

NON-COMMUTATIVE RATIONAL FUNCTIONS IN STRONGLY CONVERGENT RANDOM VARIABLES

SHENG YIN

This paper is dedicated to Professor Haagerup

Communicated by U. Franz

ABSTRACT. Random matrices like GUE, GOE and GSE have been shown that they possess a lot of nice properties. In 2005, a new property of independent GUE random matrices is discovered by Haagerup and Thorbørnsen. It is called strong convergence property and then more random matrices with this property are followed. In general, the definition can be stated for a sequence of tuples over some C^* -algebras. In this paper, we want to show that, for a sequence of strongly convergent random variables, non-commutative polynomials can be extended to non-commutative rational functions under certain assumptions. As a direct corollary, we can conclude that for a tuple $(X_1^{(n)}, \dots, X_m^{(n)})$ of independent GUE random matrices, $r(X_1^{(n)}, \dots, X_m^{(n)})$ converges in trace and in norm to $r(s_1, \dots, s_m)$ almost surely, where r is a rational function and (s_1, \dots, s_m) is a tuple of freely independent semi-circular elements which lies in the domain of r .

1. INTRODUCTION

In 1990's, a deep relation between random matrices and free probability was revealed in the paper [21] by Voiculescu. In this paper, Voiculescu proved that if $(X_1^{(n)}, \dots, X_m^{(n)})$ is a tuple of independent $n \times n$ normalized Hermitian Gaussian

Copyright 2016 by the Tusi Mathematical Research Group.

Date: Received: Feb. 23, 2017; Accepted: May 29, 2017.

2010 *Mathematics Subject Classification.* Primary 47B06; Secondary 60B20.

Key words and phrases. Strong convergence, non-commutative rational functions, random matrices.

random matrices for each $n \in \mathbb{N}$, then all the moments converge, i.e.,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left\{ \text{tr}_n(p(X_1^{(n)}, \dots, X_m^{(n)})) \right\}$$

exists for any non-commutative polynomial p , where we denote the normalized trace by tr_n . Furthermore, we can realize the limits as a tuple of freely independent semi-circular elements (s_1, \dots, s_m) in some C^* -probability space (\mathcal{A}, τ) , namely, a unital C^* -algebra with a state τ . So we can write

$$\lim_{n \rightarrow \infty} \mathbb{E} \left\{ \text{tr}_n(p(X_1^{(n)}, \dots, X_m^{(n)})) \right\} = \tau(p(s_1, \dots, s_m))$$

for any polynomial p . This result has been extended to some other random matrix models, for example, a tuple of Wigner matrices with some assumptions on moments of entries [8]. On the other hand, it is also known that this convergence for random matrices can be improved to the almost sure convergence, see Hiai, Petz [13] and Thorbørnsen [20].

Later, Haagerup and Thorbørnsen showed that the convergence of random matrices can happen in a stronger sense, that is, convergence in the norm. To be precise, in [11], they showed that for any polynomial p ,

$$\lim_{n \rightarrow \infty} \left\| p(X_1^{(n)}(\omega), \dots, X_m^{(n)}(\omega)) \right\| = \|p(s_1, \dots, s_m)\| \quad (1.1)$$

for almost every ω in the underlying probability space. Then we say that $(X_1^{(n)}, \dots, X_m^{(n)})$ *strongly converges* and (s_1, \dots, s_m) is its *strong limit*. Following their work, Schultz [18] shows that GOE and GSE also admit semi-circular elements as strong limit. Then Capitaine and Donati-Martin [4] and Anderson [1] generalize the result to certain Wigner matrices. Capitaine and Donati-Martin [4] also extend the result to Wishart matrices with free Poisson elements as strong limit.

Moreover, in the paper [16] by Male, he finds that a tuple of random matrices from GUE can be enlarged with another tuple of independent random matrices who has a strong limit. Later, in the paper [7] by Collins and Male, they show that this property also holds for Haar unitary matrices. And then in the paper [2] by Belinschi and Capitaine, they proved that this property also holds for certain Wigner matrices.

Meanwhile, in recent papers by Skoufranis [19] and Pisier [17], it is shown that the strong convergence property is preserved when adjoining two tuples of non-commutative random variables which admit strong limits and are free from each other. In other words, they proved that the reduced free product is stable with respect to strong convergence.

Therefore, these results show that the strong convergence property is stable under some algebraic operations, so it is natural to ask if the strong convergence is stable under another basic algebraic operation, namely, taking inverses. And then we can hope that the polynomials in (1.1) can be replaced by rational functions under some assumptions.

On the other hand, we know that one of the main ingredients used by Haagerup and Thorbørnsen is the so-called linearization trick, see [11, 10] for the idea

and details. Inspired by the fact that such a linearization also holds for non-commutative rational expressions or rational functions, we can expect an affirmative answer to our question. In this paper, we will show that this result is indeed true but the linearization technique is not essentially necessary when we are going from polynomials to rational functions.

In the following, we always consider the strong convergence in the faithful tracial C^* -probability space setting.

Definition 1.1. Let $(\mathcal{A}^{(n)}, \tau^{(n)})$, $n \in \mathbb{N}$ and (\mathcal{A}, τ) be some C^* -probability spaces with faithful traces. Then we assume that $x^{(n)} = (x_1^{(n)}, \dots, x_m^{(n)})$ is a tuple of elements from $\mathcal{A}^{(n)}$ for each $n \in \mathbb{N}$, and $x = (x_1, \dots, x_m)$ is a tuple of elements in (\mathcal{A}, τ) s.t. $x^{(n)}$ strongly converges to x . That is, they satisfy the following:

$$\begin{aligned} \lim_{n \rightarrow \infty} \tau^{(n)}(p(x^{(n)}, (x^{(n)})^*)) &= \tau(p(x, x^*)), \\ \lim_{n \rightarrow \infty} \|p(x^{(n)}, (x^{(n)})^*)\|_{\mathcal{A}^{(n)}} &= \|p(x, x^*)\|_{\mathcal{A}} \end{aligned}$$

for any polynomial p in $2m$ non-commuting indeterminates.

