

APPLICATIONS OF TERNARY RINGS TO C^* -ALGEBRAS

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ABSTRACT. We show that there is a functor from the category of positive admissible ternary rings to the category of $*$ -algebras, which induces an isomorphism of partially ordered sets between the families of C^* -norms on the ternary ring and its corresponding $*$ -algebra. We apply this functor to obtain Morita–Rieffel equivalence results between cross-sectional C^* -algebras of Fell bundles, and to extend the theory of tensor products of C^* -algebras to the larger category of full Hilbert C^* -modules. We prove that, like in the case of C^* -algebras, there exist maximal and minimal tensor products. As applications we give simple proofs of the invariance of nuclearity and exactness under Morita–Rieffel equivalence of C^* -algebras.

1. INTRODUCTION

An important tool in the study of C^* -algebras is Morita–Rieffel equivalence. When two C^* -algebras are Morita–Rieffel equivalent, they are related by a certain type of bimodule, from which one can see that these algebras share many properties. A Morita–Rieffel equivalence between two C^* -algebras implies that these algebras have many characteristics in common: they have the same K-theory, their spectra and primitive ideal spaces are homeomorphic, etc. In [12], Zettl introduced and studied C^* -ternary rings, and showed that these objects are essentially Morita–Rieffel equivalence bimodules. In fact, given a C^* -ternary ring

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E , there exists essentially a unique structure of Morita–Rieffel equivalence bimodule on E compatible with its structure of ternary ring (perhaps after a minor change on the ternary product).

On the other hand, when dealing with constructions such as tensor products or any sort of crossed products of C^* -algebras, in general one has to follow two steps: first one defines some $*$ -algebra, and then one takes the completion of that algebra with respect to a C^* -norm. A situation that appears frequently is that there is more than one reasonable C^* -norm to perform this second step. In many cases, for instance in several imprimitivity theorems, one is interested in finding a Morita–Rieffel equivalence between different C^* -completions of a given pair of $*$ -algebras which are related by a certain bimodule. This is the situation we study in the present paper, adopting a viewpoint similar to that in Zettl’s work, but starting from a more algebraic level.

More precisely, suppose E is an A – B bimodule, where A and B are $*$ -algebras, $\langle \cdot, \cdot \rangle_A : E \times E \rightarrow A$ and $\langle \cdot, \cdot \rangle_B : E \times E \rightarrow B$ satisfy all the algebraic properties of Hilbert bimodule inner products. In particular $\langle x, y \rangle_A z = x \langle y, z \rangle_B$, $\forall x, y, z \in E$. Then we can endow E with a $*$ -ternary ring structure by defining a ternary product $(\cdot, \cdot, \cdot) : E \times E \times E \rightarrow E$ such that $(x, y, z) = x \langle y, z \rangle_B$. We show that, under certain conditions, the partially ordered sets of C^* -norms on E and on the $*$ -algebras A and B are isomorphic to each other, in such a way that the completions with respect to corresponding C^* -norms under these isomorphisms yields a Morita–Rieffel equivalence bimodule.

We think that the best way to do it is by using the above mentioned abstract characterization of equivalence bimodules given by Zettl in [12], under the name of C^* -ternary rings. Such an object is a Banach space with a ternary product on it, which implicitly carries all the structure of an equivalence bimodule. Natural morphisms between C^* -ternary rings are linear maps that preserve ternary products. With such morphisms, one obtains a C^* -category, which is very convenient for the study of properties invariant under Morita–Rieffel equivalence.

The structure of the paper is as follows. In the next section, working in a pure algebraic level, we define the category of admissible $*$ -ternary rings, and we show there is a functor from this category to the category of $*$ -algebras or, more precisely, to the category of right basic triples (see Definition 2.6). In Section 3, given an admissible ternary ring E with associated basic triple $(E, A, \langle \cdot, \cdot \rangle_A)$, we consider the lattice of C^* -seminorms on A that satisfy the Cauchy-Schwarz inequality $\|\langle x, y \rangle_A\|^2 \leq \|\langle x, x \rangle\| \|\langle y, y \rangle\|$, $\forall x, y \in E$. Then we prove that this lattice is isomorphic to the lattice of C^* -seminorms on E . In passing we obtain some of the results of [12] and [1] regarding C^* -ternary rings. Besides, since there is also a functor to the category of left basic triples, we obtain a fortiori an isomorphism between the lattices of C^* -seminorms (satisfying the Cauchy-Schwarz property) on the $*$ -algebras associated to the left and to the right sides. The Hausdorff completions of corresponding C^* -seminorms under this isomorphism turn out to be Morita–Rieffel equivalent. In the last part of Section 3 we consider positive ternary rings, for which the C^* -seminorms on the associated $*$ -algebras automatically satisfy the Cauchy-Schwarz inequality. In Section 4 we briefly study the case of C^* -ternary rings, in which basic triples are replaced by C^* -basic triples, that is,

Hilbert modules, and the functors from C^* -ternary rings to C^* -basic triples are shown to be exact. Finally, Section 5 is devoted to applications. We first refine a result from [2] concerning cross-sectional algebras of Fell bundles over groups. Then we consider tensor products of C^* -ternary rings, which is essentially the same as tensor products of Hilbert modules. We show that the theory of tensor products of C^* -algebras extends to this larger category, in the sense that there exist a maximal and a minimal tensor products. By using this theory we obtain easy and natural proofs of the known results of the Morita–Rieffel invariance of nuclearity and exactness of C^* -algebras.

2. TERNARY RINGS

2.1. Ternary rings.

Definition 2.1. A $*$ -ternary ring is a complex linear space E with a map $\mu : E \times E \times E \rightarrow E$, called $*$ -ternary product on E , which is linear in the odd variables and conjugate linear in the second one, and such that:

$$\mu(\mu(x, y, z), u, v) = \mu(x, \mu(u, z, y), v) = \mu(x, y, \mu(z, u, v)), \quad \forall x, y, z, u, v \in E$$

A homomorphism of $*$ -ternary rings is a linear map $\phi : (E, \mu) \rightarrow (F, \nu)$ such that $\nu(\phi(x), \phi(y), \phi(z)) = \phi(\mu(x, y, z))$, $\forall x, y, z \in E$. Sometimes we will write (x, y, z) or $(x, y, z)_E$ instead of $\mu(x, y, z)$, and we will use the expression $*$ -tring instead of $*$ -ternary ring.

There is an inclusion of the category of $*$ -algebras into the category of $*$ -trings: if A is a $*$ -algebra, then $(x, y, z) \mapsto xy^*z$ is a ternary product on A , and if $\pi : A \rightarrow A'$ is a homomorphism of $*$ -algebras, then so is of $*$ -trings.

Definition 2.2. If a subspace F of a $*$ -tring E is invariant under the ternary product, we say that it is a sub- $*$ -tring of E , or just a subring of E . A subring F is said to be hermetic in E if for $x \in E$ we have $(x, x, x) \in F \iff x \in F$.

Definition 2.3. A $*$ -tring E will be called admissible if $\{0\}$ is hermetic in E . A $*$ -algebra A will be called admissible if it is admissible as a $*$ -tring.

Note that a $*$ -algebra A is admissible if and only if the condition $a^*a = 0$ implies $a = 0$.

Definition 2.4. Let E be a $*$ -tring and $F \subseteq E$ a subspace. We say that F is an ideal of E if $(E, E, F) + (E, F, E) + (F, E, E) \subseteq F$.

If $\pi : E \rightarrow F$ is a homomorphism into an admissible $*$ -tring F , then $\ker \pi$ is an hermetic ideal of E :

$$\pi((x, x, x)) = 0 \iff (\pi(x), \pi(x), \pi(x)) = 0 \iff \pi(x) = 0$$

In case F is an ideal of E , then E/F has an obvious structure of $*$ -tring for which the canonical map $q : E \rightarrow E/F$ is a homomorphism of $*$ -trings. Note that E/F is admissible whenever F is hermetic. In particular if $\pi : E \rightarrow F$ is a homomorphism into an admissible $*$ -tring F , then $E/\ker \pi$ is admissible

Suppose E is a complex vector space, and let E^* denote its complex conjugate linear space. If (E, μ) is a $*$ -tring, then $\mu^* : E^* \times E^* \times E^* \rightarrow E^*$ given

by $\mu^*(x, y, z) = \mu(z, y, x)$, $\forall x, y, z \in E^*$, is a $*$ -ternary product on E^* . We call (E^*, μ^*) the *adjoint* or *reverse $*$ -tring* of (E, μ) . If $\pi : E \rightarrow F$ is a homomorphism, then π remains a homomorphism $E^* \rightarrow F^*$, so it is clear that reversion is an autofunctor of order two of the category of $*$ -trings, which moreover sends admissible $*$ -trings into admissible $*$ -trings. If A is a $*$ -algebra considered as a $*$ -tring as above, then its reverse $*$ -tring A^* is the conjugate linear space of A^{op} considered as a $*$ -tring.

Example 2.5 (Basic triples). Suppose (E, A, \langle, \rangle) is a triple consisting of a \mathbb{C} -vector space E , a $*$ -algebra A over which E is a right module, and a sesquilinear map $\langle, \rangle : E \times E \rightarrow A$ (conjugate linear in the first variable), such that $\langle x, y \rangle a = \langle x, ya \rangle$ and $\langle x, y \rangle^* = \langle y, x \rangle$, $\forall x, y \in E$, $a \in A$. Then $(, ,) : E \times E \times E \rightarrow E$ given by $(x, y, z) \mapsto x \langle y, z \rangle$ is a ternary product. We will say that $(E, (, ,))$ is the ternary ring associated with (E, A, \langle, \rangle) .

Definition 2.6. Triples as in Example 2.5 will be referred to as (right) basic triples. A basic triple $(E, A, \langle, \rangle_A)$ will be called admissible whenever A is admissible, and full if $\text{span}\{\langle x, y \rangle_A : x, y \in E\} = A$. By a homomorphism from the basic triple $(E, A, \langle, \rangle_A)$ into the basic triple $(F, B, \langle, \rangle_B)$ we mean a pair (φ, ψ) of maps such that $\varphi : E \rightarrow F$ is linear, $\psi : A \rightarrow B$ is a homomorphism of $*$ -algebras, and $\varphi(xa) = \varphi(x)\psi(a)$, $\forall x \in E$, $a \in A$.

Similarly we can define left basic triples, using left instead of right A -modules.

We will see soon that any admissible $*$ -tring can be described in terms of basic triples as in 2.5.

Proposition 2.7. *Let (E, A, \langle, \rangle) be a basic triple.*

- (1) *If A is admissible, and $\langle x, x \rangle = 0$ implies $x = 0$, then the $*$ -tring E is admissible as well.*
- (2) *If (E, A, \langle, \rangle) is admissible and full, then E is faithful as an A -module.*

Proof. If $x \in E$ is such that $x \langle x, x \rangle = 0$, then

$$\langle x, x \rangle^* \langle x, x \rangle = \langle x, x \rangle \langle x, x \rangle = \langle x, x \langle x, x \rangle \rangle = 0.$$

Now, if A is admissible, the latter equality implies $\langle x, x \rangle = 0$, so $x = 0$. As for the second statement suppose $(E, A, \langle, \rangle_A)$ is admissible and full, and $a \in A$ is such that $a = \sum_{j=1}^n \langle y_j, z_j \rangle$ and $ya = 0$, $\forall y \in F$. Then we have $a^*a = \sum_{j=1}^n \langle y_j, z_j a \rangle = 0$, so $a = 0$. Then E is a faithful A -module. \square

Lemma 2.8. *Suppose that $(E, A, \langle, \rangle_A)$ and $(F, B, \langle, \rangle_B)$ are basic triples, with the former full, and F admissible as $*$ -tring and faithful as a B -module. Then, if $\varphi : (E, (, ,)) \rightarrow (F, (, ,))$ is a homomorphism between their associated $*$ -trings, there exists a unique homomorphism of $*$ -algebras $\psi : A \rightarrow B$ such that $\psi(\langle x, y \rangle_A) = \langle \varphi(x), \varphi(y) \rangle_B$, $\forall x, y \in E$. Besides we have $\varphi(xa) = \varphi(x)\psi(a)$, $\forall x \in E$, $a \in A$, and*

$$\ker \psi \subseteq \{a \in A : Ea \subseteq \ker \varphi\} \subseteq \{a \in A : \psi(a)^* \psi(a) = 0\}, \quad (2.1)$$

both inclusions being equalities if B is admissible. If E is also a faithful A -module and φ is injective, then so is ψ .

