{\infty},\dots,f_m^{\infty})-\mathcal{I}_\alpha(f_1^{\infty},\dots,b_jf_j^{\infty},\dots,f_m^{\infty})(x) \\
&= U'+V'+W',
\end{aligned}$$

where $\alpha_1,\dots,\alpha_m$ are not all equal to 0 or ∞ at the same time. We first deal with W'.

$$\begin{aligned}
|W'| &\le |(b_j(x)-b_B)\mathcal{I}_\alpha(f_1^{\infty},\dots,f_j^{\infty},\dots,f_m^{\infty})| \\
&\quad+|\mathcal{I}_\alpha(f_1^{\infty},\dots,(b_j-b_B)f_j^{\infty},\dots,f_m^{\infty})(x)| \\
&\le |(b_j(x)-b_B)|\sum_{k=1}^{\infty}\frac{1}{\left(\int_{2^{k+1}B}(\prod_{i=1}^{m}\omega_i)^q\right)^{1/q}}\prod_{i=1}^{m}\|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i^{p_i})} \\
&\quad+\sum_{k=1}^{\infty}\frac{|b_{2^{k+1}B}-b_B|}{\left(\int_{2^{k+1}B}(\prod_{i=1}^{m}\omega_i)^q\right)^{1/q}}\prod_{i=1}^{m}\|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i^{p_i})} \\
&\quad+\sum_{k=1}^{\infty}\frac{1}{|2^{k+1}B|^{m-\alpha/n}}\int_{(2^{k+1}B)^m}\prod_{\substack{j=1,\\ j\neq i}}^{m}|f_i(y_i)f_j(y_j)\big(b_j(y_j)-b_{2^{k+1}B}\big)|d\vec{y}.
\end{aligned}$$

By the Hölder inequality, the $A_{(\vec{p}/s,q/s)}$ condition and (2.2) we have

$$\sum_{k=1}^{\infty}\frac{1}{|2^{k+1}B|^{m-\alpha/n}}\int_{(2^{k+1}B)^m}\prod_{\substack{j=1,\\ j\neq i}}^{m}|f_i(y_i)f_j(y_j)\big(b_j(y_j)-b_{2^{k+1}B}\big)|d\vec{y}$$

$$\begin{aligned}
&\leqslant \sum_{k=1}^{\infty} \frac{1}{|2^{k+1}B|^{m/s-\alpha/n}} \left(\prod_{\substack{j=1,\\ j\neq i}}^{m} \int_{2^{k+1}B} |f_i(y_i)|^s dy_i \right)^{1/s} \\
&\quad \times \left(\int_{2^{k+1}B} |(b_j(y_j) - b_{2^{k+1}B}) f_j(y_j)|^s dy_j \right)^{1/s} \\
&\leq \|b_j\|_{BMO} \sum_{k=1}^{\infty} \frac{1}{\left(\int_{2^{k+1}B} (\prod_{i=1}^m \omega_i)^q\right)^{1/q}} \prod_{i=1}^{m} \|f_i \chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i^{p_i})}.
\end{aligned}$$

So we have

$$\begin{aligned}
|W'| &\leq |(b_j(x) - b_B)| \sum_{k=1}^{\infty} \frac{1}{\left(\int_{2^{k+1}B} (\prod_{i=1}^m \omega_i)^q\right)^{1/q}} \prod_{i=1}^{m} \|f_i \chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i^{p_i})} \\
&\quad + \sum_{k=1}^{\infty} \frac{|b_{2^{k+1}B} - b_B|}{\left(\int_{2^{k+1}B} (\prod_{i=1}^m \omega_i)^q\right)^{1/q}} \prod_{i=1}^{m} \|f_i \chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i^{p_i})} \\
&\quad + \|b_j\|_{BMO} \sum_{k=1}^{\infty} \frac{1}{\left(\int_{2^{k+1}B} (\prod_{i=1}^m \omega_i)^q\right)^{1/q}} \prod_{i=1}^{m} \|f_i \chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i^{p_i})}.
\end{aligned}$$