In the second section, we will give a concise introduction to rational functions and rational expressions and some of their relevant properties. Then, in the last section, we are going to prove the main theorem:

Theorem 1.2. *If $x^{(n)} = (x_1^{(n)}, \dots, x_m^{(n)})$ strongly converges to $x = (x_1, \dots, x_m)$, then for any rational expression r , $r(x, x^*)$ is the limit of $r(x^{(n)}, (x^{(n)})^*)$ in trace and in norm, provided that (x, x^*) lies in the domain of r .*

The basic idea behind this is that from polynomials to rational expressions, our only obstacle is due to taking the inverse. But we will see that the inverse can be approximated by polynomials uniformly in all dimensions, hence we can reduce the convergence of rational expressions to the result on polynomials and also show that $(x^{(n)}, (x^{(n)})^*)$ will lie in the domain eventually.

As an example or consequence, we can apply our main result to any random matrices which have a strong limit.

Corollary 1.3. *Let $X^{(n)} = (X_1^{(n)}, \dots, X_m^{(n)})$ be a tuple of independent $n \times n$ random matrices for each $n \in \mathbb{N}$, and $x = (x_1, \dots, x_m)$ a tuple of freely independent random variables in some faithful tracial C^* -probability space (\mathcal{A}, τ) . Assume that $X^{(n)}$ strongly converges to x almost surely. Then for any rational expression r with (x, x^*) in its domain, we have $(X^{(n)}(\omega), (X^{(n)}(\omega))^*)$ lies in the domain of r eventually and*

$$\begin{aligned} \lim_{n \rightarrow \infty} \text{tr}_n(r(X^{(n)}(\omega), (X^{(n)}(\omega))^*)) &= \tau(r(x, x^*)), \\ \lim_{n \rightarrow \infty} \|r(X^{(n)}(\omega), (X^{(n)}(\omega))^*)\| &= \|r(x, x^*)\|_{\mathcal{A}} \end{aligned}$$

for almost every ω in the underlying space.

In particular, it allows us to claim that a rational expression in independent GUE random matrices converges almost surely in trace to the same rational expression in free independent semi-circular elements. In fact, such a result is not surprising at all. In the recent paper [12] by Helton, Mai and Speicher, they

extended the method used for the calculation of the distribution of polynomials in free random variables to the rational case, based on the fact that linearization works equally well for rational expressions. From their simulation in Section 4.7 of [12], we can expect that a rational expression in independent Gaussian random matrices should almost surely converge in distribution to the same rational expression in free semi-circular elements. By our theorem this is true whenever we have random matrices which admit strong limits.

2. RATIONAL FUNCTIONS AND THEIR RECURSIVE STRUCTURE

In this section, we will give a short introduction to rational functions and rational expressions with some highlights which are necessary for our result in the next section.

It is well-known that for each integral domain, we can construct the unique quotient field, namely, the smallest field in which this integral domain can be embedded. This was generalized to certain non-commutative rings with a property called the Ore condition. This condition can allow us to construct the field in essentially the same way as in the commutative case. However, to extend such embedding results to more general cases requires new ideas.

For example, the ring of polynomial in any m ($m \geq 2$) non-commuting indeterminates doesn't satisfy Ore condition due to its non-commutative nature. So it is not quite obvious that a field of fractions of non-commutative polynomials really exists and that such a field is unique even if it exists.

From 1960's, Cohn began to study the problem of embedding non-commutative rings into fields and then he developed a matrix method to introduce the matrix ideals, as the analogue of the ideals in commutative case. He showed that the prime matrix ideals can be used to describe some "kernels" of the embeddings of rings into skew fields, as every prime ideal in a commutative ring arises as the kernel of a homomorphism into some commutative field. And this characterization allows us to derive a criteria for the embeddability of rings into fields.

In the following, we always use \mathcal{P} to denote the non-commutative polynomials ring and \mathcal{R} the field of fractions obtained from \mathcal{P} by Cohn's construction. We won't go into details of this construction but we will talk about some basic properties to show what do these rational functions look like. In fact, the only thing about Cohn's construction we shall need is the following theorem:

Theorem 2.1. *Let $r \in \mathcal{R}$ be a rational function, then there exists some $n \in \mathbb{N}$, a matrix of polynomials $A \in M_n(\mathcal{P})$, a row of polynomials $u \in M_{1,n}(\mathcal{P})$ and a column of polynomials $v \in M_{n,1}(\mathcal{P})$ s.t. A is invertible in $M_n(\mathcal{R})$ and $r = uA^{-1}v$.*

For a more general statement and the proof, see [5, Ch 7].

In fact, to represent a rational function in terms of matrices of polynomials appears not only in the context of ring theory, but also in the system and control theory, called "realization". Moreover, such a realization is usually required to be in a linear form, i.e., all the entries in the matrices in the above theorem are at most of degree 1 as polynomials. So this technique is also called linearization. But we won't talk any more about this realization or linearization technique in this paper, though it has a variety of implications in different areas.

Now we want to use this theorem to show that the field of rational functions has a recursive structure. That is, all the rational functions can be obtained by taking finitely many algebraic operations (addition, multiplication, inversion) from polynomials. This exactly meets what we would expect for rational functions intuitively but may not be obvious from the theory of Cohn.

Denote $\mathcal{R}_0 = \mathcal{P}$, and by \mathcal{R}_1 we denote the subring of \mathcal{R} generated by $\mathcal{R}_0 \cup \mathcal{R}_0^{-1}$, where \mathcal{R}_0^{-1} is the set of inverses of all nonzero polynomials. Now, suppose that we have constructed the subring $\mathcal{R}_n \subseteq \mathcal{R}$ for some $n \in \mathbb{N}$, then we let \mathcal{R}_{n+1} be the subring of \mathcal{R} generated by $\mathcal{R}_n \cup \mathcal{R}_n^{-1}$, where \mathcal{R}_n^{-1} is the set of inverses of all nonzero rational functions in \mathcal{R}_n . So we have an increasing sequence of subrings $\{\mathcal{R}_n\}_{n \geq 1}$ in \mathcal{R} . Then we set

$$\mathcal{R}_\infty = \bigcup_{n=1}^{\infty} \mathcal{R}_n.$$

We expect (and will show below) that we have $\mathcal{R}_\infty = \mathcal{R}$. The following argument is based on a similar idea for proving that \mathcal{R} is really a “free” field, i.e., every 0 identity comes from algebraic manipulations. For a reference, see [6] and also [5].