Proof. We will suppose that $(F, B, \langle, \rangle_B)$ is full: otherwise we just replace B by $\text{span}\langle F, F \rangle_B$. We concentrate in showing the existence of the map ψ , because its uniqueness is obvious. To this end suppose that x_1, \dots, x_n and y_1, \dots, y_n are elements in E such that $\sum_{j=1}^n \langle x_j, y_j \rangle_A = 0$, and therefore also $\sum_{j=1}^n \langle y_j, x_j \rangle_A = 0$. Consider the element $c := \sum_{j=1}^n \langle \varphi(x_j), \varphi(y_j) \rangle_B$ of B . All we have to do is to show that $c = 0$. Now, if $x \in E$ and $u \in F$ we have

$$\begin{aligned} (\varphi(x), uc, uc) &= \sum_k (\varphi(x), (u, \varphi(x_k), \varphi(y_k)), uc) \\ &= \sum_k ((\varphi(x), \varphi(y_k), \varphi(x_k)), u, uc) = (\varphi(x) \sum_k \langle y_k, x_k \rangle_A, u, uc) = 0. \end{aligned}$$

Hence, if $u \in F$:

$$(uc, uc, uc) = \sum_j ((u, \varphi(x_j), \varphi(y_j)), uc, uc) = \sum_j (u, \varphi(x_j), (\varphi(y_j), uc, uc)) = 0$$

Since F is admissible, it follows that $uc = 0, \forall u \in F$, so $c = 0$ because F is a faithful B -module.

Suppose now that $a \in \ker \psi$. Then $\varphi(xa) = \varphi(x)\psi(a) = 0$, so $Ea \subseteq \ker \varphi$. On the other hand, if the element $a = \sum_j \langle x_j, y_j \rangle_A$ is such that $Ea \subseteq \ker \varphi$, then $\psi(a^*a) = \psi(\sum_{i,j} \langle y_i, x_i \langle x_j, y_j \rangle_A \rangle_A) = \sum_{i,j} \langle \varphi(y_i), \varphi(x_i) \psi(\langle x_j, y_j \rangle_A) \rangle_A$, so

$$\psi(a)^* \psi(a) = \sum_i \langle \varphi(y_i), \varphi(x_i) \psi(\sum_j \langle x_j, y_j \rangle_A) \rangle_A = \sum_i \langle \varphi(y_i), \varphi(x_i a) \rangle_A = 0,$$

because $\varphi(x_i a) = 0 \forall i$. In case B is admissible we have $\psi(a)^* \psi(a) = 0$ if and only if $a \in \ker \psi$, so in this case the three considered sets agree. Finally, when E is faithful and $\ker \varphi = 0$, we have $\{a \in A : Ea \subseteq \ker \varphi\} = 0$, so $\ker \psi = 0$. \square

Given two modules E and F over a ring R , we denote by $\text{Hom}_R(E, F)$ the abelian group of R -linear maps from E into F , and just by $\text{End}_R(E)$ in case $E = F$. Let E be an admissible $*$ -tring, and suppose $T \in \text{End}_{\mathbb{C}}(E)$ is such that there exists $S \in \text{End}_{\mathbb{C}}(E)$ that satisfies $(x, Ty, z) = (Sx, y, z), \forall x, y, z \in E$. Since $\{0\}$ is hermetic in E , given $T \in \text{End}_{\mathbb{C}}(E)$, there exists at most one such endomorphism S ; in this case we say that S is the adjoint of T to the left, and we denote it by T^* . The set $\mathcal{L}_l(E)$ of \mathbb{C} -linear endomorphisms of E that have an adjoint to the left is clearly a unital subalgebra of $\text{End}_{\mathbb{C}}(E)$. Every pair of elements $y, z \in E$ gives rise to an endomorphism $\theta_{y,z} : E \rightarrow E$ given by $\theta_{y,z}(x) := (x, y, z)$. It is readily checked that $\theta_{y,z}$ is adjointable with adjoint $\theta_{z,y}$.

Proposition 2.9. *Let E be an admissible $*$ -tring. Then the map $*$: $\mathcal{L}_l(E) \rightarrow \mathcal{L}_l(E)$, given by taking the adjoint, is an involution in $\mathcal{L}_l(E)$. Moreover, the $*$ -algebra $\mathcal{L}_l(E)$ is an admissible $*$ -tring, and $\text{span}\{\theta_{y,z} : y, z \in E\}$ is a twosided ideal of $\mathcal{L}_l(E)$, which is essential in the sense that $T\theta_{y,z} = 0 \forall y, z \in E$ or $\theta_{y,z}T = 0 \forall y, z \in E$ implies $T = 0$.*

Proof. It is clear that the map $T \mapsto T^*$ is conjugate linear and antimultiplicative. On the other hand, if $T \in \mathcal{L}_l(E)$:

$$(u, (Tx, y, z), u) = (u, z, (y, T(x), u)) = (u, z, (T^*(y), x, u)) = (u, (x, T^*(y), z), u)$$

$\forall x, y, z, u \in E$ and $T \in \mathcal{L}_l(E)$, which shows that $T^{**} = T$. Now, if $x \in E$, and $T \in \mathcal{L}_l(E)$ is such that $T^*T = 0$: $(Tx, Tx, Tx) = (x, T^*Tx, Tx) = 0$, so $T(x) = 0$, and therefore $T = 0$. Finally, if $T \in \mathcal{L}_l(E)$ and $x, y, z \in E$: $\theta_{y,z}T(x) = (Tx, y, z) = (x, T^*y, z) = \theta_{T^*y,z}(x)$. Thus $T\theta_{y,z} = (\theta_{z,y}T^*)^* = \theta_{y,Tz}$. This shows that $\text{span}\{\theta_{y,z} : y, z \in E\}$ is an ideal of $\mathcal{L}_l(E)$. If $\theta_{y,z}T = 0 \forall y, z \in E$, then $0 = \theta_{Tx,Tx}T(x) = (Tx, Tx, Tx), \forall x \in E$. Then $Tx = 0 \forall x \in E$ because E is admissible, so $T = 0$. \square

The next result shows that any admissible $*$ -tring E gives rise to an admissible and full *right* basic triple $(E, E_0^r, \langle \cdot, \cdot \rangle_r)$. In the same way one could show that E also defines a *left* basic triple $(E, E_0^l, \langle \cdot, \cdot \rangle_l)$.

Theorem 2.10. *Let E and F be admissible $*$ -trings. Then:*

- (1) *There exists a pair $(E_0^r, \langle \cdot, \cdot \rangle_r)$ such that $(E, E_0^r, \langle \cdot, \cdot \rangle_r)$ is an admissible and full basic triple, whose associated $*$ -tring is E .*
- (2) *If $\pi : E \rightarrow F$ is a homomorphism of $*$ -trings, and $(E_0^r, \langle \cdot, \cdot \rangle_r)$ and $(F_0^r, \langle \cdot, \cdot \rangle_r)$ are pairs like above for E and F respectively, there exists a unique homomorphism of $*$ -algebras $\pi_0^r : E_0^r \rightarrow F_0^r$ such that*

$$\pi_0^r(\langle x, y \rangle_r) = \langle \pi(x), \pi(y) \rangle_r, \quad \forall x, y \in E.$$

Moreover, $\pi(xb) = \pi(x)\pi_0^r(b), \forall x \in E, b \in E_0^r$, that is, the pair (π, π_0^r) is a homomorphism of basic triples.

- (3) *The pair $(E_0^r, \langle \cdot, \cdot \rangle_r)$ is the unique (up to canonical isomorphisms) such that the triple $(E, E_0^r, \langle \cdot, \cdot \rangle_r)$ is a full and admissible with E as associated $*$ -tring.*

Proof. Note that E is a faithful right $\mathcal{L}_l(E)^{\text{op}}$ -module with $xT := T(x)$. Consider the ideal $E_0^r := \text{span}\{\theta_{y,z} : y, z \in E\}$ of $\mathcal{L}_l(E)^{\text{op}}$ and let $\langle \cdot, \cdot \rangle_r : E \times E \rightarrow E_0^r$ be given by $\langle x, y \rangle_r := \theta_{x,y}$. It is routine to verify that $(E, E_0^r, \langle \cdot, \cdot \rangle_r)$ is a full and admissible basic triple whose associated $*$ -tring is E . The second statement follows at once from 2.8 and 2.7, while the last assertion of the theorem follows immediately from the second one. \square

Corollary 2.11. *The assignment*

$$(E \xrightarrow{\pi} F) \longmapsto (E, E_0^r, \langle \cdot, \cdot \rangle_r) \xrightarrow{(\pi, \pi_0^r)} (F, F_0^r, \langle \cdot, \cdot \rangle_r)$$

defines a functor from the category of admissible $$ -trings into the category of admissible and full basic triples.*

Corollary 2.12. *Let $(E, A, \langle \cdot, \cdot \rangle_A)$ be a basic triple such that E is faithful as an A -module and E is admissible as a $*$ -tring. Then there exists a unique homomorphism $\psi : E_0^r \rightarrow A$ such that $\langle x, y \rangle_r = \langle x, y \rangle_A, \forall x, y \in E$. The homomorphism ψ is injective, and it is an isomorphism if $(E, A, \langle \cdot, \cdot \rangle_A)$ is full.*

Proof. Let $(E, E_0^r, \langle \cdot, \cdot \rangle_r)$ be the full and admissible basic triple provided by Theorem 2.10. The identity map on E is an injective homomorphism of $*$ -trings, so by 2.8 there exists a unique homomorphism $\psi : E_0^r \rightarrow A$ such that $\langle x, y \rangle_r = \langle x, y \rangle_A, \forall x, y \in E$, which is injective because E is faithful as E_0^r -module. It is clear that ψ is also surjective when the given basic triple is full. \square

Corollary 2.13. *Let $(E, A, \langle \cdot, \cdot \rangle_A)$ be a full basic triple such that E is faithful as a A -module. Then A is admissible if E is admissible.*

Proof. Just note that if E is admissible, then $E_0^r \cong A$ by 2.12, and E_0^r is admissible according to 2.10. \square

Corollary 2.14. *Let F be an ideal of the admissible $*$ -tring E , $(E, E_0^r, \langle \cdot, \cdot \rangle_E)$ and $(F, F_0^r, \langle \cdot, \cdot \rangle_F)$ the full and admissible basic triples associated, respectively, with E and F (given by Theorem 2.10). If $A := \text{span}\{\langle x, y \rangle_E : x, y \in F\}$, then A is a $*$ -ideal of E_0^r , and the basic triples $(F, F_0^r, \langle \cdot, \cdot \rangle_F)$ and $(F, A, \langle \cdot, \cdot \rangle_E)$ are isomorphic.*

Proof. The triple $(F, A, \langle \cdot, \cdot \rangle_r)$ is admissible and full, with F as induced $*$ -tring. Then F is a faithful A -module by 2.7. According to 2.12, there exists a unique map $\psi : F_0^r \rightarrow A$ such that (id, ψ) is a homomorphism from $(F, F_0^r, \langle \cdot, \cdot \rangle_F)$ to $(F, A, \langle \cdot, \cdot \rangle_E)$, and ψ is an isomorphism of $*$ -algebras. It follows that (id, ψ^{-1}) is the inverse homomorphism of (id, ψ) . \square

From now on if F is an ideal in the admissible $*$ -tring E , we will think of F_0^r as a $*$ -ideal of E_0^r via the identification provided by 2.14:

$$F_0^r \cong \text{span}\{\langle x, y \rangle_E : x, y \in F\}. \quad (2.2)$$

For the next result recall that an ideal F of the $*$ -tring E is hermetic if and only if E/F is admissible.

Proposition 2.15. *Let $\pi : E \rightarrow F$ be a homomorphism between the admissible $*$ -trings E and F , such that $\ker \pi$ is hermetic. If $I_{\ker \pi} := \{a \in E_0^r : Ea \subseteq \ker \pi\}$, then:*

$$(\ker \pi)_0^r \subseteq \ker \pi_0^r \subseteq I_{\ker \pi}$$

Proof. Taking into account (2.2) above and the second part of 2.10, the first inclusion is clear. The second inclusion follows from the admissibility of $E/\ker \pi$ and (2.1) in Lemma 2.8. \square

Remark 2.16. Suppose F is an hermetic ideal of the admissible $*$ -tring E . Let $q : E \rightarrow E/F$ be the quotient map, $I_F := \{a \in E_0^r : Ea \subseteq F\}$, $p : E_0^r \rightarrow E_0^r/I_F$ the canonical projection and $\bar{q}_0^r : E_0^r/I_F \rightarrow (E/F)_0^r$ the isomorphism induced by q_0^r , so the following diagram commutes:

$$\begin{array}{ccc} E_0^r & \xrightarrow{q_0^r} & (E/F)_0^r \\ & \searrow p & \nearrow \bar{q}_0^r \\ & E_0^r/I_F & \end{array}$$

Then:

$$\bar{q}_0^r(p(\langle x, y \rangle_E)) = q_0^r(\langle x, y \rangle_E) = \langle q(x), q(y) \rangle_{E/F}, \quad \forall x, y \in E.$$

Therefore the pair $((E/F)_0^r, \langle \cdot, \cdot \rangle_{E/F})$ associated with E/F in Theorem 2.10 may be replaced by the pair $(E_0^r/I_F, [\cdot, \cdot]_{E/F})$, where $[q(x), q(y)]_{E/F} = p(\langle x, y \rangle_E)$, $\forall x, y \in E$ and the action of E_0^r/I_F on E/F is given by $q(x)p(a) = q(xa)$, $\forall x \in E, a \in A$.

Proposition 2.17. *Let $\pi : E \rightarrow F$ be a homomorphism between admissible $*$ -trings. Then:*

- (1) π is injective if and only if $\pi_0^r : E_0^r \rightarrow F_0^r$ is injective.
- (2) If π is onto, or an isomorphism, then so is $\pi_0^r : E_0^r \rightarrow F_0^r$.

