Adv. Oper. Theory 2 (2017), no. 3, 287–292
http://doi.org/10.22034/aot.1702-1121
ISSN: 2538-225X (electronic)
http://aot-math.org

SEMIGROUP HOMOMORPHISMS ON MATRIX ALGEBRAS

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Communicated by L. Molnár

Abstract. We explore the connection between ring homomorphisms and semigroup homomorphisms on matrix algebras over rings or C^*-algebras.

1. Introduction

It is an interesting question what possibly small portion of information distinguishes multiplicative semigroup homomorphism between rings and ring homomorphisms between rings. Rings on which every semigroup isomorphism is automatically additive are called to have unique addition and there exists a vast literature on this topic, see R.E. Johnson [4] and many others. It is not possible to characterize semigroups which are the multiplicative semigroup of a ring axiomatically, see S.R. Kogalovskij [5]. There is also an extensive investigation when the occurring rings happen to be matrix rings of the form $M_n(R)$, and here one is interested in classifying all semigroup homomorphisms between them. Confer J. Landin and I. Reiner [6], M. Jodeit Jr. and T.Y. Lam [3] and many others. The prototype answer appears to be that every semigroup homomorphism $\phi : GL_n(K) \to GL_m(K)$ with K a division ring and $m < n$ is of the form

$$\phi(x) = \psi(\det(x))$$

for a semigroup homomorphism $\psi : R^*/[R^*, R^*] \to GL_m(K)$ and Dieudonné's determinant, see D.Ž. Djoković [1]. For integral domains R, $M_n(R)$ has unique addition [3]. Most investigations on semigroup homomorphisms of matrix algebras have ground rings principal ideal domains, fields or division rings.

Date: Received: Feb. 15, 2017; Accepted: Apr. 26, 2017.

2010 *Mathematics Subject Classification.* Primary 46L05; Secondary 20M25, 15B33.

Key words and phrases. semigroup, ring, matrix, multiplicative, additive, unique addition, C^*-algebra.

Since a ring R is Morita equivalent to its matrix ring $M_n(R)$ it is often no big restriction if one considers matrix rings. For example K-theory cannot distinguish between the ring and its stablization by matrix. Similar things can be said for C^*-algebras and their notion of Morita equivalence and topological K-theory.

In this short note we show that a semigroup homomorphisms $\phi : M_2(R) \to S$ for rings R and S is a ring homomorphism if and only if it satisfies the single relation $\phi(e_{11}) + \phi(e_{22}) = \phi(1)$. An analogous statement holds for C^*-algebras. See Proposition 2.1, Corollary 2.4 and Proposition 3.3 .

Import and much deeper related results are the classification of $*$-semigroup endomorphisms on the C^*-algebra $B(H)$ for H an infinite Hilbert space by L. Molnár [7] and of bijective semigroup homomorphisms between standard operator algebras of Banach spaces by P. Šemrl [8] and J. Molnár. Notice that $B(H) \cong M_n(B(H))$ is matrix-stable for which our observation applies. When finishing this note we came also accross the strongly related paper [2] by J. Hakeda, but it considers bijective $*$-semigroup isomorphisms between $*$-algebras.

We will also investigate how group homomorphisms (of the form $\phi \otimes id_{M_{16}}$) on unitary and general linear groups of matrix C^*-algebras can be extended to ring or $*$-homomorphisms, see Propositions 3.1 and 3.2. It seems to be an interesting and widely open question which group homomorphisms between groups $GL(M_n(R))$ for typically noncommutative non-division rings R with zero divisors even exist, if not restrictions of ring homomorphisms on $M_n(R)$. For example $\begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$ is an invertible matrix for an orthogonal projection $p \in B(H)$, but no entry is invertible and Dieudonné's determinant is not applicable.

2. Algebra homomorphisms and semigroup homomorphisms

For a ring A we shall denote $M_n(A)$ also by $A \otimes M_n$. For algebras A latter denotes the algebra tensor product. We write $e_{ij} \in M_n$ and $e_{ij} := 1 \otimes e_{ij} \in A \otimes M_n$ for the usual matrix units. We also use the notation ϕ_n for $\phi \otimes id_{M_n} : A \otimes M_n \to B \otimes M_n$, and reversely allow this notation sloppily also for any function ϕ for the canonical matrix map ϕ_n. A $*$-semigroup homomorphism between C^*-algebras means an involution respecting semigroup homomorphism. For unital algebras we write λ for $\lambda 1$, where λ is a scalar. We say ϕ is K-homogeneous if $\phi(\lambda x) = \lambda \phi(x)$ for all x and scalars $\lambda \in K$.