For W', we just consider this case: $\alpha_i = \infty$ for $i = 1, \ldots, l$ and $\alpha_j = 0$ for $j = l+1, \ldots, m$. But there exist two cases

$$\begin{aligned}
&b_j(x) \mathcal{I}_\alpha(f_1^\infty, \ldots, f_j^\infty, \ldots, f_l^\infty, f_{l+1}^0, \ldots, f_m^0) \\
&\qquad - \mathcal{I}_\alpha(f_1^\infty, \ldots, b_j f_j^\infty, \ldots, f_l^\infty, f_{l+1}^0, \ldots, f_m^0)(x)
\end{aligned}$$

or

$$\begin{aligned}
&b_j(x) \mathcal{I}_\alpha(f_1^\infty, \ldots, f_l^\infty, f_{l+1}^0, \ldots, f_j^0, \ldots, f_m) \\
&\qquad - \mathcal{I}_\alpha(f_1^\infty, \ldots, f_l^\infty, f_{l+1}^0, \ldots, b_j f_j^0, \ldots, f_m^0)(x).
\end{aligned}$$

We just consider

$$\begin{aligned}
&|b_j(x) \mathcal{I}_\alpha(f_1^\infty, \ldots, f_j^\infty, \ldots, f_l^\infty, f_{l+1}^0, \ldots, f_m^0) - \\
&\quad \mathcal{I}_\alpha(f_1^\infty, \ldots, b_j f_j^\infty, \ldots, f_l^\infty, f_{l+1}^0, \ldots, f_m^0)(x)| \\
&\leq |(b_j(x) - b_B) \mathcal{I}_\alpha(f_1^\infty, \ldots, f_j^\infty, \ldots, f_l^\infty, f_{l+1}^0, \ldots, f_m^0)| \\
&\quad + |\mathcal{I}_\alpha(f_1^\infty, \ldots, (b_j - b_B) f_j^\infty, \ldots, f_l^\infty, f_{l+1}^0, \ldots, f_m^0)(x)| \\
&\leq |(b_j(x) - b_B)| \sum_{k=1}^{\infty} \frac{1}{\left(\int_{2^{k+1}B} (\prod_{i=1}^m \omega_i)^q\right)^{1/q}} \prod_{i=1}^{m} \|f_i \chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i^{p_i})} \\
&\quad + \sum_{k=1}^{\infty} \frac{|b_{2^{k+1}B} - b_B|}{\left(\int_{2^{k+1}B} (\prod_{i=1}^m \omega_i)^q\right)^{1/q}} \prod_{i=1}^{m} \|f_i \chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i^{p_i})} \\
&\quad + \sum_{k=1}^{\infty} \frac{\prod_{i=l+1}^m \int_{2B} |f_i(y_i)| dy_i}{|2^{k+1}B|^{m-\alpha/n}} \int_{(2^{k+1}B)^m} \prod_{\substack{j=1,\\ j\neq i}}^{l} |f_i(y_i) f_j(y_j) \big(b_j(y_j) - b_{2^{k+1}B}\big)| d\vec{y}.
\end{aligned}$$

The other case is completely analogous. The last term is similar to (3.7), we get