Proof. Since the second statement is clear we prove only the first one. Now if π_0^r is injective and $x \in E$, the admissibility of E and F implies that:

$$\pi(x) = 0 \iff \langle \pi(x), \pi(x) \rangle_r = 0 \iff \pi_0^r(\langle x, x \rangle_r) = 0 \iff x = 0,$$

so π is injective as well. On the other hand the injectivity of π implies that of π_0^r by 2.8. \square

3. CORRESPONDENCE BETWEEN C^* -SEMINORMS.

3.1. C^* -seminorms.

Definition 3.1. A C^* -seminorm on a $*$ -tring (E, μ) is a seminorm such that:

- (1) $\|\mu(x, y, z)\| \leq \|x\| \|y\| \|z\|, \forall x, y, z \in E$.
- (2) $\|\mu(x, x, x)\| = \|x\|^3, \forall x \in E$.

If $\|\cdot\|$ is a norm, we call it a C^* -norm, and we say that $(E, \|\cdot\|)$ is a pre- C^* -ternary ring. If $(E, \|\cdot\|)$ is also a Banach space, we say that it is a C^* -ternary ring, or just a C^* -tring.

If E is a $*$ -tring, the set of C^* -seminorms on E will be denoted by $\mathcal{SN}(E)$, and $\mathcal{N}(E)$ will denote the set of C^* -norms on E . The set $\mathcal{SN}(E)$ is partially ordered by: $\gamma_1 \leq \gamma_2$ if $\gamma_1(x) \leq \gamma_2(x), \forall x \in E$.

Definition 3.2. A $*$ -tring E will be called C^* -closable, or just closable, in case $\mathcal{N}(E) \neq \emptyset$. Similar terminology will be used for $*$ -algebras.

Observe that any C^* -closable $*$ -tring is admissible.

In the next proposition, whose easy proof is left to the reader, we record some basic facts about $*$ -trings.

Proposition 3.3. *Let E be a $*$ -tring. Then:*

- (1) $N_\gamma := \{x \in E : \gamma(x) = 0\}$ is an hermetic ideal of E , for all $\gamma \in \mathcal{SN}(E)$.
- (2) The intersection of hermetic subtrings is also hermetic.
- (3) The quotient E/N is admissible, where $N := \cap\{N_\gamma : \gamma \in \mathcal{SN}(E)\}$ and N_γ is as in 1.
- (4) If $\mathcal{SN}(E)$ separates points of E , then E is admissible.
- (5) If $\mathcal{SN}(E)$ separates points of E and is bounded, then E is C^* -closable.

If H and K are Hilbert spaces and $B(H, K)$ denotes the corresponding space of bounded linear maps, a subspace E of $B(H, K)$ closed under the ternary product $(R, S, T) \mapsto RS^*T \in E, \forall R, S, T \in E$, is a $*$ -tring with that product. In case E is also closed it is called a ternary ring of operators (TRO). Note that if (E, μ) is a C^* -tring, then $(E, -\mu)$ also is a C^* -tring, called the opposite of (E, μ) and denoted by E^{op} . The opposite of a TRO is called anti-TRO.

New C^* -ternary rings can be obtained by direct sums: if $(E, \|\cdot\|_E, \mu_E)$ and $(F, \|\cdot\|_F, \mu_F)$ are C^* -trings, then $(E \oplus F, \max\{\|\cdot\|_E, \|\cdot\|_F\}, \mu_E \oplus \mu_F)$ is a C^* -tring. We denote it just by $E \oplus F$.

Suppose that E is a full right Hilbert A -module, and define the ternary product on E : $\mu_E(x, y, z) := x\langle y, z \rangle$. Then (E, μ_E) is a C^* -tring with the norm $\|x\| = \sqrt{\langle x, x \rangle}$. Now, if F is a full right Hilbert B -module, then $E \oplus F^{\text{op}}$ is also a C^* -tring. This is the fundamental example of C^* -tring, as shown by Zettl in [12, 3.2] (see also Corollary 3.10 below).

Zettl also showed that there exist unique sub- C^* -trings E_+ and E_- of E such that $E = E_+ \oplus E_-$, and E_+ is isomorphic to a TRO, while E_- is isomorphic to an anti-TRO (see [12]). The decomposition above is called the *fundamental decomposition* of E .

Definition 3.4. We say that a C^* -tring E is positive (negative) if $E = E_+$ (respectively: if $E = E_-$).

If E is a C^* -tring, we define $E_p := E_+ \oplus E_-^{\text{op}}$. Then E_p is a positive C^* -tring.

Let E^* be the reverse $*$ -tring of (the $*$ -tring) E . It is clear that a norm on E is a C^* -norm if and only if is a C^* -norm on E^* . Moreover, E is a (positive) C^* -tring if and only if so is E^* .

3.2. From pre- C^* -trings to pre- C^* -algebras. In what follows we will examine an intermediate situation between the $*$ -algebraic context of 2.10 and the C^* -context originally considered by Zettl.

If α is a seminorm on the vector space X , then $N_\alpha := \{x \in X : \alpha(x) = 0\}$ is a closed subspace of X , so X/N_α is a normed space with the norm $\bar{\alpha}$ induced by α : $\bar{\alpha}(x + N_\alpha) = \alpha(x)$. The completion $(X_\alpha, \bar{\alpha})$ of $(X/N_\alpha, \bar{\alpha})$ will be referred to as the *Hausdorff completion* of the seminormed space (X, α) , and the map $x \mapsto x + N_\alpha$ will be called the canonical map.

In case γ is a C^* -seminorm on the ternary ring E , then E/N_γ is a pre- C^* -tring with the induced norm $\bar{\gamma}$. Thus the corresponding Hausdorff completion E_γ of E is a C^* -tring.

Proposition 3.5. *Suppose E is an admissible $*$ -tring and $\gamma \in \mathcal{SN}(E)$. Let $\gamma^r : E_0^r \rightarrow [0, \infty)$ be the operator seminorm on E_0^r , that is:*

$$\gamma^r(a) := \sup\{\gamma(xa) : \gamma(x) \leq 1\}. \tag{3.1}$$

Then $\gamma^r \in \mathcal{SN}(E_0^r)$, and $\gamma^r \in \mathcal{N}(E_0^r) \iff \gamma \in \mathcal{N}(E)$. Moreover the following relations hold:

$$\gamma(xa) \leq \gamma(x)\gamma^r(a), \forall x \in E, a \in E_0^r \tag{3.2}$$

$$\gamma^r(\langle x, y \rangle_r) \leq \gamma(x)\gamma(y), \forall x, y \in E \tag{3.3}$$

$$\gamma(x)^2 = \gamma^r(\langle x, x \rangle_r), \forall x \in E \tag{3.4}$$

Proof. Given $a = \sum_{i=1}^n \langle x_i, y_i \rangle \in E_0^r$ the linear map $x \mapsto xa$ is bounded because $\gamma(xa) \leq \gamma(x) \sum_{i=1}^n \gamma(x_i)\gamma(y_i)$. Then (3.2) and (3.3) follow immediately and Definition 3.1 implies (3.4). With $a \in E_r^0$ as before and $x \in E$ we have

$$(xa, xa, xa) = \sum_{i=1}^n ((x, x_i, y_i), xa, xa) = \sum_{i=1}^n (x, (xa, y_i, x_i), xa),$$

so

$$\gamma(xa)^3 = \gamma(x, xaa^*, xa) \leq \gamma^r(aa^*) \gamma^r(a) \gamma(x)^3,$$

from where it follows that $\gamma^r(a)^2 \leq \gamma^r(aa^*) \leq \gamma^r(a)\gamma^r(a^*)$. From the computations above is clear that $\gamma^r \in \mathcal{N}(E_0^r) \iff \gamma \in \mathcal{N}(E)$. In particular E_0^r is a C^* -closable algebra whenever E is a C^* -closable tring. \square

Definition 3.6. Suppose $(E, A, \langle, \rangle_A)$ is a basic triple such that (E, γ) is a C^* -tring and a Banach module over the C^* -algebra (A, α) , and that $\langle, \rangle_A : E \times E \rightarrow A$ is continuous. Then the triple is said to be a C^* -basic triple. We say that it is full if the ideal $\text{span}\{\langle x, y \rangle_A : x, y \in E\}$ of A is dense in A .

The next two results will be useful for studying the relation between a C^* -basic triple $(E, A, \langle, \rangle_A)$ and the basic triple $(E, E_0^r, \langle, \rangle_r)$. What we will show first, in 3.9, is that $(E, E_0^r, \langle, \rangle_r)$ can be embedded in $(E, A, \langle, \rangle_A)$.

Proposition 3.7. *Let A be a Banach $*$ -algebra and I a $*$ -ideal of A , not necessarily closed. Then any C^* -seminorm on I can be extended to a C^* -seminorm on A . If I is dense, such extension is unique.*

Proof. Consider $\alpha \in \mathcal{SN}(I)$, $\alpha \neq 0$. Let I_α be the Hausdorff completion of (I, α) , $p : I \rightarrow I_\alpha$ the canonical map, and let $\pi : I_\alpha \rightarrow B(H)$ be a faithful representation. Now, according to [4, VI-19.11], the representation $\pi p : I \rightarrow B(H)$ can be extended to a representation ρ of A . Then $a \mapsto \|\rho(a)\|$ defines a C^* -seminorm on A that extends α . Note that the continuity of ρ implies the continuity of α , from which the uniqueness of the extension follows in case I is dense in A . \square

Corollary 3.8. *Let I be a $*$ -ideal of the C^* -algebra A . Then the unique C^* -norm one can define in I is the restriction to I of the norm of A .*

Proposition 3.9. *Let $(E, A, \langle, \rangle_A)$ be a full C^* -basic triple, and γ and α the corresponding norms of E and A . Then (A, α) is the completion of (E_0^r, γ^r) , and \langle, \rangle_A is the continuous extension of \langle, \rangle_r .*

Proof. Note that E is admissible for it is a C^* -tring. On the other hand E is a faithful A -module: if $a \in A$ is such that $xa = 0 \forall x \in E$, then $\langle x, y \rangle_A a = 0 \forall x, y \in E$, so it follows that $ba = 0$ for every b in the dense ideal $\text{span}\{\langle x, y \rangle_A : x, y \in E\}$ of A , which implies $a = 0$. Thus there exists, by 2.7, a unique homomorphism $\psi : E_0^r \rightarrow A$ such that $\psi(\langle x, y \rangle_r) = \langle x, y \rangle_A, \forall x, y \in E$. Besides ψ is injective and $\psi(E_0^r) = \text{span}\{\langle x, y \rangle_A : x, y \in E\}$ (thus we may suppose E_0^r is a dense ideal of A). Now 3.8 implies γ_0^r is the restriction of α to $\psi(E_0^r)$ and, since the latter is dense in A , we conclude that A is the completion of E_0^r . \square

As a consequence we obtain the following result, due to H. Zettl:

Corollary 3.10 (cf. [12, Proposition 3.2]). *Let (E, γ) be a C^* -tring and E^r the completion of E_0^r with respect to γ^r . Then $(E, E^r, \langle \cdot, \cdot \rangle_r)$ is, up to isomorphism, the unique full C^* -basic triple whose first component is E .*

Proposition 3.11. *Let $\pi : E_1 \rightarrow E_2$ be a homomorphism of $*$ -trings between the C^* -trings E_1 and E_2 . Then there exists a unique homomorphism $\pi^r : E^r \rightarrow F^r$ such that $\pi^r(\langle x, y \rangle_E) = \langle \pi(x), \pi(y) \rangle_{F^r}$, $\forall x, y \in E$, and $\pi(xa) = \pi(x)\pi^r(a)$ $\forall x \in E$, $a \in E^r$. Consequently π is always contractive, and is isometric if and only if it is injective.*

Proof. It is clear that, if the homomorphism π^r exists, it must be an extension of $\pi_0^r : E_0^r \rightarrow F_0^r$. Let $\rho : F^r \rightarrow B(H)$ be a faithful representation. Then $\rho\pi_0^r$ is a representation of E_0^r . Now, since $(E, E^r, \langle \cdot, \cdot \rangle_r)$ is a C^* -triple, E_0^r is a $*$ -ideal in E^r . Therefore $\rho\pi_0^r$ can be uniquely extended to a representation $\bar{\rho} : E^r \rightarrow B(H)$ ([4, VI.19.11]). Since $\rho(F^r)$ is closed and $\bar{\rho}(E^r)$ is a subset of the closure of $\rho\pi_0^r(E_0^r)$, we have $\bar{\rho}(E^r) \subseteq \rho(F^r)$. Then take $\pi^r := \rho^{-1}\bar{\rho}$. Note that $\|\pi(x)\|^2 = \|\pi^r(\langle x, x \rangle)\| \leq \|\langle x, x \rangle\| = \|x\|^2$, with equality if π^r is injective. This shows that π is contractive. Finally, if π is injective, so is π_0^r and, as in the proof of 3.8, this implies that π^r also is injective, thus an isometry. \square