First, for reader’s convenience, we give a short proof for a well-known lemma about Schur complements for matrices in a unital algebra setting.

Lemma 2.2. *Suppose that \mathcal{A} is a complex and unital algebra. Let $k, l \in \mathbb{N}$, $A \in M_k(\mathcal{A})$, $B \in M_{k \times l}(\mathcal{A})$, $C \in M_{l \times k}(\mathcal{A})$ and $D \in M_l(\mathcal{A})$ s.t. D is invertible. Then the matrix*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is invertible in $M_{k+l}(\mathcal{A})$ iff the Schur complement $A - BD^{-1}C$ is invertible in $M_k(\mathcal{A})$. In this case, we will have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -D^{-1}C & 1 \end{pmatrix} \begin{pmatrix} (A - BD^{-1}C)^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} 1 & -BD^{-1} \\ 0 & 1 \end{pmatrix}. \quad (2.1)$$

Proof. It’s easy to check that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & BD^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix}$$

holds whenever D is invertible. Since the matrices

$$\begin{pmatrix} 1 & BD^{-1} \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix}$$

are clearly invertible in $M_{k+l}(\mathcal{A})$, the equivalence of invertibilities of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $A - BD^{-1}C$ follows immediately. And (2.1) follows from a simple calculation

$$\begin{aligned} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} &= \begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix}^{-1} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix}^{-1} \begin{pmatrix} 1 & BD^{-1} \\ 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 0 \\ -D^{-1}C & 1 \end{pmatrix} \begin{pmatrix} (A - BD^{-1}C)^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} 1 & -BD^{-1} \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

□

With the help of the above lemma, we can show the following lemma, which is crucial for our statement on $\mathcal{R}_\infty = \mathcal{R}$.

Lemma 2.3. *If an n -by- n matrix $A \in M_n(\mathcal{R}_\infty)$ is invertible in $M_n(\mathcal{R})$, then $A^{-1} \in M_n(\mathcal{R}_\infty)$.*

Proof. We are going to prove this by induction on the size of matrices. First, let $r \in M_1(\mathcal{R}_\infty)$, then we can view it as a rational function in \mathcal{R}_∞ , which implies that there is some $k \in \mathbb{N}$ s.t. $r \in \mathcal{R}_k$. Thus, r is invertible in $M_1(\mathcal{R}) = \mathcal{R}$ means that $r \neq 0$, and so we have $r^{-1} \in \mathcal{R}_k^{-1} \subseteq \mathcal{R}_{k+1} \subseteq \mathcal{R}_\infty$.

Now assume that the claim is true for matrices of size $n - 1$. Let $A \in M_n(\mathcal{R}_\infty)$ be invertible in $M_n(\mathcal{R})$, then, WLOG, we can write

$$A = \begin{pmatrix} B & u \\ v & p \end{pmatrix}$$

with $p \neq 0$, because we can multiply by a permutation matrix to achieve this. Hence, we see that $B - up^{-1}v \in M_{n-1}(\mathcal{R}_\infty)$ is invertible in $M_{n-1}(\mathcal{R})$ by the previous lemma, then it follows that $(B - up^{-1}v)^{-1} \in M_{n-1}(\mathcal{R}_\infty)$ by the induction. Since

$$A^{-1} = \begin{pmatrix} I_{n-1} & 0 \\ -p^{-1}u & 1 \end{pmatrix} \begin{pmatrix} (B - up^{-1}v)^{-1} & 0 \\ 0 & p^{-1} \end{pmatrix} \begin{pmatrix} I_{n-1} & -vp^{-1} \\ 0 & 1 \end{pmatrix}$$

by (2.1), we can see clearly that $A^{-1} \in M_n(\mathcal{R}_\infty)$ since each matrix in the right hand side lies in $M_n(\mathcal{R}_\infty)$. This completes the proof. \square

Theorem 2.4. *We have*

$$\mathcal{R} = \mathcal{R}_\infty.$$

Proof. Let $r \in \mathcal{R}$ be a rational function, then, by Theorem 2.1, there exists a matrix of polynomials $A \in M_n(\mathcal{P})$, a row $u \in M_{1,n}(\mathcal{P})$ and a column $v \in M_{n,1}(\mathcal{P})$ for some $n \in \mathbb{N}$ s.t. A is invertible in $M_n(\mathcal{R})$ and $r = uA^{-1}v$. By the previous lemma, and since $\mathcal{P} \subseteq \mathcal{R}_\infty$, we see that $A^{-1} \in M_n(\mathcal{R}_\infty)$ and thus $r \in \mathcal{R}_\infty$. \square

It is well-known that in the commutative case, every rational function can be written in a form like pq^{-1} , where p and q are polynomials. This means that we will have $\mathcal{R} = \mathcal{R}_1 = \mathcal{R}_n$ for all $n \geq 1$. But it is not true any more for non-commutative rational functions due to its noncommutativity. For example, we can't write $xy^{-1}x \in \mathcal{R}_1$ as the product pq^{-1} with two polynomials p, q . And the rational function $(x^{-1} + y^{-1} + z^{-1})^{-1}$ lies in \mathcal{R}_2 but not in \mathcal{R}_1 .