Proposition 2.1. *Let A, B be rings where A is unital and $\varphi : A \otimes M_2 \to B$ an arbitrary function. Then the following are equivalent:*

(a) *φ is a ring homomorphism.*
(b) *φ is a semigroup homomorphism such that*

$$\varphi(1) = \varphi(e_{11}) + \varphi(e_{22}). \tag{2.1}$$

Proof. Clearly (a) implies (b). Assume (b). We have $\varphi(x_{ij} \otimes e_{ij})\varphi(y_{kl} \otimes e_{kl}) = \varphi(x_{ij} y_{kl} \otimes e_{il})\delta_{j,k}$ for $1 \le i, j, k, l \le 2$. One has

$$\varphi(x) = \varphi(1)\varphi(x)\varphi(1) = \sum_{i,j=1}^{2} \varphi(x_{ij} \otimes e_{ij})$$

for all $x = \sum_{i,j=1}^{2} x_{ij} \otimes e_{ij} \in A \otimes M_2$ by (2.1). Now notice that

$$\begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a+b & 0 \\ 0 & 0 \end{pmatrix}$$

for all $a, b \in A$. Applying here φ, using its multiplicativity and observing the upper left corner we obtain $\varphi(a \otimes e_{11}) + \varphi(b \otimes e_{11}) = \varphi(a \otimes e_{11} + b \otimes e_{11})$ and similar so for all other corners. We conclude that φ is additive. □

Corollary 2.2. *Let A and B be rings where A is unital. Then $\varphi : A \to B$ is a ring homomorphism if and only if $\varphi \otimes id_{M_2}$ is a semigroup homomorphism sending zero to zero.*

Corollary 2.3. *Let A, B be algebras over a field K where A is unital and $\varphi : M_2 \otimes A \to B$ an arbitrary function. Then the following are equivalent:*

(a) *φ is an algebra homomorphism.*
(b) *φ is a semigroup homomorphism which is linear on $Ke_{11} + Ke_{22}$.*

If $K = \mathbb{C}$ then we may also add

(c) *φ is a semigroup homomorphism which is linear on $\mathbb{R}1$ and satisfies*

$$\varphi(i) = i\varphi(e_{11}) + i\varphi(e_{22}). \tag{2.2}$$

Proof. (a) ⇒ (b) ⇒ (c) are trivial. (b) ⇒ (a): φ is K-homogeneous and by Proposition 2.1 a ring homomorphism. (c) ⇒ (a): φ is $\mathbb{R}$-linear by $\varphi(\lambda 1 x) = \lambda\varphi(1)\varphi(x)$ ($\lambda \in \mathbb{R}$). If we take (2.2) to the four then we get (2.1). Combining (2.2) and (2.1) gives $\varphi(i) = i\varphi(1)$ and thus φ is $\mathbb{C}$-linear. The assertion follows then from Proposition 2.1. □

Corollary 2.4. *Let A, B be C^*-algebras where A is unital and $\varphi : M_2 \otimes A \to B$ an arbitrary function. Then the following are equivalent:*

(a) *φ is a $*$-homomorphism.*
(b) *φ is a $*$-semigroup homomorphism satisfying identity (2.2).*

Proof. Taking (2.2) to the four yields (2.1). Hence φ is a ring homomorphism by Proposition 2.1 and thus $\mathbb{Q}$-linear. By [9, Theorem 3.7] φ is continuous and thus $\mathbb{C}$-linear by combining (2.2) and (2.1). □

Corollary 2.5. *Let A and B be C^*-algebras where A is unital. Then $\varphi : A \to B$ is a $*$-homomorphism if and only if $\varphi \otimes id_{M_2}$ is a $*$-semigroup homomorphism and $\varphi(0) = 0$ and $\varphi(i) = i\varphi(1)$.*