Corollary 3.12 (cf. [1][Proposition 4.1]). *The assignment*

$$(E \xrightarrow{\pi} F) \longmapsto (E, E^r, \langle \cdot, \cdot \rangle_r) \xrightarrow{(\pi, \pi^r)} (F, F^r, \langle \cdot, \cdot \rangle_r)$$

defines a functor from the category of C^ -trings to the category of full C^* -basic triples.*

It follows from Proposition 3.5 that any C^* -seminorm on E_0^r induced by a C^* -seminorm on E by means of (3.1) must satisfy the Cauchy-Schwarz condition (3.3). So it is natural to restrict our attention to the following subsets of C^* -seminorms on E_0^r :

$$\begin{aligned} \mathcal{SN}_{cs}(E_0^r) &:= \{\alpha \in \mathcal{SN}(E_0^r) : \alpha(\langle x, y \rangle_r)^2 \leq \alpha(\langle x, x \rangle_r) \alpha(\langle y, y \rangle_r)\} \\ \mathcal{N}_{cs}(E_0^r) &:= \mathcal{SN}_{cs}(E_0^r) \cap \mathcal{N}(E_0^r). \end{aligned}$$

In fact it will be convenient to place ourselves in a slightly more general setting:

Definition 3.13. Let $(E, A, \langle \cdot, \cdot \rangle)$ be a basic triple. We define

$$\mathcal{SN}_{cs}^{(\cdot)}(A) := \{\alpha \in \mathcal{SN}(A) : \alpha(\langle x, y \rangle)^2 \leq \alpha(\langle x, x \rangle)\alpha(\langle y, y \rangle), \forall x, y \in E\}.$$

Proposition 3.14. *Let $(E, A, \langle \cdot, \cdot \rangle)$ be a basic triple, and consider E with the $*$ -tring structure induced by $\langle \cdot, \cdot \rangle$. Given $\alpha \in \mathcal{SN}_{cs}^{(\cdot)}(A)$, let $\tilde{\alpha} : E \rightarrow [0, \infty)$ be defined by:*

$$\tilde{\alpha}(x) := \alpha(\langle x, x \rangle)^{1/2} \tag{3.5}$$

Then

- (1) $\tilde{\alpha}(xa) \leq \tilde{\alpha}(x)\alpha(a)$.
- (2) $\tilde{\alpha} \in \mathcal{SN}(E)$
- (3) *If E is a faithful A -module and $\tilde{\alpha} \in \mathcal{N}(E)$, then $\alpha \in \mathcal{N}_{cs}^{(\cdot)}(A)$.*
- (4) *If $\alpha \in \mathcal{N}_{cs}^{(\cdot)}(A)$ and $\langle x, x \rangle = 0$ implies $x = 0$, then $\tilde{\alpha} \in \mathcal{N}(E)$*

Proof. Since the Cauchy-Schwarz inequality (3.3) holds for α , it follows as usual that $\tilde{\alpha}$ satisfies the triangular inequality and, since homogeneity is obvious, $\tilde{\alpha}$ is a seminorm on E . On the other hand, since α is a C^* -seminorm and satisfies (3.3) we have, for all $x, y, z \in E, a \in A$:

$$\begin{aligned}\tilde{\alpha}(xa) &= \alpha(a^* \langle x, x \rangle a)^{1/2} \leq \alpha(a) \tilde{\alpha}(x) \\ \tilde{\alpha}(\langle x, y, z \rangle) &= \tilde{\alpha}(x \langle y, z \rangle) \leq \tilde{\alpha}(x) \alpha(\langle y, z \rangle) \leq \tilde{\alpha}(x) \tilde{\alpha}(y) \tilde{\alpha}(z) \\ \tilde{\alpha}(\langle x, x, x \rangle) &= \alpha(\langle x, x \rangle^3)^{1/2} = \alpha(\langle x, x \rangle)^{3/2} = \tilde{\alpha}(x)^3,\end{aligned}$$

so $\tilde{\alpha}$ is a C^* -seminorm on E . The first of the above inequalities implies that α is a norm whenever $\tilde{\alpha}$ so is and E is a faithful A -module. Finally, if α is a norm, it follows directly from (3.5) that $\tilde{\alpha}$ also is a norm when the condition $\langle x, x \rangle = 0$ implies $x = 0$. \square

Corollary 3.15. *If E is an admissible $*$ -tring and $\gamma \in \mathcal{SN}(E)$, $\alpha \in \mathcal{SN}_{cs}(E_0^r)$, then $\tilde{\gamma}^r = \gamma$ and $\tilde{\alpha}^r \leq \alpha$.*

Proof. The first statement follows immediately from (3.4) and (3.5). As for the second one we have $\tilde{\alpha}^r(a) = \sup\{\tilde{\alpha}(xa) : \tilde{\alpha}(x) \leq 1\} \leq \alpha(a)$ by 1. of 3.14. \square

Corollary 3.16. *Let $(E, A, \langle \cdot, \cdot \rangle)$ be a full basic triple, and $\alpha \in \mathcal{SN}_{cs}^{(\cdot)}(A)$. If $\tilde{\alpha} \in \mathcal{SN}(E)$ is given by (3.5), then $I_{N_{\tilde{\alpha}}} = N_{\alpha}$, where $I_{N_{\tilde{\alpha}}} := \{a \in A : Ea \subseteq N_{\tilde{\alpha}}\}$.*

Proof. The inclusion $N_{\alpha} \subseteq I_{N_{\tilde{\alpha}}}$ is clear because $\tilde{\alpha}(xa) \leq \tilde{\alpha}(x)\alpha(a)$, $\forall x \in E, a \in A$. Conversely, suppose that $a \in A$ is such that $\tilde{\alpha}(xa) = 0$, $\forall x \in E$. Then $\alpha(a^* \langle x, y \rangle a) = \alpha(\langle xa, ya \rangle) \leq \tilde{\alpha}(xa) \tilde{\alpha}(ya) = 0$, $\forall x, y \in E$. Now, since the triple is full, we can write $aa^* = \sum_j \langle x_j, y_j \rangle$, for certain $x_j, y_j \in E$, so we have:

$$0 \leq \alpha(a)^4 = \alpha(a^* a)^2 = \alpha(a^* aa^* a) = \alpha(a^* \sum_j \langle x_j, y_j \rangle a) \leq \sum_j \alpha(a^* \langle x_j, y_j \rangle a) = 0,$$

hence $a \in N_{\alpha}$. \square

Proposition 3.17. *Let $(E, A, \langle \cdot, \cdot \rangle)$ be a full basic triple, and $\alpha \in \mathcal{SN}_{cs}^{(\cdot)}(A)$. Let $\gamma := \tilde{\alpha} \in \mathcal{SN}(E)$, $\tilde{\alpha}$ given by (3.5). Then E_{γ} is a C^* -tring, $(E_{\gamma}^r, \tilde{\gamma}^r) = (A_{\alpha}, \bar{\alpha})$ and $\tilde{\alpha}^r = \alpha$.*

Proof. Denote by $q : E \rightarrow E/N_{\gamma} \subseteq E_{\gamma}$ and $p : A \rightarrow A/N_{\alpha} \subseteq A_{\alpha}$ the corresponding canonical maps. We define $E/N_{\gamma} \times A/N_{\alpha} \rightarrow E/N_{\gamma}$ and $[\cdot, \cdot] : E/N_{\gamma} \times E/N_{\gamma} \rightarrow A/N_{\alpha}$ such that $q(x)p(a) := q(xa)$ and $[q(x), q(y)] := p(\langle x, y \rangle)$ respectively. Let us see that these operations are continuous in the norms $\tilde{\gamma}$ and $\bar{\alpha}$. The action of A/N_{α} on E/N_{γ} is continuous, for if $x, y \in E$ and $a \in A$:

$$\tilde{\gamma}(q(x)p(a)) = \tilde{\gamma}(q(xa)) = \gamma(xa) \leq \gamma(x)\alpha(a) = \tilde{\gamma}(q(x))\bar{\alpha}(p(a))$$

And the sesquilinear map $[\cdot, \cdot]_{E/N_{\gamma}}$ also is continuous, because:

$$\bar{\alpha}([q(x), q(y)]_{E/N_{\gamma}}) = \bar{\alpha}(p(\langle x, y \rangle_E)) = \alpha(\langle x, y \rangle_E) \leq \gamma(x)\gamma(y) = \tilde{\gamma}(q(x))\tilde{\gamma}(q(y)).$$

Therefore these operations extend to continuous maps $E_{\gamma} \times A_{\alpha} \rightarrow E_{\gamma}$ and $[\cdot, \cdot] : E_{\gamma} \times E_{\gamma} \rightarrow A_{\alpha}$, so we obtain a full C^* -basic triple $(E_{\gamma}, A_{\alpha}, [\cdot, \cdot])$. Therefore $(A_{\alpha}, \alpha) = (E_{\gamma}^r, \tilde{\gamma}^r)$ by 3.9. As for the last assertion, we have to prove that $\gamma^r = \alpha$

or, equivalently, that $\bar{\gamma}^r = \bar{\alpha}$. So it is enough to show that $\gamma^r = \bar{\gamma}^r p$. But, if $a \in A$:

$$\bar{\gamma}^r(p(a)) = \sup\{\bar{\gamma}(q(x)p(a)) : \bar{\gamma}(q(x)) \leq 1\} = \sup\{\bar{\gamma}(q(xa)) : \gamma(x) \leq 1\} = \gamma^r(a).$$

□

Propositions 3.5 and 3.14 allow us to define maps $\Phi_r : \mathcal{SN}(E) \rightarrow \mathcal{SN}_{cs}(E_0^r)$ and $\Psi_r : \mathcal{SN}_{cs}(E_0^r) \rightarrow \mathcal{SN}(E)$ such that $\Phi_r(\gamma) = \gamma^r$, given by (3.1), and $\Psi_r(\alpha) = \tilde{\alpha}$, given by (3.5). We want to show that in fact Φ_r and Ψ_r are mutually inverse maps that preserve the order.

Theorem 3.18. *Let E be an admissible $*$ -tring. Then the maps $\Phi_r : \mathcal{SN}(E) \rightarrow \mathcal{SN}_{cs}(E_0^r)$ and $\Psi_r : \mathcal{SN}_{cs}(E_0^r) \rightarrow \mathcal{SN}(E)$ are mutually inverse isomorphisms of lattices. Moreover $\Phi_r(\mathcal{N}(E)) = \mathcal{N}_{cs}(E_0^r)$ and $\Psi_r(\mathcal{N}_{cs}(E_0^r)) = \mathcal{N}(E)$.*

Proof. By Corollary 3.15 we have $\Psi_r \Phi_r = Id_{\mathcal{SN}(E)}$, and Proposition 3.17 shows that $\Phi_r \Psi_r = Id_{\mathcal{SN}_{cs}(E_0^r)}$, so the maps Φ_r and Ψ_r are mutually inverse. Besides, it follows from 3.5 that $\Phi_r(\gamma)$ is a norm if and only if so is γ . On the other hand is clear that Ψ_r preserves the order, thus it remains to be shown that Φ_r also preserves the order. To this end consider $\gamma_1 \leq \gamma_2$ in $\mathcal{SN}(E)$. Since $id : (E, \gamma_2) \rightarrow (E, \gamma_1)$ is continuous, it induces a homomorphism $\pi : E_{\gamma_2} \rightarrow E_{\gamma_1}$, which in turn induces, according with Proposition 3.11, a homomorphism $\pi^r : E_{\gamma_2}^r \rightarrow E_{\gamma_1}^r$, which is necessarily contractive. Thus if $a \in E_0^r$, we have:

$$\gamma_1^r(a) = \bar{\gamma}_1^r(\pi^r(a + N_{\gamma_2^r})) \leq \bar{\gamma}_2^r(a + N_{\gamma_1^r}) = \gamma_2^r(a),$$

which shows that $\gamma_1^r \leq \gamma_2^r$. □

All we have done to the right side can be done also to the left side. For example, every admissible $*$ -tring E induces a (left) admissible and full basic triple $(E, E_0^l, \langle \cdot, \cdot \rangle_l)$, we have an isomorphism of posets $\Phi_l : \mathcal{SN}(E) \rightarrow \mathcal{SN}_{cs}(E_0^l)$ with inverse $\Psi_l : \mathcal{SN}_{cs}(E_0^l) \rightarrow \mathcal{SN}(E)$, given by $\Phi_l(\gamma) = \gamma^l$ and $\Psi(\alpha) = \tilde{\alpha}$, where $\gamma^l(a) := \sup\{\gamma(ax) : \gamma(x) \leq 1\}$ and $\tilde{\alpha}(x) := \alpha(\langle x, x \rangle_l)^{1/2}$, etc. Then we obtain the following consequences:

Corollary 3.19. *Let E be an admissible $*$ -tring. Then $\Phi_r \Psi_l : \mathcal{SN}_{cs}(E_0^l) \rightarrow \mathcal{SN}_{cs}(E_0^r)$ is an isomorphism of lattices such that $\Phi_r \Psi_l(\mathcal{N}_{cs}(E_0^l)) = \mathcal{N}_{cs}(E_0^r)$. The inverse of $\Phi_r \Psi_l$ is $\Phi_l \Psi_r$.*

As mentioned at the end of 3.1 in [12][Theorem 3.1], Zettl proved that any C^* -tring is of the form $E = E_+ \oplus E_-$, where E_+ and E_-^{op} are isomorphic to a TRO. In fact we have $E_+ := \{x \in E : \langle x, x \rangle_r \text{ is positive}\}$, $E_- := \{x \in E : -\langle x, x \rangle_r \text{ is positive}\}$, and E_+ and E_- are ideals of E such that $\langle E_+, E_- \rangle = 0$. If $E_p := E_+ \oplus E_-^{op}$, we will have that $E_p^r = E^r$ and $E_p^l = E^l$, and now E_p is a Morita–Rieffel equivalence between E^l and E^r . Thus we have:

Corollary 3.20. *Let E be an admissible $*$ -tring and $\gamma \in \mathcal{SN}(E)$. Then E_γ^l and E_γ^r are Morita–Rieffel equivalent C^* -algebras.*

In general we will have to deal with algebras that strictly contain E_0^r , but whose C^* -seminorms are essentially the same, as the following results show.