On the other hand, we should note that such a representation is not unique. For a simple example,

$$r(x, y) = (xy)^{-1} = y^{-1}x^{-1} \in \mathcal{R}_1,$$

we can see that we can use one polynomial xy or two polynomials x, y to represent the same rational function r . This causes a problem when we try to evaluate a rational function and to define its domain over some algebra. For example, let us consider the evaluation of the above rational function $r(x, y)$ on some unital algebra \mathcal{A} . From the first representation $(xy)^{-1}$, it gives a domain

$$D_1 = \{(a, b) \in \mathcal{A}^2 | ab \text{ is invertible in } \mathcal{A}\},$$

on which the function r is well-defined. But from the second one $y^{-1}x^{-1}$, its domain is

$$D_2 = \{(a, b) \in \mathcal{A}^2 \mid a, b \text{ are invertible in } \mathcal{A}\}.$$

Clearly $D_2 \subseteq D_1$, but in general, we won't have $D_1 \subseteq D_2$. For example, if $\mathcal{A} = B(H)$ for some infinitely dimensional Hilbert space, and l is the one-sided left-shift operator, then l^* is the right-shift operator and we have the property $l \cdot l^* = 1$ but $l^* \cdot l \neq 1$. Therefore, we see that $(l, l^*) \notin D_2$ since both of them are not invertible but $(l, l^*) \in D_1$.

Furthermore, if we want to evaluate a rational function r which has two different representations \hat{r}_1 and \hat{r}_2 , then we need to guarantee that for each element in the intersection of the domains of \hat{r}_1 and \hat{r}_2 , their evaluations will agree. But this is also not true in general. To see this, we can consider the following example,

$$r(x, y) = 1 = y(xy)^{-1}x.$$

Let l, l^* be the left-shift and right-shift operators again, then we see that $l^*(ll^*)^{-1}l = l^*l \neq 1$.

Thanks to the insights of Cohn, we can avoid such a problem by considering an algebra \mathcal{A} which is stably finite, i.e., for each $n \in \mathbb{N}$, any $A, B \in M_n(\mathcal{A})$, we have that $AB = 1$ implies $BA = 1$. In fact, an algebra \mathcal{A} is stable finite if and only if all such representations of the zero function on the algebra give zero evaluation. See Theorem 7.8.3 in the book [5]. It is clear that $M_n(\mathbb{C})$ is stably finite for any $n \in \mathbb{N}$, so we can plug in our random matrices when they are in the domain. And fortunately, any C^* -probability space with a faithful trace is also stably finite (for a proof of this fact, see Lemma 2.2 in [12]). So in this case, the evaluation is well-defined if the elements are in the domain of the considered representation.

In some sense, the above representations of rational functions are the “irreducible” ones. That is, for a rational function $r \in \mathcal{R}$, we can always take more times of algebraic operations than we really need. For example, we can write

$$\mathcal{R}_0 \ni 1 = x^{-1}x = (x + yy^{-1})^{-1}(x + zz^{-1}) = \dots$$

In order to obtain the maximal domain of a rational function, it's much safer that we take the union of all the domains given by all possible representations that can be “reduced” to the same rational function.

Now we want to give a formal definition of such representations or expressions, and show that they have a similar recursive structure as rational functions \mathcal{R} . Then we can define the domains of these rational expressions and hence the domains of rational functions.

Denoting $\mathfrak{R}_0 = \mathcal{P}$, we define \mathfrak{R}_1 to be the free complex algebra with generating set $\mathfrak{R}_0 \cup \mathfrak{R}_0^{-1}$, i.e., we view the polynomials and their inverses as letters instead of rational functions in \mathcal{R} . In particular, 0^{-1} is also a valid non-empty word though it is meaningless when we try to consider it as rational functions. Then we build the free algebra with all words over this alphabet $\mathfrak{R}_0 \cup \mathfrak{R}_0^{-1}$. As a remark, we should note that for a polynomial, says x , the words $x^{-1} \cdot x$, $x \cdot x^{-1}$ and 1 are different words in \mathfrak{R}_1 , and 0 is a non-empty word in \mathfrak{R}_1 .

Therefore, we can construct a sequence of free algebra \mathfrak{R}_n , $n \in \mathbb{N}$ recursively, that is, each \mathfrak{R}_n is just the free algebra generated by the alphabet $\mathfrak{R}_{n-1} \cup \mathfrak{R}_{n-1}^{-1}$, $n \geq 1$. It is clear that we have a natural inclusion map $i_n : \mathfrak{R}_n \rightarrow \mathfrak{R}_{n+1}$, $n \in \mathbb{N}$ and hence we have their direct limit, denoted by \mathfrak{R}_∞ .

Now we define $\phi_0 : \mathfrak{R}_0 \rightarrow \mathcal{R}_0$ as the identity map on polynomials. Then we can define a homomorphism $\phi_1 : \mathfrak{R}_1 \rightarrow \mathcal{R}_1$ through extending the map

$$\phi_1(\alpha) = \begin{cases} \phi_0(\alpha) & \alpha \text{ is a letter in the set } \mathfrak{R}_0, \\ (\phi_0(\beta))^{-1} & \alpha = \beta^{-1} \text{ is a letter in the set } \mathfrak{R}_0^{-1}, \beta \neq 0, \\ 0 & \alpha = 0^{-1}. \end{cases}$$

Therefore, we can define a sequence of homomorphisms $\{\phi_n\}_{n \in \mathbb{N}}$ recursively, that is, by extending the map

$$\phi_n(\alpha) = \begin{cases} \phi_{n-1}(\alpha) & \alpha \text{ is a letter in } \mathfrak{R}_{n-1}, \\ (\phi_{n-1}(\beta))^{-1} & \alpha = \beta^{-1} \text{ is a letter in } \mathfrak{R}_{n-1}^{-1}, \beta \notin \ker \phi_{n-1}, \\ 0 & \alpha = \beta^{-1} \text{ is a letter in } \mathfrak{R}_{n-1}^{-1}, \beta \in \ker \phi_{n-1}. \end{cases}$$

Thus, we see that there is a homomorphism $\Phi : \mathfrak{R}_\infty \rightarrow \mathcal{R}_\infty = \mathcal{R}$. In other words, we have commutative diagrams as following: for every $n \in \mathbb{N}$,

$$\begin{array}{ccccccccc} \mathcal{P} & \longrightarrow & \mathfrak{R}_1 & \longrightarrow & \mathfrak{R}_2 & \longrightarrow & \cdots & \longrightarrow & \mathfrak{R}_n & \longrightarrow & \mathfrak{R}_\infty \\ \downarrow \phi_0 & & \downarrow \phi_1 & & \downarrow \phi_2 & & & & \downarrow \phi_n & & \downarrow \Phi \\ \mathcal{P} & \longrightarrow & \mathcal{R}_1 & \longrightarrow & \mathcal{R}_2 & \longrightarrow & \cdots & \longrightarrow & \mathcal{R}_n & \longrightarrow & \mathcal{R}_\infty \end{array}$$