Proof. By Corollary 2.2 $\varphi \otimes id_{M_2}$ is a ring homomorphism, thus $\mathbb{Q}$-linear and continuous by [9, Theorem 3.7]. □

Proposition 2.6. *Let A, B be C^*-algebras where A is unital and $\varphi : \overline{GL(M_2 \otimes A)} \to B$ an arbitrary function (norm closure). Then the following are equivalent:*

(a) *φ extends to a $*$-homomorphism $M_2 \otimes A \to B$.*
(b) *φ is a $*$-semigroup homomorphism satisfying identity (2.2).*

Similarly, φ extends to an algebra homomorphism if and ony if φ is a semigroup homomorphism which is continuous on $\mathbb{R}1$ and satisfies identity (2.2).

Proof. Assume that φ is a semigroup homomorphism satisfying (2.2). Consider the matrices

$$\gamma_c := \begin{pmatrix} c & \lambda \\ \lambda & 0 \end{pmatrix}, \ \alpha_a := \begin{pmatrix} 1 & a \\ 0 & \lambda \end{pmatrix}, \ \beta_b := \begin{pmatrix} b & \lambda \\ 1 & 0 \end{pmatrix}$$

for $\lambda \in \mathbb{R}\backslash\{0\}$ and $a, b, c \in A$. They are invertible; just notice that they are evidently bijective operators on $H \oplus H$ for a representation of A on a Hilbert space H. Letting $\lambda \to 0$ we see that all single matrix entries $a_{ij} \otimes e_{ij}$ (for all $a_{ij} \in A$) and all matrices indicated in the proof of Proposition 2.1 are in $\overline{GL(M_2 \otimes A)}$. Taking (2.2) to the four yields identity (2.1). By the proof of Proposition 2.1 we see that $\varphi(a) = \sum_{i,j=1}^n \varphi(a_{ij} \otimes e_{ij})$ for all $a \in \overline{GL(M_2 \otimes A)}$, which we use now as a definition for φ for all $a \in A \otimes M_2$. Also by the proof of Proposition 2.1 we have $\varphi(a_{ij} \otimes e_{ij} + b_{ij} \otimes e_{ij}) = \varphi(a_{ij} \otimes e_{ij}) + \varphi(b_{ij} \otimes e_{ij})$ for all $a_{ij}, b_{ij} \in A$, which shows that the extended φ is additive.

Since every element in a C^*-algebra can be written as a finite sum of invertible elements of the form λu with $\lambda \in \mathbb{C}$ and u unitary we see that φ is multiplicative. If the originally given φ respects involution this also shows that the new φ does so; in this case we are done with Corollary 2.4.

Otherwise, for proving the second equivalence we proceed: Since φ is a ring homomorphism it is $\mathbb{Q}$-linear. By continuity $\varphi(\lambda 1) = \lambda\varphi(1)$ for all $\lambda \in \mathbb{R}$ and for $\lambda = i$ by (2.1) and (2.2). Hence φ is $\mathbb{C}$-linear. □

3. C^*-HOMOMORPHISMS AND GROUP HOMOMORPHISMS

The methods of this section applies analogously to rings A and B, or Banach algebras where we use topology, if every element in such rings allows to be written as a finite sum of invertible elements. This is true for C^*-algebras.

Proposition 3.1. *Let $\varphi : A \to B$ be an arbitrary function between unital C^*-algebras A and B.*

Then φ is a unital $*$-homomorphism if and only if φ is $\mathbb{C}$-homogeneous and $\varphi \otimes id_{M_{16}}$ restricts to a group homomorphism

$$U(A \otimes M_{16}) \to U(B \otimes M_{16}).$$

Proof. Since φ_{16} is unital, necessarily φ is unital. Embedding $U(M_n(A)) \subseteq U(M_{16}(A))$ via $u \mapsto \operatorname{diag}(u, 1)$ it is clear that φ_n restricts to group homomorphims between the unitary groups too for $1 \le n \le 16$. As $\varphi(u)\varphi(u^*) = 1$ for $u \in U(A)$, $\varphi(u^*) = \varphi(u)^*$.