Proposition 3.21. *Let I be a selfadjoint ideal of a $*$ -algebra A , and suppose that $\alpha \in \mathcal{SN}(I)$. Let $\alpha' : A \rightarrow [0, \infty]$ be given by $\alpha'(a) := \sup\{\alpha(ax) : x \in I, \alpha(x) \leq 1\}$. For every $a \in A$ consider $L_a : I \rightarrow I$, such that $L_a(x) = ax, \forall x \in I$. Then the following statements are equivalent:*

- (1) $\alpha'(a) < \infty, \forall a \in A$.
- (2) L_a is bounded, $\forall a \in A$.
- (3) α can be extended to a C^* -seminorm on A .

Suppose that one of the conditions above holds true. Then:

- (a) α' is a C^* -seminorm on A , and $\alpha' \leq \beta$ for every $\beta \in \mathcal{SN}(A)$ that extends α .
- (b) If α is a norm, then α' is a norm if and only if $\text{Ann}_A(I) = 0$, where $\text{Ann}_A(I) := \{a \in A : ax = 0, \forall x \in I\}$.

Proof. Since $\|L_a\| = \alpha'(a)$, we have that conditions 1. and 2. are equivalent. It is also clear that 3. \Rightarrow 1. Suppose now that $\alpha'(a) < \infty, \forall a \in A$. Let show that α' is a C^* -seminorm on A that extends α . It is easy to check that $\alpha'(ab) \leq \alpha'(a)\alpha'(b), \forall a, b \in A$. Moreover:

$$\begin{aligned} \alpha'(a^*a) &= \sup\{\alpha(a^*ax) : x \in I, \alpha(x) \leq 1\} \geq \sup\{\alpha(x^*a^*ax) : x \in I, \alpha(x) \leq 1\} \\ &\geq \sup\{\alpha(ax)^2 : x \in I, \alpha(x) \leq 1\} = \alpha'(a)^2. \end{aligned}$$

Therefore $\alpha' \in \mathcal{SN}(A)$. The fact that α' extends α , as well as assertion (a), are consequences of the fact that for every C^* -seminorm β on A one has that $\beta(a) = \sup\{\beta(ab) : \beta(b) \leq 1\}$. Finally, suppose that α is a norm on I . Then $\alpha'(a) = 0 \iff \alpha(ax) = 0, \forall x \in I$, that is $\alpha'(a) = 0 \iff a \in \text{Ann}_A(I)$. \square

Theorem 3.22. *Let $(E, A, \langle \cdot, \cdot \rangle)$ be an admissible basic triple, with E a faithful A -module, and admissible as $*$ -tring. Suppose that any C^* -seminorm on E_0^r can be extended in a unique way to a C^* -seminorm on A (recall Corollary 2.12). Then the lattices $\mathcal{SN}(E)$ and $\mathcal{SN}_{cs}^{\langle \cdot, \cdot \rangle}(A)$ are isomorphic. If in addition $\text{Ann}_A(E_0^r) = 0$, the posets $\mathcal{N}(E)$ and $\mathcal{N}_{cs}^{\langle \cdot, \cdot \rangle}(A)$ are isomorphic as well.*

Proof. Since any C^* -seminorm on E_0^r can be uniquely extended to a C^* -seminorm on A , we are allowed to identify $\mathcal{SN}(A)$ and $\mathcal{SN}(E_0^r)$ as lattices, and it is clear that this yields also an identification between $\mathcal{SN}_{cs}^{\langle \cdot, \cdot \rangle}(A)$ and $\mathcal{SN}_{cs}(E_0^r)$, and the latter is isomorphic to $\mathcal{SN}(E)$ by 3.18. If moreover $\text{Ann}_A(E_0^r) = 0$, the same argument applies to $\mathcal{N}(E)$ and $\mathcal{N}_{cs}(A)$. \square

In case A is a Banach $*$ -algebra, any C^* -seminorm on a $*$ -ideal can be extended to a C^* -seminorm defined on the whole algebra. Moreover we have:

Proposition 3.23. *Let A be an admissible Banach $*$ -algebra and I a dense $*$ -ideal of A , not necessarily closed. Then any C^* -norm on I can be uniquely extended to a C^* -norm on A .*

Proof. Let $\alpha \in \mathcal{N}(I)$. By 3.7 α has a unique extension to a C^* -seminorm on A , and by 3.21 this extension must be α' such that $\alpha'(a) = \sup\{\alpha(ax) : x \in I, \alpha(x) \leq 1\}$. Suppose $a \in \text{Ann}_A(I)$. Then $aa^* = 0$, because I is dense in A and $ax = 0, \forall x \in I$. Thus $a = 0$ for A is admissible. Then α' is a norm by 3.21. \square

Corollary 3.24. *Let $(E, A, \langle \cdot, \cdot \rangle_E)$ be an admissible basic triple with A a Banach $*$ -algebra and E a faithful A -module. Suppose in addition that E is an admissible $*$ -tring such that E_0^r is a dense ideal of A (recall Corollary 2.12). Then the lattices $\mathcal{SN}(E)$ and $\mathcal{SN}_{cs}(A)$ are isomorphic, as well as the partially ordered sets $\mathcal{N}(E)$ and $\mathcal{N}_{cs}(A)$.*

Proof. Just combine Theorem 3.22 with Proposition 3.7 and Proposition 3.23. \square

Corollary 3.25. *Let $(E, A, \langle \cdot, \cdot \rangle_A)$ and $(E, B, \langle \cdot, \cdot \rangle_B)$ be respectively left and right admissible basic triples, with A and B Banach $*$ -algebras such that E is an $(A - B)$ -bimodule with the given structure, and $\langle x, y \rangle_{Az} = x \langle y, z \rangle_B$, $\forall x, y, z \in E$. If E is faithful as a left A -module and as a right B -module, and E_0^l and E_0^r are dense in A and B respectively, then there is an isomorphism of lattices between $\mathcal{SN}_{cs}^{(\cdot, \cdot)^A}(A)$ and $\mathcal{SN}_{cs}^{(\cdot, \cdot)^B}(B)$, that restricts to an isomorphism between the posets $\mathcal{N}_{cs}^{(\cdot, \cdot)^A}(A)$ and $\mathcal{N}_{cs}^{(\cdot, \cdot)^B}(B)$.*

3.3. Positive modules. In general is not a simple task to decide if a given C^* -seminorm satisfies the Cauchy-Schwarz property with respect to a certain sesquilinear map. However this is always the case for the positive modules we introduce next.

Let α be a C^* -seminorm on the $*$ -algebra A , and let $p_\alpha : A \rightarrow A_\alpha$ be the canonical map, where A_α is the Hausdorff completion of A . If $\Lambda \subseteq \mathcal{SN}(A)$, then $A_\Lambda^+ := \bigcap_{\alpha \in \Lambda} p_\alpha^{-1}(A_\alpha^+)$ is a cone. When $\Lambda = \mathcal{SN}(A)$, we write A^+ instead of A_Λ^+ . Therefore A^+ is the set of elements of A that are positive in any C^* -Hausdorff completion of A . Of course the map $\Lambda \mapsto A_\Lambda^+$ is order reversing.

Definition 3.26. Given $\Lambda \subseteq \mathcal{SN}(A)$, we say that $a \in A$ is positive in (A, Λ) , or that it is Λ -positive, if $a \in A_\Lambda^+$. The elements of A^+ are just called the positive elements of A .

It is clear that A^+ contains the cone $C_A := \{\sum_{i,j=1}^n a_i^* a_j : n \in \mathbb{N}, a_i \in A, i = 1, \dots, n\}$, and that $p_\alpha(C_A)$ is dense in A_α^+ , $\forall \alpha \in \mathcal{SN}(A)$. Also note that if $\phi : A \rightarrow B$ is a homomorphism between $*$ -algebras, then $\phi(A^+) \subseteq B^+$ and $\phi(C_A) \subseteq C_B$.

If $\mathcal{SN}(A)$ is bounded, with $\alpha := \max \mathcal{SN}(A)$, then a is positive in A if and only if a is positive in (A, α) . In particular, if A is a Banach $*$ -algebra, then $a \in A^+$ if and only if $\iota(a) \in C^*(A)^+$, where $\iota : A \rightarrow C^*(A)$ is the natural map of A into its C^* -enveloping algebra $C^*(A)$.

Lemma 3.27. *Let A be C^* -closable. Then $A^+ = \bigcap \{p_\alpha^{-1}(A_\alpha^+) : \alpha \in \mathcal{N}(A)\}$.*

Proof. Clearly we have that $A^+ \subseteq \bigcap \{p_\alpha^{-1}(A_\alpha^+) : \alpha \in \mathcal{N}(A)\}$. Let $\beta \in \mathcal{SN}(A)$. Since the maximum of two C^* -seminorms is again a C^* -seminorm, and since A is C^* -closable, we may pick $\beta' \in \mathcal{N}(A)$ such that $\beta' \geq \beta$. Then the identity map on A induces a homomorphism $\phi : A_{\beta'} \rightarrow A_\beta$, determined by $\phi(p_{\beta'}(a)) = p_\beta(a)$, $\forall a \in A$. If $a \in \bigcap \{p_\alpha^{-1}(A_\alpha^+) : \alpha \in \mathcal{N}(A)\}$ then $p_{\beta'}(a) \in A_{\beta'}^+$, and therefore $p_\beta(a) \in A_\beta^+$. This proves the converse inclusion. \square

Once we have a cone of positive elements on a $*$ -algebra A , we are able to define a notion similar to that of Hilbert module.

Definition 3.28. Let A be a $*$ -algebra, E a right A -module, and $\Lambda \subseteq \mathcal{SN}(A)$. We say that a map $\langle \cdot, \cdot \rangle : E \times E \rightarrow A$ is a Λ -semi-pre-inner product on E if:

- (1) $\langle x, \lambda_1 y + \lambda_2 z \rangle = \lambda_1 \langle x, y \rangle + \lambda_2 \langle x, z \rangle, \forall x, y, z \in E, \lambda_1, \lambda_2 \in \mathbb{C}$.
- (2) $\langle x, ya \rangle = \langle x, y \rangle a, \forall x, y \in E, a \in A$.
- (3) $\langle y, x \rangle = \langle x, y \rangle^*, \forall x, y \in E$.
- (4) $\langle x, x \rangle$ is Λ -positive, $\forall x \in E$.

The pair $(E, \langle \cdot, \cdot \rangle)$ is then called a right positive Λ -module. In case $\Lambda = \mathcal{SN}(A)$ we say that $(E, \langle \cdot, \cdot \rangle)$ is a right positive A -module.

Similarly we define left semi-pre-inner-products and left positive modules.

Definition 3.29. An admissible $*$ -tring E is right (left) positive if $(E, \langle \cdot, \cdot \rangle_r)$ is a positive E_0^r -module (respectively: $(E, \langle \cdot, \cdot \rangle_l)$ is a positive E_0^l -module). It is said positive if it is both left and right positive.

Observe that if E is a C^* -tring, which is positive as an admissible $*$ -tring, then it is obviously a positive C^* -tring. Conversely, it is readily checked that any positive C^* -tring is a positive admissible $*$ -tring.

Proposition 3.30. Let (A, α) be a C^* -seminormed algebra and $(E, \langle \cdot, \cdot \rangle)$ a right positive (A, α) -module. Let $\tilde{\alpha} : E \rightarrow [0, \infty)$ be given by $\tilde{\alpha}(x) = \sqrt{\alpha(\langle x, x \rangle)}$, $\forall x \in E$. Consider E as a $*$ -tring with $(x, y, z) := x \langle y, z \rangle, \forall x, y, z \in E$. Then:

- (1) We have $\alpha(a) \leq \alpha(b)$ whenever a and $b - a$ are positive elements of A .
- (2) $\tilde{\alpha}(x)^2 \langle y, y \rangle - \langle x, y \rangle^* \langle x, y \rangle$ is positive in (A, α) , and $\alpha(\langle x, y \rangle) \leq \tilde{\alpha}(x) \tilde{\alpha}(y), \forall x, y \in E$ (Cauchy-Schwarz).
- (3) $\alpha(\langle x, x \rangle) a^* a - a^* \langle x, x \rangle a \geq 0, \forall x \in E, a \in A$.
- (4) $\tilde{\alpha}(xa) \leq \tilde{\alpha}(x) \alpha(a), \forall x \in E, a \in A$.
- (5) $\tilde{\alpha} \in \mathcal{SN}(E)$.