It is clear that Φ is surjective, so for a rational function r in \mathcal{R} , each element in its preimage $\Phi^{-1}(r)$ is a representation of r , and we call it a *rational expression* of r . As a word over some alphabet, the evaluation of a rational expression at a tuple of elements in an algebra is clear, and thus the *domain* of a rational expression is the set of any tuple that makes the evaluation possible. As mentioned previously, if an algebra \mathcal{A} is stably finite, then the evaluation of a rational expression depends only on the corresponding rational function. We define the *domain* of a rational function r as the union of the domains of all possible rational expressions in $\Phi^{-1}(r)$.

As a remark, we can see that the elements in $\ker \Phi$ arise as the representations which can be “reduced” to 0, such as

$$y^{-1} (x^{-1} + y^{-1})^{-1} x^{-1} - (x + y)^{-1},$$

or which can be “reduced” to 0^{-1} , like

$$[1 - y(xy)^{-1}x]^{-1},$$

which make no sense when we evaluate them on algebras, and thus always have the empty domain.

Now all the ingredients for rational functions are ready. But before we move on to the convergence problem, we make just two more remarks about rational

expressions and functions, which may be helpful for better understanding on this subject.

First, the rational expressions also give us another way to rediscover the rational functions. Let $\mathcal{A} = \bigsqcup_{n=1}^{\infty} [(M_n(\mathbb{C}))^r]$ be the algebra consisting of all r -tuples of matrices of all sizes. Then we can define that two rational expressions \hat{r}_1 and \hat{r}_2 are “equivalent” if $\hat{r}_1(a) = \hat{r}_2(a)$ for each $a \in \text{dom}(\hat{r}_1) \cap \text{dom}(\hat{r}_2) \subseteq \mathcal{A}$. Then it can be shown that these equivalence classes of rational expressions coincide with the rational functions (for details, see [15]).

At last, we want to emphasize again that these rational functions or expressions are not just abstract objects from non-commutative ring theory, but also appear in system and control theory, from the theory of finite automata and formal languages to robust control and linear matrix inequalities. In fact, they already use rational expressions to consider related problems for about 50 years there. For example, in the “regular” case, i.e., the rational expression with non zero value at point 0, the language of power series is applied and first appeared in the theory of formal languages and finite automata quite long ago. For a good exposition on this, see the monograph by Berstel and Reutenauer [3].

3. CONVERGENCE OF THE NORM AND TRACE FOR RATIONAL EXPRESSIONS

Now we know enough to move on to our strong convergence problem of rational functions. Equivalently, we will just consider rational expressions due to our discussion in the last section. First of all, for a given rational expression r and a given tuple $x = (x_1, \dots, x_m)$ in some C^* -probability space (\mathcal{A}, τ) with faithful trace τ , an assumption that (x, x^*) lies in the domain of r is reasonable. However, if there is a sequence of tuples $x^{(n)} = (x_1^{(n)}, \dots, x_m^{(n)})$ from faithful tracial C^* -probability spaces $(\mathcal{A}^{(n)}, \tau^{(n)})$ s.t. $x^{(n)}$ strongly converges to x , then it's not necessary to assume that $(x^{(n)}, (x^{(n)})^*)$ also lies the domain of r . It turns out that we can deduce this well-definedness of $r(x^{(n)}, (x^{(n)})^*)$ for sufficiently large n .

Theorem 3.1. *Suppose that $x^{(n)}$ strongly converges to x and the tuple (x, x^*) lies in the domain of a rational function $r \in \mathcal{R}$. Then we have*

- (1) $(x^{(n)}, (x^{(n)})^*)$ lies in the domain of r eventually;
- (2) the convergence of norms, i.e.,

$$\lim_{n \rightarrow \infty} \|r(x^{(n)}, (x^{(n)})^*)\|_{\mathcal{A}^{(n)}} = \|r(x, x^*)\|_{\mathcal{A}}.$$

Proof. We will prove our main theorem in a recursive way based on the description of rational expressions in the last section. That is, we want to prove the above statement by induction on \mathfrak{R}_k , $k = 0, 1, 2, \dots$. For $k = 0$, it is the convergence for polynomials, which is just our assumption. Thus, we suppose that the above two statements hold for any rational expression $r \in \mathfrak{R}_k$ and we are going to prove them for \mathfrak{R}_{k+1} .

First, we need to check the domain problem. Since each rational expression in \mathfrak{R}_{k+1} can be represented as a finite sum of products of some rational expressions in \mathfrak{R}_k and their inverses, we only need to prove that, for any $\hat{r} \in \mathfrak{R}_k$ with (x, x^*)

in the domain of $\hat{r}^{-1} \in \mathfrak{R}_{k+1}$, $(x^{(n)}, (x^{(n)})^*)$ lies in the domain of \hat{r}^{-1} eventually. Or in other words, if $\hat{r}(x, x^*)$ is invertible as an operator in \mathcal{A} , then $\hat{r}(x^{(n)}, (x^{(n)})^*)$ is invertible in $\mathcal{A}^{(n)}$ for sufficiently large n .