To simplify notation, let us say that a is a scaled unitary in A if $a \in \mathbb{C}U(A)$. The set of scaled unitaries forms a monoid. Since φ is $\mathbb{C}$-homogeneous, the maps $\varphi \otimes id_{M_n}$ restrict also to monoid homomorphisms $\mathbb{C}U(A \otimes M_n) \to \mathbb{C}U(B \otimes M_n)$ between the set of scaled unitaries.

Let a, b be scaled unitaries in A. Define scaled unitaries

$$u = \begin{pmatrix} 1 & a \\ -a^* & 1 \end{pmatrix}, \qquad v = \begin{pmatrix} b & -1 \\ 1 & b^* \end{pmatrix}. \tag{3.1}$$

Their product is the scaled unitary

$$uv = \begin{pmatrix} a+b & -1+ab^* \\ 1-ba^* & a^*+b^* \end{pmatrix}.$$

Applying here $\varphi \otimes id$ using $\varphi_2(uv) = \varphi_2(u)\varphi_2(v)$ and observing the upper left corner we obtain $\varphi(a+b) = \varphi(a) + \varphi(b)$.

Now write a' for the product uv and define b' to be also the product uv, however, with a and b replaced by other scaled unitaries c and d, respectively, in A.

Now consider the same matrices u and v as in (3.1) above, but with a replaced by a' and b by b'. These are four times four matrices. Consider again the product uv of these newly defined matrices u and v. It has $a' + b'$ in the 2×2 upper left corner and so the entry $a + b + c + d$ in the 1×1 upper left corner. Applying $\varphi \otimes id_{M_4}$ to this identity of 4×4-matrices and using

$$\varphi_4(uv) = \varphi_4(u)\varphi_4(v) = \begin{pmatrix} \varphi_2(1) & \varphi_2(a') \\ \varphi_2(-a'^*) & \varphi_2(1) \end{pmatrix} \begin{pmatrix} \varphi_2(b') & \varphi_2(-1) \\ \varphi_2(1) & \varphi_2(b'^*) \end{pmatrix}$$

yields $\varphi(a+b+c+d) = \varphi(a+b) + \varphi(c+d) = \varphi(a) + \varphi(b) + \varphi(c) + \varphi(d)$ by comparing the upper left corner.

Repeating this recursive procedure two more times we get additivity of φ of sixteen scaled unitaries. Since we may write any element of A as the sum of four scaled unitaries it is obvious that φ is a $*$-homomorphism. □

Proposition 3.2. *Let $\varphi : A \to B$ be an arbitrary function between unital C^*-algebras A and B.*

Then φ is a unital ring homomorphism if and only if $\varphi \otimes id_{M_2}$ restricts to a group homomorphism

$$GL(A \otimes M_2) \to GL(B \otimes M_2).$$

Proof. In more general rings where elements can be written as sums of invertible elements we may prove this exactly by the same recursive procedure as in the last proof. All we have to do is to replace scaled unitaries by invertible elements. Observe that a and b can also be chosen to be zero in (3.1), where we replace a^* and b^* by a^{-1} and b^{-1}. In the C^*-case we observe, however, that we can choose any elements a and b in (3.1) (simply check surjectivity and injectivity of the involved matrices for a representation on Hilbert space), so that we end up with additivity and hence multiplicativity of φ after one step of the procedure. □

Proposition 3.3. *Let $\varphi : GL(A \otimes M_2) \to B$ be an arbitrary function where A and B are C^*-algebras and A is unital.*

Then φ extends to a $*$-homomorphism $A \otimes M_2 \to B$ if and only if φ is a $*$-semigroup homomorphism which is uniformly continuous and satisfies the identity

$$\varphi \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} = \lim_{\lambda \to 0,\, \lambda \in \mathbb{R}_+} i\varphi \begin{pmatrix} 1 & \lambda \\ \lambda & 0 \end{pmatrix} + i\varphi \begin{pmatrix} 0 & \lambda \\ \lambda & 1 \end{pmatrix}.$$

An analogous equivalence holds true without the star prefixes (omitting $*$-).

Proof. Since φ is uniformly continuous it is clear that it maps Cauchy sequences to Cauchy sequences. We can thus extend it continuously to $\overline{GL(A \otimes M_2)}$ via the limits of Cauchy sequences. The assertion follows then from Proposition 2.6. □

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