Proof. Let $p_\alpha : A \rightarrow A/I_\alpha =: A_\alpha$ be the natural map, where I_α is the ideal $I_\alpha := \{a \in A : \alpha(a) = 0\}$, and let $\bar{\alpha}$ be the quotient norm on A_α . Now let $F := \text{span}\{xb \in E : x \in E, b \in I_\alpha\}$. Then $E I_\alpha \subseteq F$, so E/F is an A/I_α -module. Moreover, $\langle E, F \rangle \subseteq I_\alpha$ and $\langle F, E \rangle \subseteq I_\alpha$, so we can consider the map $[\cdot, \cdot] : E/F \times E/F \rightarrow A/I_\alpha$ given by $[q(x), q(y)] = p_\alpha(\langle x, y \rangle)$, which satisfies properties 1.–4. of Definition 3.28 above. If a and $b - a$ are positive in A , then $0 \leq p_\alpha(a) \leq p_\alpha(b)$ in A_α , and therefore $\bar{\alpha}(p_\alpha(a)) \leq \bar{\alpha}(p_\alpha(b))$, that is $\alpha(a) \leq \alpha(b)$. This proves 1. Now, the first part of the second statement follows from the proof of [6, Proposition 1.1], since $p_\alpha(\tilde{\alpha}(x)^2 \langle y, y \rangle - \langle y, x \rangle \langle x, y \rangle) = \bar{\alpha}([q(x), q(x)] [q(y), q(y)] - [q(y), q(x)] [q(x), q(y)])$ in A_α . The second part of 2. follows from the first one and from 1. To see 3. just observe that by applying p_α to the element $\alpha(\langle x, x \rangle) a^* a - a^* \langle x, x \rangle a$ of A we get the positive element $\bar{\alpha}([x, x]) p_\alpha(a)^* p_\alpha(a) - p_\alpha(a)^* [x, x] p_\alpha(a)$ of A_α . Assertion 4. easily follows from 1. and 3: by 3. we have $a^* \langle x, x \rangle a \leq \tilde{\alpha}(x)^2 a^* a$, then $\tilde{\alpha}(xa)^2 = \alpha(\langle xa, xa \rangle) = \alpha(a^* \langle x, x \rangle a)$, and by 1. this is less or equal to $\alpha(\tilde{\alpha}(x)^2 a^* a) = \tilde{\alpha}(x)^2 \alpha(a)^2$. It is clear that $\tilde{\alpha}(\lambda x) = |\lambda| \tilde{\alpha}(x), \forall x \in E, \lambda \in \mathbb{C}$, and from the Cauchy-Schwarz inequality just proved it readily follows that $\tilde{\alpha}$ also satisfies the triangle inequality, so it is a seminorm on E . Now, if $x, y, z \in E$: $\tilde{\alpha}(x \langle y, z \rangle)^2 = \alpha(\langle y, z \rangle^* \langle x, x \rangle \langle y, z \rangle)$. Thus, in the case $x = y = z$:

$$\tilde{\alpha}(\langle x, x, x \rangle) = \alpha(\langle x, x \rangle^3)^{1/2} = \alpha(\langle x, x \rangle^{1/2})^3 = \tilde{\alpha}(x)^3.$$

According to 3. we have $\langle y, z \rangle^* \langle x, x \rangle \langle y, z \rangle \leq \alpha(\langle x, x \rangle) \langle y, z \rangle^* \langle y, z \rangle$ in (A, α) . From this fact, together with 4. and the Cauchy-Schwarz inequality we conclude that

$$\tilde{\alpha}(x \langle y, z \rangle)^2 \leq \tilde{\alpha}(x)^2 \alpha(\langle y, z \rangle)^2 \leq (\tilde{\alpha}(x) \tilde{\alpha}(y) \tilde{\alpha}(z))^2$$

so $\tilde{\alpha}$ is a C^* -seminorm on E . \square

Corollary 3.31. *If E is a right positive $*$ -tring, then $\mathcal{SN}_{cs}(E_0^r) = \mathcal{SN}(E_0^r)$, and $\mathcal{SN}(E) \cong \mathcal{SN}(E_0^r)$ and $\mathcal{N}(E) \cong \mathcal{N}(E_0^r)$ as ordered sets.*

Proposition 3.32. *Let E be an admissible $*$ -tring and $\gamma \in \mathcal{SN}(E)$. If E is a right positive (E_0^r, γ^r) -module, then E is also a left positive (E_0^l, γ^l) -module. Therefore E is right positive if and only if is left positive.*

Proof. Let E_γ be the Hausdorff completion of (E, γ) . Since E_γ is a right Hilbert module over E_γ^r , it turns out that E_γ is a positive C^* -tring, and therefore a left Hilbert module over E_γ^l , so E is a left positive (E_0^l, γ^l) -module. \square

Proposition 3.33. *Let B be an admissible Banach $*$ -algebra and suppose E is a right closed ideal of B such that $\text{span}\{x^*y : x, y \in E\}$ is dense in B . Let A be the closure in B of $\text{span}\{xy^* : x, y \in E\}$. If xx^* is positive in A , $\forall x \in E$, then the restriction map $\varphi : \mathcal{SN}(B) \rightarrow \mathcal{SN}(A)$, $\beta \mapsto \beta|_A$, is a lattice isomorphism such that $\varphi(\mathcal{N}(B)) = \mathcal{N}(A)$, and for each $\beta \in \mathcal{SN}(B)$ the Hausdorff completion B_β of B is Morita–Rieffel equivalent to the Hausdorff completion $A_{\varphi(\beta)}$ of A . In particular, the corresponding enveloping C^* -algebras $C^*(B)$ and $C^*(A)$ of B and A are Morita–Rieffel equivalent C^* -algebras.*

Proof. Let $\langle \cdot, \cdot \rangle_B : E \times E \rightarrow B$ and $\langle \cdot, \cdot \rangle_A : E \times E \rightarrow A$ be such that $\langle x, y \rangle_B = x^*y$ and $\langle x, y \rangle_A = xy^*$. Then E is both a positive B -module and a positive A -module. Since B is admissible, so are E and A . Besides E is a faithful B -module, for if $xb = 0 \forall x \in E$, then $\sum_j x_j^* y_j b = 0 \forall x_j, y_j \in E$, so $b^*b = 0$, and this implies $b = 0$ because B is admissible. Similarly, E is a faithful A -module. It follows by 2.12 that we can identify E_0^r with $\text{span}\{x^*y : x, y \in E\}$ and E_0^l with $\text{span}\{xy^* : x, y \in E\}$. Now the proof ends with an invocation to Corollary 3.25 \square

4. C^* -TERNARY RINGS

As previously mentioned, Zettl found a unique decomposition $E = E_+ \oplus E_-$ of any C^* -tring E , E_+ being isomorphic to a TRO and E_- being isomorphic to an anti-TRO (see the discussion preceding Corollary 3.20). Of course, because of the uniqueness of the fundamental decomposition, there is a left version of the situation above: $E_+ := \{x \in E : \langle x, x \rangle_l \in E_+^l\}$, $E_- := \{x \in E : \langle x, x \rangle_l \in -E_+^l\}$, $\langle E_+, E_- \rangle_l = 0$, $E^l = E_+^l \oplus E_-^l$, and $(E_+, -\langle \cdot, \cdot \rangle_l)$ and $(E_-, -\langle \cdot, \cdot \rangle_l)$ are full left Hilbert E_+^l and E_-^l modules respectively. This way, E is an $(E^l - E^r)$ Banach bimodule that satisfies

$$\langle x, y \rangle_l z = \mu(x, y, z) = x \langle y, z \rangle_r, \quad \forall x, y, z \in E.$$

If E is a C^* -tring, we define $E_p := E_+ \oplus E_-^{\text{op}}$. Then E_p is a positive C^* -tring, and $E_p^r = E^r$, $E_p^l = E^l$. Therefore E_p is a $(E^l - E^r)$ -imprimitivity bimodule, so in

particular E^l and E^r are Morita–Rieffel equivalent. Note also that if $\phi : E \rightarrow F$ is a homomorphism of C^* -trings, then $\phi(E_+) \subseteq F_+$ and $\phi(E_-) \subseteq F_-$, because $\langle \phi(x), \phi(x) \rangle = \phi^r(\langle x, x \rangle)$. Therefore $\phi : E_p \rightarrow F_p$ is also a homomorphism of C^* -trings. Thus $E \mapsto E_p$ is a functor.

Let E^* be the reverse $*$ -tring of E . It is clear that a norm on E is a C^* -norm if and only if it is a C^* -norm on E^* . Moreover, E is a (positive) C^* -tring if and only if so is E^* , and $E^l = (E^*)^r$, $E^r = (E^*)^l$. Note that E and E^* are essentially the same object as C^* -trings. Thus the properties of E^r and E^l deduced from properties of E will be the same.

Definition 4.1. By a left (right) ideal of the C^* -ternary ring E we mean a closed subspace F of E such that $(E, E, F) \subseteq F$ (respectively: $(F, E, E) \subseteq F$). An ideal of E is both a left and a right ideal of E . We denote by $L(E)$, $R(E)$, and $I(E)$ the families of left, right and twosided ideals of E .

Our definition of ideal, for a closed subspace F of E , is equivalent to the definition which just requires the condition $(E, F, E) \subset F$ to be satisfied. Note that E_+ and E_- are ideals in every C^* -tring E . Moreover, since E_+ and E_- are orthogonal, it easily follows that a closed subspace F of E is an ideal of E if and only if it is an ideal in E_p . Thus the ideal structures of E and of E_p are the same.

If A is a C^* -algebra, we will denote by $I(A)$ and $H(A)$ respectively the families of (closed) twosided ideals and hereditary C^* -subalgebras of A .

As in the algebraic case, if E is a C^* -tring and F is a sub- C^* -tring of E , then the subalgebra $\overline{\text{span}}\langle F, F \rangle_r$ of E^r may be taken to represent the C^* -algebra F^r . With this choice of F^r we have the following result:

Proposition 4.2. *The map $L(E) \rightarrow H(E^r)$ given by $F \mapsto F^r$ is a bijection, with inverse given by $A \mapsto EA$. When restricted to $I(E)$, the map $F \mapsto F^r$ is a bijection onto $I(E^r)$. Moreover, all of these maps are lattice isomorphisms.*

Proof. We prove that the map $L(E) \rightarrow H(E^r)$ is a bijection. Recalling that we may replace E by E_p (which can be seen as a full right Hilbert E^r -module), the rest of the proof follows from [8, 3.22]. If A is a C^* -subalgebra of E^r : $(E, E, EA) = E\langle E, EA \rangle = E\langle E, E \rangle A = (E, E, E)A \subseteq EA$, so EA is a left ideal in E . Conversely, if F is a left ideal in E :

$$\langle F, F \rangle \langle E, E \rangle \langle F, F \rangle = \langle E \langle F, F \rangle, E \langle F, F \rangle \rangle = \langle (E, F, F), (E, F, F) \rangle \subseteq \langle F, F \rangle.$$

Thus, taking the closed linear spans in both sides of the above inclusion we have: $F^r E^r F^r = F^r$, which shows that F^r is hereditary. To see that the correspondences are mutually inverses, note that if F is a C^* -tring, then $F = FF^r$. On the other hand, if A is a hereditary C^* -subalgebra of E^r , then $EA = \overline{\text{span}}\langle EA, EA \rangle_r = \overline{\text{span}}A \langle E, E \rangle_r A = AE^r A = A$. \square

Corollary 4.3. *Let $\pi : E \rightarrow F$ be a homomorphism of $*$ -trings, where E and F are C^* -trings. Then $(\ker \pi)^r = \ker(\pi^r)$.*

Proof. It is clear that $\ker \pi \supseteq E \ker \pi^r$, so $(\ker \pi)^r \supseteq \ker \pi^r$. On the other hand $(\ker \pi)^r = \overline{\text{span}}\{\langle x, y \rangle_r : x, y \in \ker \pi\} \subseteq \ker \pi^r$. \square

Remark 4.4. By Proposition 2.17 if $\pi : E \rightarrow F$ is a surjective homomorphism between C^* -trings, then $\pi_0^r : E_0^r \rightarrow F_0^r$ is also surjective, so also is $\pi^r : E^r \rightarrow F^r$ for the image of π^r is closed. However the converse is false: consider the Hilbert space inclusion $\mathbb{C} \xrightarrow{\iota} \mathbb{C}^2$; then ι is not onto, although ι^r is the identity on \mathbb{C} .

For a proof of the next result the reader is referred to [8, 3.25].