For a rational expression, say \hat{r} , we always denote $\hat{r}^{(\infty)} = \hat{r}(x, x^*)$, $\hat{r}^{(n)} = \hat{r}(x^{(n)}, (x^{(n)})^*)$. Because $\hat{r}^{(\infty)}(\hat{r}^{(\infty)})^*$ is positive and invertible, we have

$$\|R^{(\infty)} - \hat{r}^{(\infty)}(\hat{r}^{(\infty)})^*\| < R^{(\infty)}$$

where $R^{(\infty)} = \|\hat{r}^{(\infty)}(\hat{r}^{(\infty)})^*\| > 0$. By the assumption, we know

$$\begin{aligned} \|\hat{r}^{(\infty)}(\hat{r}^{(\infty)})^*\| &= \|R^{(\infty)} - (\hat{r}(\hat{r})^*)^{(\infty)}\| \\ &= \lim_{n \rightarrow \infty} \|R^{(\infty)} - (\hat{r}(\hat{r})^*)^{(n)}\| \end{aligned} \quad (3.1)$$

because $R^{(\infty)} - \hat{r}(\hat{r})^*$ is a rational expression in \mathfrak{R}_k . Then, denoting $R^{(n)} = \|\hat{r}^{(n)}(\hat{r}^{(n)})^*\|$, from the inverse triangle inequality

$$\left| \|R^{(n)} - \hat{r}^{(n)}(\hat{r}^{(n)})^*\| - \|R^{(\infty)} - \hat{r}^{(n)}(\hat{r}^{(n)})^*\| \right| \leq |R^{(n)} - R^{(\infty)}|$$

and

$$R^{(\infty)} = \lim_{n \rightarrow \infty} R^{(n)},$$

it follows that

$$\lim_{n \rightarrow \infty} \|R^{(n)} - \hat{r}^{(n)}(\hat{r}^{(n)})^*\| = \lim_{n \rightarrow \infty} \|R^{(\infty)} - \hat{r}^{(n)}(\hat{r}^{(n)})^*\|.$$

Hence, combining with (3.1), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} (R^{(n)} - \|R^{(n)} - \hat{r}^{(n)}(\hat{r}^{(n)})^*\|) &= R^{(\infty)} - \lim_{n \rightarrow \infty} \|R^{(n)} - \hat{r}^{(n)}(\hat{r}^{(n)})^*\| \\ &= R^{(\infty)} - \lim_{n \rightarrow \infty} \|R^{(\infty)} - \hat{r}^{(n)}(\hat{r}^{(n)})^*\| \\ &= R^{(\infty)} - \|R^{(\infty)} - \hat{r}^{(\infty)}(\hat{r}^{(\infty)})^*\| \\ &> 0. \end{aligned}$$

This implies that

$$\|R^{(n)} - \hat{r}^{(n)}(\hat{r}^{(n)})^*\| < R^{(n)}$$

for n large enough, which is equivalent to say $\hat{r}^{(n)}(\hat{r}^{(n)})^*$ is invertible eventually. Recall that $(\mathcal{A}^{(n)}, \tau^{(n)})$ is stable finite, so we can easily deduce that $\hat{r}^{(n)}$ is also invertible because it has a right inverse $(\hat{r}^{(n)})^*(\hat{r}^{(n)}(\hat{r}^{(n)})^*)^{-1}$ when n is large enough.

Moreover, denoting by $\sigma(a)$ the spectrum of an operator a , we can see that

$$\begin{aligned}
\|(\hat{r}^{(\infty)})^{-1}\| &= \sqrt{\|(\hat{r}^{(\infty)}(\hat{r}^{(\infty)})^*)^{-1}\|} \\
&= \sqrt{(\min \sigma(\hat{r}^{(\infty)}(\hat{r}^{(\infty)})^*))^{-1}} \\
&= \sqrt{(R^{(\infty)} - \|R^{(\infty)} - \hat{r}^{(\infty)}(\hat{r}^{(\infty)})^*\|)^{-1}} \\
&= \lim_{n \rightarrow \infty} \sqrt{(R^{(n)} - \|R^{(n)} - \hat{r}^{(n)}(\hat{r}^{(n)})^*\|)^{-1}} \\
&= \lim_{n \rightarrow \infty} \sqrt{(\min \sigma(\hat{r}^{(n)}(\hat{r}^{(n)})^*))^{-1}} \\
&= \lim_{n \rightarrow \infty} \sqrt{\|(\hat{r}^{(n)}(\hat{r}^{(n)})^*)^{-1}\|} \\
&= \lim_{n \rightarrow \infty} \|(\hat{r}^{(n)})^{-1}\|. \tag{3.2}
\end{aligned}$$

Now, considering a rational expression $\hat{r} \in \mathfrak{R}_{k+1}$ s.t. its domain contains (x, x^*) , then, by the above argument, we can see that $(x^{(n)}, (x^{(n)})^*)$ lies in the domain of \hat{r} eventually. That is, there is $N \in \mathbb{N}$ s.t. $(x^{(n)}, (x^{(n)})^*)$ is in the domain of \hat{r} for all $n > N$. Setting

$$\mathcal{M} = \left\{ (a, a^{(N+1)}, a^{(N+2)}, \dots) \in \mathcal{A} \times \prod_{n>N} \mathcal{A}^{(n)} \mid \max \left\{ \|a\|, \sup_{n>N} \|a^{(n)}\| \right\} < \infty \right\},$$

then \mathcal{M} is C^* -algebra with the norm

$$\|(a, a^{(N+1)}, \dots)\| = \sup \left\{ \|a\|, \sup_{n>N} \|a^{(n)}\| \right\}.$$

We put

$$X_i = (x_i, x_i^{(N+1)}, \dots),$$

for $1 \leq i \leq m$, then $X_i \in \mathcal{M}$. Moreover, denoting $X = (X_1, \dots, X_m)$, we have (X, X^*) lies in the domain of \hat{r} over \mathcal{M}^{2m} , namely,

$$\hat{r}(X, X^*) = (\hat{r}^{(\infty)}, \hat{r}^{(N+1)}, \dots)$$

is well defined. Furthermore, we can see $\hat{r}(X, X^*)$ is also in \mathcal{M} . In fact, recall that \hat{r} can be written as a finite sum of products consisting of rational expressions in \mathfrak{R}_k and of their inverses, which are all bounded because of (3.2), i.e., for each $\hat{s} \in \mathfrak{R}_k$, $\|(\hat{s}^{(\infty)})^{-1}\| = \lim_{n \rightarrow \infty} \|(\hat{s}^{(n)})^{-1}\| < \infty$. It follows that

$$\max \left\{ \|\hat{r}^{(\infty)}\|, \sup_{n>N} \|\hat{r}^{(n)}\| \right\} < \infty,$$

which means that $\hat{r}(X, X^*) \in \mathcal{M}$.