$$\begin{aligned}
&\sum_{k=1}^{\infty}\frac{\prod_{i=l+1}^{m}\int_{2B}|f_i(y_i)|dy_i}{|2^{k+1}B|^{m-\alpha/n}}\int_{(2^{k+1}B)^m}\prod_{\substack{j=1,\\ j\neq i}}^{l}|f_i(y_i)f_j(y_j)\big(b_j(y_j)-b_{2^{k+1}B}\big)|d\vec{y}\\
&\quad\le \sum_{k=1}^{\infty}\frac{1}{|2^{k+1}B|^{m-\alpha/n}}\prod_{\substack{j=1,\\ j\neq i}}^{m}\int_{2^{k+1}B}|f_i(y_i)|dy_i\int_{2^{k+1}B}|b_j(y_j)-b_{2^{k+1}B}|f(y_j)dy_j\\
&\quad\le \|b_j\|_{BMO}\sum_{k=1}^{\infty}\frac{1}{\big(\int_{2^{k+1}B}(\prod_{i=1}^{m}\omega_i)^q\big)^{1/q}}\prod_{i=1}^{m}\|f\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i^{p_i})},
\end{aligned}$$

so we have

$$\begin{aligned}
|\mathcal{I}^j_{\vec{b},\alpha}(\vec{f})(x)| \le\ & |\mathcal{I}^j_{\vec{b},\alpha}(\vec{f}\chi_{2B})(x)|\\
&+|(b_j(x)-b_B)|\sum_{k=1}^{\infty}\frac{1}{\big(\int_{2^{k+1}B}(\prod_{i=1}^{m}\omega_i)^q\big)^{1/q}}\prod_{i=1}^{m}\|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i^{p_i})}\\
&+\sum_{k=1}^{\infty}\frac{|b_{2^{k+1}B}-b_B|}{\big(\int_{2^{k+1}B}(\prod_{i=1}^{m}\omega_i)^q\big)^{1/q}}\prod_{i=1}^{m}\|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i^{p_i})}\\
&+\|b_j\|_{BMO}\sum_{k=1}^{\infty}\frac{1}{\big(\int_{2^{k+1}B}(\prod_{i=1}^{m}\omega_i)^q\big)^{1/q}}\prod_{i=1}^{m}\|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i^{p_i})}.
\end{aligned}\tag{3.14}$$

Take $L^q\left(\prod_{i=1}^{m}\omega_i^q\right)$ norm on the ball $B(y,r)$ in both sides of (3.14). And using Lemma 2.4, (2.2), (2.3) and (1.5), we have

$$\begin{aligned}
&\|\mathcal{I}^j_{\vec{b},\alpha}(\vec{f})(x)\|_{L^q(\prod_{i=1}^{m}\omega_i^q)}\\
&\quad\lesssim \|b_j\|_{BMO}\prod_{i=1}^{m}\|f_i\chi_{B(y,2r)}\|_{L^{p_i}(\omega_i)}\\
&\qquad+\|b_j\|_{BMO}\sum_{k=1}^{\infty}\frac{k\left(\int_B(\prod_{i=1}^{m}\omega_i)^q\right)^{1/q}}{\big(\int_{2^{k+1}B}(\prod_{i=1}^{m}\omega_i)^q\big)^{1/q}}\prod_{i=1}^{m}\|f_i\chi_{2^{k+1}B}\|_{L^{p_i}(\omega_i^{p_i})}.
\end{aligned}\tag{3.15}$$

Multiplying both sides of (3.15) by $\left(\prod_{i=1}^{m}\omega_i^q\right)(B)^{1/\beta-1/q-1/\gamma}$, we obtain

$$\begin{aligned}
&v_{\vec{\omega}}(B)^{1/\beta-1/q-1/\gamma}\|\mathcal{I}^j_{\vec{b},\alpha}(\vec{f})\chi_{B(y,r)}\|_{L^q(\prod_{i=1}^{m}\omega_i^q)}\\
&\lesssim \sum_{k=0}^{\infty}\frac{(k+1)\|b_j\|_{BMO}}{2^{nk\delta(1/\beta-1/\gamma)}}\left(\prod_{i=1}^{m}\omega_i^q\right)(2^{k+1}B)^{1/\beta-1/q-1/\gamma}\prod_{i=1}^{m}\|f_i\chi_{B(y,2^{k+1}r)}\|_{L^{p_i}(\omega_i^{p_i})}.
\end{aligned}$$

This completes the proof.

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