Proposition 4.5. *Let F be an ideal of a C^* -tring E , and consider the quotient E/F with its natural structure of $*$ -tring. Then E/F is a C^* -tring with the quotient norm, and $(E/F)^r = E^r/F^r$.*

Corollary 4.6. *Let E and G be C^* -trings, and $\pi : E \rightarrow G$ a homomorphism of $*$ -trings. Consider $F = \ker(\pi)$, and let $p : E^r \rightarrow E^r/F^r$ be the quotient map. Then there exists a unique homomorphism of C^* -algebras $\overline{\pi^r} : E^r/F^r \rightarrow G^r$ such that $\overline{\pi^r}p = \pi^r$. The homomorphism $\overline{\pi^r}$ is injective. In particular, if $\pi : E \rightarrow E/F$ is the quotient map, where F is an ideal of E , then $\overline{\pi^r} : E^r/F^r \rightarrow (E/F)^r$ is a natural isomorphism.*

Proof. Proposition 3.11 provides a unique homomorphism of C^* -algebras $\pi^r : E^r \rightarrow G^r$ such that $\langle \pi(x), \pi(y) \rangle = \pi^r(\langle x, y \rangle)$, $\forall x, y \in E$. The existence and uniqueness of $\overline{\pi^r}$, as well as its injectivity, follow now from the quotient universal property, together with the fact that $\ker(\pi^r) = F^r$ by Corollary 4.3. Finally, if F is an ideal of E , by Proposition 4.5 we have that E/F is a C^* -tring, and the projection $\pi : E \rightarrow E/F$ is a homomorphism of $*$ -trings. \square

Corollary 4.7. *The functor $E \mapsto E^r$, $\pi \mapsto \pi^r$, from the category of C^* -trings into the category of C^* -algebras, is exact. More precisely: if*

$$0 \longrightarrow F_1 \xrightarrow{\phi} F_2 \xrightarrow{\psi} F_3 \longrightarrow 0$$

is an exact sequence of C^ -trings, then the sequence:*

$$0 \longrightarrow F_1^r \xrightarrow{\phi^r} F_2^r \xrightarrow{\psi^r} F_3^r \longrightarrow 0$$

also is exact.

Corollary 4.8. *If $\pi : E \rightarrow F$ is a homomorphism of C^* -trings, then $\pi(E)$ is closed in F . The ideals of a C^* -tring E are exactly the kernels of the homomorphisms defined on E .*

5. APPLICATIONS

5.1. C^* -algebras associated with Fell bundles. The proof of Theorem 1.1 of [2] relies on the existence of a certain inner product (see Corollary 5.3 below), although no proof is included there of the fact that such inner product is indeed positive. In the following lines we provide such a proof, and we refine the above mentioned result.

Recall that a right ideal $\mathcal{E} = (E_t)_{t \in G}$ of a Fell bundle $\mathcal{B} = (B_t)_{t \in G}$ is a sub-Banach bundle of \mathcal{B} such that $\mathcal{E}\mathcal{B} \subseteq \mathcal{E}$.

Given a right Hilbert B -module X , let denote by D_X the cone of finite sums $\sum_i \langle x_i, x_i \rangle \subseteq B^+$. It is clear that if $\{X_\lambda\}_{\lambda \in \Lambda}$ is a family of right Hilbert B -modules and $X := \oplus_\lambda X_\lambda$ (direct sum of Hilbert modules), then $\sum_\lambda D_{X_\lambda} \subseteq D_X$ -with equality if Λ is finite- and $\sum_\lambda D_{X_\lambda}$ is dense in D_X .

Similarly, for the right ideal \mathcal{E} of the Fell bundle \mathcal{B} , we define $D_\mathcal{E} := \{\sum_{i=1}^n c_i^* c_i : n \in \mathbb{N}, c_i \in \mathcal{E}, \forall i\} \subseteq B_e^+$. Then we have:

Lemma 5.1. *Let $\mathcal{E} = (E_t)_{t \in G}$ be a right ideal of the Fell bundle $\mathcal{B} = (B_t)_{t \in G}$. Then $\text{span}(\mathcal{E}^* \mathcal{E} \cap B_e)$ is dense in B_e if and only if the cone $D_\mathcal{E}$ satisfies the following property:*

$$\forall b \in B_e, \epsilon > 0, \text{ there exists } d \in D_\mathcal{E} \text{ such that } \|d\| \leq 1 \text{ and } \|b - bd\| < \epsilon. \quad (5.1)$$

Proof. Suppose that $b \in B_e$ is such that for any $\epsilon > 0$ there exists $d \in D_\mathcal{E}$ such that $\|b - bd\| < \epsilon$. Since $D_\mathcal{E} \subseteq \text{span}(\mathcal{E}^* \mathcal{E} \cap B_e)$ and the latter is an ideal in B_e , we conclude that $b \in \overline{\text{span}}(\mathcal{E}^* \mathcal{E} \cap B_e)$. Then $\text{span}(\mathcal{E}^* \mathcal{E} \cap B_e)$ is dense in B_e whenever $D_\mathcal{E}$ satisfies (5.1). Note now that $D_\mathcal{E} = \sum_{t \in G} D_{E_t}$, which is dense in D_E , where $E := \oplus_{t \in G} E_t$. Thus $D_\mathcal{E}$ satisfies (5.1) if and only if that property holds for D_E . Assume that $\text{span}(\mathcal{E}^* \mathcal{E} \cap B_e)$ is dense in B_e . Then E is a full Hilbert module over B_e , and therefore it satisfies (5.1) by [6, (ii) of Lemma 7.2]. \square

Lemma 5.2. *Let $\mathcal{B} = (B_t)_{t \in G}$ be a Fell bundle over the locally compact group G , $\mathcal{A} = (A_t)$ a sub-Fell bundle of \mathcal{B} , and $\mathcal{E} = (E_t)$ a right ideal of \mathcal{B} such that $\mathcal{A} \subseteq \mathcal{E}$, $\mathcal{E} \mathcal{E}^* \subseteq \mathcal{A}$ and $\text{span}(\mathcal{E}^* \mathcal{E} \cap B_e)$ is dense in B_e . If $\xi \in L^1(\mathcal{E})$, then $\xi * \xi^*$ can be arbitrarily approximated in $L^1(\mathcal{A})$ by a finite sum $\sum_{j=1}^m \eta_j * \eta_j^*$, where $\eta_j \in L^1(\mathcal{A})$, $\forall j = 1, \dots, m$.*

Proof. We will suppose that $\xi \in C_c(\mathcal{E})$, which is clearly enough. Since $C_0(\mathcal{E})$ is a nondegenerate right Banach B_e -module, given a positive integer n there exists $b_n \in B_e$ such that $\|\xi - \xi b_n\| < 1/n$ and $0 \leq b_n \leq 1$. Then we can find $c_n \in D_\mathcal{E}$ such that $\|b_n^{1/2} - b_n^{1/2} c_n\| < 1/n$. Set $d_n := b_n^{1/2} c_n b_n^{1/2}$ and note that $d_n \in D_\mathcal{E}$ because \mathcal{E} is a right ideal. The continuity of the operations imply $\|b_n - d_n\| \rightarrow 0$ and $\|\xi - \xi d_n\|_1 \rightarrow 0$. Thus $\|\xi * \xi^* - \xi d_n * \xi^*\|_1 \rightarrow 0$.

Now for every n there exist $u_1, \dots, u_{m_n} \in \mathcal{E}$ such that $d_n = \sum_{j=1}^{m_n} u_j^* u_j$. Thus $\xi d_n * \xi^* = \sum_{j=1}^{m_n} (\xi u_j^* u_j) * \xi^* = \sum_{j=1}^{m_n} (\xi u_j^*) * (\xi u_j^*)^*$ and, as \mathcal{E} is a right ideal, $\xi u_j^* \in C_c(\mathcal{A})$. This completes the proof. \square

Corollary 5.3. *Under the assumptions of Lemma 5.2, let $\|\cdot\|_\mathcal{A} : L^1(\mathcal{A}) \rightarrow [0, \infty)$ be the maximal C^* -norm of $L^1(\mathcal{A})$. Then $L^1(\mathcal{E}) \times L^1(\mathcal{E}) \rightarrow L^1(\mathcal{A})$ given by $(\xi, \eta) \mapsto \xi * \eta^*$ is an inner product.*

Corollary 5.4. *Under the assumptions of Lemma 5.2, the map $\varphi : \mathcal{SN}(L^1(\mathcal{B})) \rightarrow \mathcal{SN}(L^1(\mathcal{B}))$ given by $\beta \mapsto \beta|_{L^1(\mathcal{A})}$ is an isomorphism of partially ordered sets that sends the maximal and reduced norms on $L^1(\mathcal{B})$ to the maximal and reduced norms on $L^1(\mathcal{A})$ respectively, and such that $\varphi(\mathcal{N}(L^1(\mathcal{B}))) = \mathcal{N}(L^1(\mathcal{A}))$. Moreover, the Hausdorff completions of $L^1(\mathcal{B})$ and $L^1(\mathcal{A})$ with respect to β and $\varphi(\beta)$ respectively are Morita–Rieffel equivalent.*

Proof. We only have to prove the correspondence between the reduced C^* -norms, but this is the content of [2]. \square

5.2. Tensor products of C^* -trings. In the present section we apply the previous results to the study of tensor products of C^* -trings. Maximal and minimal tensor product for TROs were constructed in [5] using linking algebras, but we define tensor products of C^* -trings E and F using the tensor products of E^r and F^r . The main result is Theorem 5.12.

From now on the algebraic tensor product of the \mathbb{C} -vector spaces E_1, \dots, E_n will be denoted by $E_1 \odot \dots \odot E_n$, or just by $\odot_{j=1}^n E_j$. Let E_{ij}, F_i be complex vector spaces, $\forall i = 1, \dots, m, j = 1, \dots, n$, and suppose that $\alpha_i : \prod_{j=1}^n E_{ij} \rightarrow F_i$ is a n -linear map, for each $i = 1, \dots, m$. Then it is clear that there exists a unique n -linear map $\alpha := \alpha_1 \odot \dots \odot \alpha_m : \prod_{j=1}^n \odot_{i=1}^m E_{ij} \rightarrow \odot_{i=1}^m F_i$ such that $\alpha(\odot_{i=1}^m e_{i1}, \dots, \odot_{i=1}^m e_{in}) = \odot_{i=1}^m \alpha_i(e_{i1}, \dots, e_{in})$. Using this fact we have the following result, whose straightforward proof is left to the reader.

Proposition 5.5. *If $(E, \mu), (F, \nu)$ are $*$ -trings, then $(E \odot F, \mu \odot \nu)$ is also a $*$ -tring. Furthermore, if $(E, A, \langle, \rangle_A)$ and $(F, B, \langle, \rangle_B)$, are full basic triples associated to (E, μ) and (F, ν) , respectively, then $(E \odot F, A \odot B, \langle, \rangle_{A \odot B})$ is a full basic triple associated to $(E \odot F, \mu \odot \nu)$.*

Definition 5.6. A C^* -tensor product of two $*$ -trings $(E, \mu, \|\cdot\|)$ and $(F, \nu, \|\cdot\|)$ is a completion of the corresponding algebraic tensor product $(E \odot F, \mu \odot \nu)$ with respect to a C^* -norm. If γ is such a C^* -norm, we denote by $E \otimes_{\gamma} F$ the corresponding C^* -tensor product.

Definition 5.7. We say that a C^* -tring E is nuclear if for every C^* -tring F there exists just one C^* -tensor product $E \otimes F$.

We will see next that $\mathcal{SN}(E \odot F) = \mathcal{SN}(E_p \odot F)$, which implies, in particular, that a C^* -tring E is nuclear if and only if E_p is nuclear.

Proposition 5.8. *Let E be a $*$ -tring, and F_1, F_2 ideals of E such that $E = F_1 \oplus F_2$. If $\gamma \in \mathcal{SN}(E)$, and $x = y + z$, with $y \in F_1$ and $z \in F_2$, then $\gamma(x) = \max\{\gamma(y), \gamma(z)\}$.*

Proof. Since $\gamma(x) = \sup\{\gamma((x, u, u)) : u \in E, \gamma(u) \leq 1\}$, it follows that $\gamma(x) \geq \gamma(y)$ and $\gamma(x) \geq \gamma(z)$, so $\gamma(x) \geq \max\{\gamma(y), \gamma(z)\}$. To prove the converse inequality, let us first introduce the following notation. For $u \in E$ let $u_0 := z$, $u_n := (u_{n-1}, u_{n-1}, u_{n-1})$ if $n \geq 1$. Then we have that $\gamma(u_n) = \gamma(u_{n-1})^3, \forall n \geq 1$, so $\gamma(u_n) = \gamma(u)^{3^n}, \forall n \geq 0$. Since $(E, F_1, F_2) = 0$, it follows that $x_n = y_n + z_n$. Thus: $\gamma(x) = \gamma(x_n)^{1/3^n} = \gamma(y_n + z_n)^{1/3^n} \leq (\gamma(y_n) + \gamma(z_n))^{1/3^n} = (\gamma(y)^{3^n} + \gamma(z)^{3^n})^{1/3^n} \xrightarrow{n} \max\{\gamma(y), \gamma(z)\}$, whence $\gamma(x) \leq \max\{\gamma(y), \gamma(z)\}$. \square

Corollary 5.9. *Let E and F be C^* -trings. Then $\mathcal{SN}(E \odot F) = \mathcal{SN}(E_p \odot F)$ and $\mathcal{N}(E \odot F) = \mathcal{N}(E_p \odot F)$. Consequently a C^* -tring E is nuclear if and only if E_p is nuclear.*

Our aim is to prove that there is an isomorphism between $\mathcal{N}(E \odot F)$ and $\mathcal{N}(E^r \odot F^r)$. The key step is to show that each C^* -norm on $E_0^r \odot F_0^r$ has unique extension to a C^* -norm on $E^r \odot F^r$.