Therefore, $\hat{r}(X, X^*)$ lies in the sub C^* -algebra of \mathcal{M} generated by (X, X^*) because an invertible element is still invertible in any sub C^* -algebra containing it (see Proposition 4.1.5 in [14]). Thus, for any $\varepsilon > 0$, we can find a polynomial p s.t.

$$\|p(X, X^*) - \hat{r}(X, X^*)\| < \varepsilon. \tag{3.3}$$

In particular, we have

$$\|p^{(\infty)} - \hat{r}^{(\infty)}\| < \varepsilon$$

and

$$\|p^{(n)} - \hat{r}^{(n)}\| < \varepsilon$$

for all $n > N$. Hence,

$$\begin{aligned} \left| \|\hat{r}^{(n)}\| - \|\hat{r}^{(\infty)}\| \right| &\leq \left| \|\hat{r}^{(n)}\| - \|p^{(n)}\| \right| + \left| \|p^{(n)}\| - \|p^{(\infty)}\| \right| + \left| \|p^{(\infty)}\| - \|\hat{r}^{(\infty)}\| \right| \\ &\leq \|\hat{r}^{(n)} - p^{(n)}\| + \left| \|p^{(n)}\| - \|p^{(\infty)}\| \right| + \|p^{(\infty)} - \hat{r}^{(\infty)}\| \\ &\leq 2\varepsilon + \left| \|p^{(n)}\| - \|p^{(\infty)}\| \right| \end{aligned}$$

for any $n > N$. Combining this with the fact that

$$\lim_{n \rightarrow \infty} \|p^{(n)}\| = \|p^{(\infty)}\|,$$

we have

$$\limsup_{n \rightarrow \infty} \left| \|\hat{r}^{(n)}\| - \|\hat{r}^{(\infty)}\| \right| < 2\varepsilon.$$

Since ε is arbitrary, we obtain the result of convergence of norm. \square

An immediate consequence of the theorem is that we also have the convergence in trace for rational functions.

Corollary 3.2. *Suppose that $x^{(n)}$ strongly converges to x and the tuple (x, x^*) lies in the domain of a rational function $r \in \mathcal{R}$, then we have*

$$\lim_{n \rightarrow \infty} \tau^{(n)}(r(x^{(n)}, (x^{(n)})^*)) = \tau(r(x, x^*)).$$

Proof. We can see that a similar argument as in the proof of previous theorem also works for the convergence in trace. Assume that \hat{r} is a rational expression, (X, X^*) and polynomial p are as above s.t. (3.3) holds. Then

$$\begin{aligned} & \left| \tau^{(n)}(\hat{r}^{(n)}) - \tau(\hat{r}^{(\infty)}) \right| \\ & \leq \left| \tau^{(n)}(\hat{r}^{(n)} - p^{(n)}) \right| + \left| \tau^{(n)}(p^{(n)}) - \tau(p^{(\infty)}) \right| + \left| \tau(p^{(\infty)} - \hat{r}^{(\infty)}) \right| \\ & \leq \|\hat{r}^{(n)} - p^{(n)}\| + \left| \tau^{(n)}(p^{(n)}) - \tau(p^{(\infty)}) \right| + \|\hat{r}^{(\infty)} - p^{(\infty)}\| \\ & \leq 2\varepsilon + \left| \tau^{(n)}(p^{(n)}) - \tau(p^{(\infty)}) \right| \end{aligned}$$

for n large enough. From the fact that $\lim_{n \rightarrow \infty} \tau^{(n)}(p^{(n)}) = \tau(p^{(\infty)})$, it follows, by letting ε tend to 0, that

$$\lim_{n \rightarrow \infty} \tau^{(n)}(\hat{r}^{(n)}) = \tau(\hat{r}^{(\infty)}).$$

\square

Finally, we give two remarks on possible further investigations.

First, as mentioned in the Introduction, the strong convergence is stable under taking reduced free products ([19] and [17]), that is, if $x^{(n)}$ and $y^{(n)}$ are $*$ -free for each $n \in \mathbb{N}$ and have strong limits x and y respectively, then (x, y) is the strong limit of $(x^{(n)}, y^{(n)})$. The analogue for weak convergence is also true, that is, the convergence in distribution is also stable under the reduced free product,

namely, if $x^{(n)}$ and $y^{(n)}$ are $*$ -free and have x and y as their limits in distribution respectively, then (x, y) is the limit of $(x^{(n)}, y^{(n)})$ in distribution. Some similar results for strongly convergent random matrices are mentioned in the Introduction ([16], [7] and [2]), where we can adjoin two asymptotic free tuples of random matrices. And the analogue for convergence in distribution, also holds under certain conditions for random matrices ([13]).

Therefore, as we have seen that strong convergence is stable under taking inverses, it is natural to ask if convergence in distribution is also stable under taking inverses. So assume that $x^{(n)} = (x_1^{(n)}, \dots, x_m^{(n)})$ converges in distribution to $x = (x_1, \dots, x_m)$, i.e.,

$$\lim_{n \rightarrow \infty} \tau^{(n)}(p(x^{(n)}, (x^{(n)})^*)) = \tau(p(x, x^*))$$

for any polynomial p , the question is whether we can from this conclude that

$$\lim_{n \rightarrow \infty} \tau^{(n)}(r(x^{(n)}, (x^{(n)})^*)) = \tau(r(x, x^*))$$

for a rational function r , under certain assumptions but without assuming strong convergence. To consider this convergence for random matrices does make sense because it is well known that some random matrices converge in distribution but not strongly. For example, a Wigner matrix $A = (a_{ij})_{i,j=1}^n$ whose entries are uniformly bounded i.i.d. random variable s.t. $\mathbb{E}(a_{11}) = \mu > 0$, has its largest eigenvalue asymptotically outside the support of the semi-circular law (for a reference, see [9]).

Unfortunately, it seems that outliers make the convergence in distribution unstable with respect to inverses. Here is a simple example: let $X^{(n)} \in M_n(\mathbb{C})$ be a sequence of matrices that strongly converges to x , which lies in some faithful tracial C^* -probability space (\mathcal{A}, τ) . We assume that x is invertible, then by our main theorem, we have $X^{(n)}$ is invertible eventually, and

$$\lim_{n \rightarrow \infty} \text{tr}_n((X^{(n)})^{-1}) = \tau(x^{-1}).$$

Now put

$$Y^{(n+1)} = \begin{pmatrix} \frac{1}{n+1} & 0 \\ 0 & X^{(n)} \end{pmatrix} \in M_{n+1}(\mathbb{C}),$$

then it is clear that $Y^{(n)}$ also converges in distribution to x and $Y^{(n)}$ is invertible as $X^{(n)}$ is invertible eventually. However, we can see that

$$\lim_{n \rightarrow \infty} \text{tr}_n((Y^{(n)})^{-1}) = 1 + \tau(x^{-1}).$$

Secondly, if we consider in the one-variable case, a sequence of self-adjoint random variables $\{x^{(n)}\}_{n \geq 1}$ which strongly converges to a self-adjoint random variable x , then for any continuous function f defined on a neighborhood of the interval $[-\|x\|, \|x\|]$, we can see that $f(x^{(n)})$ will be eventually well-defined since the support of $x^{(n)}$ is approaching to $[-\|x\|, \|x\|]$. On the other hand, since we can find some polynomials $\{p_k\}$ uniformly converging to f on this neighborhood, we can use the same argument as above to show that $f(x^{(n)})$ converges to $f(x)$ in trace and in norm. However, for the general multivariable case, it is not clear whether one can go beyond the case of rational functions. Nevertheless, it

is tempting to hope to be able to extend our investigation to the case of non-commutative analytic functions.

Acknowledgments. The author acknowledges the support from the ERC Advanced Grant “Non-Commutative Distributions in Free Probability” (Grant No. 339760).

The author wants to express his thanks to several people who really help a lot on this work. First, the author wants to thank his supervisor, Prof. Roland Speicher, for suggesting this interesting project and his continued support. And the author is grateful to Guillaume Cébron for the really fruitful discussion and some crucial ideas on this article. Finally, the author wants to thank Tobias Mai and Felix Leid, from whom the author learn a lot about rational functions.

REFERENCES

1. G. W. Anderson, *Convergence of the largest singular value of a polynomial in independent Wigner matrices*, Ann. Probab. **41** (2013), no. 3B, 2103–2181.
2. S. Belinschi and M. Capitaine, *Spectral properties of polynomials in independent Wigner and deterministic matrices*, arXiv:1611.07440.
3. J. Berstel and C. Reutenauer, *Les séries rationnelles et leurs langages* (French) [[Rational power series and their languages]] Études et Recherches en Informatique. [Studies and Research in Computer Science] Masson, Paris, 1984.
4. M. Capitaine and C. Donati-Martin, *Strong asymptotic freeness for Wigner and Wishart matrices*, Indiana Univ. Math. J. **56** (2007), 767–803.
5. P. M. Cohn, *Free ideal rings and localization in general rings*, Cambridge University Press, 2006.
6. P. M. Cohn and C. Reutenauer, *On the construction of the free field*, Internat. J. Algebra Comput. **9** (1999), no. 3-4, 307–323.
7. B. Collins and C. Male, *The strong asymptotic freeness of Haar and deterministic matrices*, Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 1, 147–163.
8. K. Dykema, *On certain free product factors via an extended matrix model*, J. Funct. Anal. **112** (1993), 31–60.
9. Z. Füredi and J. Komlós, *The eigenvalues of random symmetric matrices*, Combinatorica **1** (1981), no. 3, 233–241.
10. U. Haagerup, H. Schultz, and S. Thorbørnsen, *A random matrix approach to the lack of projections in $C_{red}^*(F_2)$* , Adv. Math. **204** (2006), 1–83.
11. U. Haagerup and S. Thorbørnsen, *A new application of random matrices: $Ext(C_{red}^*(F_2))$ is not a group*, Ann. of Math. (2) **162** (2005), no. 2, 711–775.
12. J. W. Helton, T. Mai, and R. Speicher, *Applications of realizations (aka linearizations) to free probability*, arXiv:1511.05330.
13. F. Hiai and D. Petz, *Asymptotic freeness almost everywhere for random matrices*, Acta Sci. Math. (Szeged) **66** (2000), no. 3-4, 809–834.
14. R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras. Vol. I. Elementary theory.*, Pure and Applied Mathematics, 100. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983.
15. D. S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov, *Non-commutative rational functions, their difference-differential calculus and realizations*, Multidimens. Syst. Signal Process. **23** (2012), no. 1-2, 49–77.
16. C. Male, *The norm of polynomials in large random and deterministic matrices with an appendix by Dimitri Shlyakhtenko*, Probab. Theory Related Fields **154** (2012), no. 3-4, 477–532.

17. G. Pisier, *Strong convergence for reduced free products*, *Infin. Dimens. Anal. Quantum. Probab. Relat. Top.*, 19 (2016), no. 2, 1650008, 22 pp.
18. H. Schultz, *Non-commutative polynomials of independent Gaussian random matrices. The real and symplectic cases*, *Probab. Theory Related Fields* **131** (2005), no. 2, 261–309.
19. P. Skoufranis, *On a notion of exactness for reduced free products of C^* -algebras*. *J. Reine Angew. Math.* **700** (2015), 129–153.
20. S. Thorbørnsen, *Mixed moments of Voiculescu's Gaussian random matrices*, *J. Funct. Anal.* **176** (2000), no. 2, 213–246.
21. D. V. Voiculescu, *Limit laws for random matrices and free products*, *Invent. Math.* **104** (1991), no. 1, 201–220.

FACULTY OF MATHEMATICS, SAARLAND UNIVERSITY, D-66123, SAARBRUECKEN, GERMANY.

E-mail address: yin@math.uni-sb.de