Lemma 5.10. *Let I and J be $*$ -ideals (not necessarily closed) of the C^* -algebras A and B , respectively. Then the map $\Theta : \mathcal{N}(A \odot B) \rightarrow \mathcal{N}(I \odot J), \gamma \mapsto \gamma|_{I \odot J}$, is*

an order preserving surjection. If, in addition, I and J are dense in A and B , respectively, then Θ is a bijection.

Proof. Clearly Θ is order preserving. Fix $\delta \in \mathcal{N}(I \odot J)$. Given $a \in A$ and $z = \sum_{j=1}^n x_i \odot y_j \in I \odot J$, define $w := \sum_{j=1}^n (\|a\|^2 - a^*a)^{1/2} x_i \odot y_j \in A \odot B$. In case A is unital it is clear that $w \in I \odot J$. If A is not unital, I is an ideal of the unitization of A , so $w \in I \odot J$ in any case. Then

$$\|a\|^2 z^* z - \left(\sum_{j=1}^n a x_i \odot y_j\right)^* \left(\sum_{j=1}^n a x_i \odot y_j\right) = w^* w \in (I \otimes_{\delta} J)^+$$

and $\delta(\sum_{j=1}^n a x_i \odot y_j) \leq \|a\| \delta(\sum_{j=1}^n x_i \odot y_j)$. Similarly, if $b \in B$, we also have $\delta(\sum_{j=1}^n x_i \odot b y_j) \leq \|b\| \delta(\sum_{j=1}^n x_i \odot y_j)$. Thus $\delta((a \odot b)z) \leq \|a\| \|b\| \delta(z)$, $\forall a \in A$, $b \in B$ and $z \in I \odot J$. Therefore, according to 3.21, the map $\delta' : A \odot B \rightarrow \mathbb{R}$ such that $\delta'(c) := \sup\{\delta(cz) : \delta(z) \leq 1\}$ is a C^* -seminorm on $A \odot B$ that extends δ . In case I and J are dense in A and B , respectively, $I \odot J$ is dense in $A \odot B$ with respect to any C^* -norm [10, Corollary T.6.2]. Thus Θ is injective. \square

Proposition 5.11. *Let E and F be positive C^* -trings and consider the admissible full basic triples $(E, E_0^r, \langle \cdot, \cdot \rangle_r^E)$ and $(F, F_0^r, \langle \cdot, \cdot \rangle_r^F)$ given by Theorem 2.10. Then the full basic triple $(E \odot F, E_0^r \odot F_0^r, \langle \cdot, \cdot \rangle_r^E \odot \langle \cdot, \cdot \rangle_r^F)$ is admissible. Furthermore, $E \odot F$ is positive and*

$$\mathcal{SN}_{cs}^{\langle \cdot, \cdot \rangle_r^E \odot \langle \cdot, \cdot \rangle_r^F}(E_0^r \odot F_0^r) = \mathcal{SN}(E_0^r \odot F_0^r)$$

Proof. To simplify our notation we denote $[\cdot, \cdot]$ the map $\langle \cdot, \cdot \rangle_r^E \odot \langle \cdot, \cdot \rangle_r^F$. Note $E_0^r \odot F_0^r$ -module is admissible because it is a $*$ -subalgebra of the C^* -closable $*$ -algebra $E^r \odot F^r$. We will show that $E \odot F$ is a positive $E_0^r \odot F_0^r$ -module. Lemma 5.10 implies there is a maximal C^* -norm on $E_0^r \odot F_0^r$, namely the restriction of the maximal C^* -norm of $E^r \odot F^r$. The comments preceding Lemma 3.27 imply that, to show $E \odot F$ is positive, it suffices to prove that $[u, u] \geq 0$ in the maximal tensor product $E^r \otimes_{\max} F^r$. Given $u = \sum_{j=1}^n x_j \otimes y_j \in E \odot F$ we have

$$[u, u] = \sum_{j,k=1}^n \langle x_j, x_k \rangle_r^E \odot \langle y_j, y_k \rangle_r^F.$$

Then Lemmas 4.2 and 4.3 of [6] give the desired result.

To show $[u, u] = 0$ implies $u = 0$ we use the linking algebras $\mathbb{L}(E)$ and $\mathbb{L}(F)$ and the linear maps

$$\begin{aligned} \alpha : E \odot F &\rightarrow \mathbb{L}(E) \odot \mathbb{L}(F), \quad x \odot y \mapsto \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \odot \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}, \\ \beta : \mathbb{L}(E) \odot \mathbb{L}(F) &\rightarrow E \odot F, \quad \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \odot \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \mapsto x_{12} \odot y_{12}, \\ \gamma : E \odot F &\rightarrow \mathbb{L}(E) \odot \mathbb{L}(F), \quad a \odot b \mapsto \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \odot \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}. \end{aligned}$$

Then $\alpha(u)^* \alpha(u) = \gamma([u, u]) = 0$, so $\alpha(u) = 0$ and $u = \beta(\alpha(u)) = 0$. \square

Theorem 5.12. *Let E and F be C^* -ternary rings. Then every set among the partially ordered sets $\mathcal{N}(E^l \odot F^l)$, $\mathcal{N}(E \odot F)$ and $\mathcal{N}(E^r \odot F^r)$ is isomorphic to*

each other. Besides, if $\gamma \in \mathcal{N}(E \odot F)$ and γ^l and γ^r are the corresponding C^* -norms on $\mathcal{N}(E^l \odot F^l)$ and $\mathcal{N}(E^r \odot F^r)$ respectively, then $E \otimes_{\gamma} F$ is a Morita–Rieffel equivalence bimodule between $E^l \otimes_{\gamma^l} F^l$ and $E^r \otimes_{\gamma^r} F^r$.

Proof. Proposition 5.11 together with Corollary 3.31 imply $\mathcal{N}(E \odot F)$ is isomorphic (as a partially ordered set) to $\mathcal{N}(E \odot F)_0^r$. By 5.5 the posets $\mathcal{N}(E \odot F)_0^r$ and $\mathcal{N}(E_0^r \odot F_0^r)$ are isomorphic, and the latter is isomorphic to $\mathcal{N}(E^r \odot F^r)$ by Lemma 5.10. Thus $\mathcal{N}(E \odot F) \cong \mathcal{N}(E^r \odot F^r)$. Similarly we have $\mathcal{N}(E \odot F) \cong \mathcal{N}(E^l \odot F^l)$. \square

Corollary 5.13. *Let E and F be C^* -trings. Then there exist a maximum C^* -norm $\|\cdot\|_{\max}$ on $E \odot F$, and a minimum C^* -norm $\|\cdot\|_{\min}$ on $E \odot F$, and*

$$\begin{aligned} (E \otimes_{\max} F)^l &= E^l \otimes_{\max} F^l, & (E \otimes_{\max} F)^r &= E^r \otimes_{\max} F^r, \\ (E \otimes_{\min} F)^l &= E^l \otimes_{\min} F^l, & (E \otimes_{\min} F)^r &= E^r \otimes_{\min} F^r. \end{aligned}$$

Corollary 5.14 (cf. [5, Theorem 6.5]). *The following assertions are equivalent for a C^* -tring E :*

- (1) E is a nuclear C^* -tring (5.7).
- (2) E^l is a nuclear C^* -algebra.
- (3) E^r is a nuclear C^* -algebra.

The equivalence between 2. and 3. in 5.14 is exactly the following well-known result ([3], [11]): *if A and B are two Morita–Rieffel equivalent C^* -algebras then A is nuclear if and only if so is B .*

5.3. Exact C^* -trings. To end the section we introduce the notion of exact C^* -tring, extending the notion of exact TRO of [5], and we prove a result similar to Corollary 5.14. The reader is referred to [9] for the theory of exact C^* -algebras.

Suppose that $0 \longrightarrow F_1 \xrightarrow{\phi} F_2 \xrightarrow{\psi} F_3 \longrightarrow 0$ is an exact sequence of C^* -trings, that is, ϕ and ψ are homomorphisms of C^* -trings, ϕ is injective, ψ is surjective, and $\ker \psi = \phi(F_1)$. Let E be a C^* -tring. Then the sequence

$$0 \longrightarrow E \odot F_1 \xrightarrow{id \odot \phi} E \odot F_2 \xrightarrow{id \odot \psi} E \odot F_3 \longrightarrow 0$$

also is exact. We have an inclusion

$$(E \odot F_2)/(E \odot F_1) \hookrightarrow (E \otimes_{\min} F_2)/(E \otimes_{\min} F_1)$$

and the latter quotient is a C^* -tring. Then there exists a C^* -norm γ on $E \odot F_3$ such that

$$0 \longrightarrow E \otimes_{\min} F_1 \xrightarrow{id \otimes \phi} E \otimes_{\min} F_2 \xrightarrow{id \otimes \psi} E \otimes_{\gamma} F_3 \longrightarrow 0$$

is exact. Since γ is greater or equal to the minimum norm, the identity map on $E \odot F_3$ extends to a surjective homomorphism $E \otimes_{\gamma} F_3 \rightarrow E \otimes_{\min} F_3$.

Definition 5.15. We say that a C^* -tring E is exact if for each exact sequence

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

of C^* -trings we have that

$$0 \longrightarrow E \otimes_{\min} F_1 \longrightarrow E \otimes_{\min} F_2 \longrightarrow E \otimes_{\min} F_3 \longrightarrow 0$$

also is exact.

Proposition 5.16. *Let E and F be C^* -trings, and suppose that G is an ideal of F (Definition 4.1). Then*

$$0 \longrightarrow E \otimes_{\min} G \longrightarrow E \otimes_{\min} F \longrightarrow E \otimes_{\min} (F/G) \longrightarrow 0$$

is exact if and only if the following sequence is exact:

$$0 \longrightarrow E^r \otimes_{\min} G^r \longrightarrow E^r \otimes_{\min} F^r \longrightarrow E^r \otimes_{\min} (F^r/G^r) \longrightarrow 0$$

Proof. Suppose first that the sequence below is exact:

$$0 \longrightarrow E \otimes_{\min} G \longrightarrow E \otimes_{\min} F \longrightarrow E \otimes_{\min} (F/G) \longrightarrow 0$$

By Corollaries 5.13 and 4.7, we have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & (E \otimes_{\min} G)^r & \longrightarrow & (E \otimes_{\min} F)^r & \longrightarrow & (E \otimes_{\min} (F/G))^r \longrightarrow 0 \\ & & \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ 0 & \longrightarrow & E^r \otimes_{\min} G^r & \longrightarrow & E^r \otimes_{\min} F^r & \longrightarrow & E^r \otimes_{\min} (F/G)^r \longrightarrow 0 \\ & & = \downarrow & & = \downarrow & & \cong \downarrow \\ 0 & \longrightarrow & E^r \otimes_{\min} G^r & \longrightarrow & E^r \otimes_{\min} F^r & \longrightarrow & E^r \otimes_{\min} F^r/G^r \longrightarrow 0 \end{array}$$

Since the upper two rows are exact, the third one also is exact.

To prove the converse, note first that

$$0 \longrightarrow E \otimes_{\min} G \longrightarrow E \otimes_{\min} F \longrightarrow (E \otimes_{\min} F)/(E \otimes_{\min} G) \longrightarrow 0$$

is exact, and $(E \otimes_{\min} F)/(E \otimes_{\min} G)$ is a C^* -completion of the ternary ring $E \odot (F/G)$. Denoting the corresponding C^* -norm by γ , we have a surjective homomorphism $\phi : E \otimes_{\gamma} (F/G) \rightarrow E \otimes_{\min} (F/G)$ which extends the identity on $E \odot (F/G)$. Now, applying the exact functor $E \mapsto E^r$ we obtain the commutative diagram with exact rows that follows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & E^r \otimes_{\min} G^r & \longrightarrow & E^r \otimes_{\min} F^r & \longrightarrow & E^r \otimes_{\gamma} F^r/G^r \longrightarrow 0 \\ & & = \downarrow & & = \downarrow & & \phi^r \downarrow \\ 0 & \longrightarrow & E^r \otimes_{\min} G^r & \longrightarrow & E^r \otimes_{\min} F^r & \longrightarrow & E^r \otimes_{\min} F^r/G^r \longrightarrow 0 \end{array}$$

It follows that the homomorphism ϕ^r is an isomorphism. \square

Corollary 5.17 (cf. [5, Theorem 6.1]). *A C^* -tring E is exact (5.15) if and only if E^r is an exact C^* -algebra.*

Proof. Immediate from Proposition 5.16 \square

As previously for nuclear C^* -algebras, we easily obtain from 5.17 the following known result([7]): *if A and B are Morita–Rieffel equivalent C^* -algebras, then A is exact if and only if B is exact.*

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