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A NOTE ON O-FRAMES FOR OPERATORS

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Abstract. A sufficient condition for a boundedly complete O-frame and a necessary condition for an unconditional O-frame are given. Also, a necessary and sufficient condition for an absolute O-frame is obtained. Finally, it is proved that if two operators have an absolute O-frame, then their product also has an absolute O-frame.

1. Introduction

The notion of frames for Hilbert spaces was formally introduced by Duffin and Schaeffer [5] in the context of nonharmonic analysis. Daubechies, Grossmann and Meyer [4] revived interest in the theory in the early stages of the development of wavelet theory. Frames are a generalization of orthonormal bases. Frames have become a central tool in many areas of mathematics, such as image processing, wireless communications, sigma - delta quantization, filter bank theory, etc. For a comprehensive survey of frames and related concepts, we refer to the textbooks by Christensen [3], Heil [8] and the survey article of Casazza [1].

Han and Larson [7] defined a Schauder frame for a Banach space E to be a compression of a Schauder basis for E. Schauder frames were further studied in [2, 9, 10, 12, 13]. The notion of an O-frame for an operator $T \in B(E, F)$ was introduced and studied by O. Reinov [11] as a generalization of Schauder frames. In the particular case when the operator $T = I$, the notion of an O-frame is equivalent to that of a Schauder frame.

The convergence (and mode of convergence) of series associated with redundant

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building blocks is important in applied mathematics. For example, the series associated with frames (with frame operator S), i.e., $f = \sum_{k=1}^{\infty}\langle f, S^{-1}f_k\rangle f_k = \sum_{k=1}^{\infty}\langle S^{-1}f, f_k\rangle f_k$ is unconditionally convergent. It would be interesting to know about various modes of convergence, of the series associated with an O-frame for a given operator, in a Banach space. In this paper, we obtain some results related to the mode of convergence of the series associated with an O-frame for an operator in Banach spaces.

We organize the paper as follows: In Section 2, we study O-frames for operators and give a sufficient condition for an O-frame to be boundedly complete. Also, we discuss O-frames in finite dimensional Banach spaces and obtain some new results. In Section 3, we study unconditional convergence of series associated with O-frames in Banach spaces and give a necessary condition for the unconditional convergence of the series related to the O-frame. In Section 4, we introduce the notion of an absolute O-frame for an operator in a Banach space and obtain a necessary and sufficient condition for it. Finally, we prove that if two operators have an absolute O-frame, then their product also has an absolute O-frame.

2. O-frames for Operators

Throughout this paper E will denote a separable Banach space and E^* the dual space of E.

Han and Larson [7] introduced the notion of Schauder frames in Banach spaces. They gave the following definition:

Definition 2.1. Let E be a Banach space. A pair of sequences $(\{f_k\}, \{f_k^*\}) \subset E \times E^*$ is called a Schauder frame for E if each $f \in E$ has the representation

$$f = \sum_{k=1}^{\infty} f_k^*(f) f_k, \tag{2.1}$$

where the series in (2.1) converges in the norm topology of E.

O. Reinov [11] introduced the notion of an O-frame for an operator and gave the following definition:

Definition 2.2. Let E and F be infinite dimensional separable Banach spaces over the scalar field ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and let $(\{f_k^*\}, \{g_k\}) \subset E^* \times F$ and $T \in B(E, F)$ be given. We say that the pair $(\{f_k^*\}, \{g_k\})$ is an O-frame for the operator T if

$$Tf = \sum_{k=1}^{\infty} f_k^*(f) g_k, \quad \text{for all} \quad f \in E, \tag{2.2}$$

where the series in (2.2) converges in the norm topology of F.

Remark 2.3. An O-frame $(\{f_k^*\}, \{g_k\}) \subset E^* \times F$ for $T = I$ is a Schauder frame for E. Also, if $(\{f_k^*\}, \{f_k\})$ is a Schauder frame for E and $T \in B(E)$, then $(\{f_k^*\}, \{Tf_k\})$ is an O-frame for T. Indeed, if $(\{f_k^*\}, \{f_k\})$ is a Schauder frame for E, then for each $f \in E$, we have

$$f = \sum_{k=1}^{\infty} f_k^*(f) f_k,$$

and for all $T \in B(E)$ we have

$$Tf = \sum_{k=1}^{\infty} f_k^*(f) T f_k, \text{ for all } f \in E.$$

Thus, the pair $(\{f_k^*\}, (\{Tf_k\})$ is an O-frame for T.

In the following example, we see that a pair of sequences $(\{f_k^*\}, \{g_k\}) \subset E^* \times E$ that is not a Schauder frame can be an O-frame for some operator T.

Example 2.4. Let $E = F = L^2(\mathbb{N}, \mu)$ be the discrete signal spaces, where μ is counting measure. Let $\{\chi_k\}$ be the sequence of standard unit vectors in E. Define sequences $\{f_k^*\} \subset E^*$ and $\{g_k\} \subset F$ by

$$f_k^*(f) = \xi_k, \ f = \{\xi_j\} \in E \ (k \in \mathbb{N})$$

and $g_k = \chi_{k+1}, \ k \in \mathbb{N}$. Then, we can easily verify that $(\{f_k^*\}, \{g_k\})$ is not a Schauder frame for E. However, if we consider the shift operator $T : E \to E$ given by

$$T(f) = \{0, \xi_1, \xi_2, ..., \}, \ f = \{\xi_j\} \in \ E,$$

then, $T \in B(E)$ and for each $f \in E$ we have

$$Tf = \sum_{k=1}^{\infty} f_k^*(f) g_k.$$

Thus, the pair $(\{f_k^*\}, \{g_k\})$ is an O-frame for T.

Definition 2.5. [11] Let $T \in B(E, F)$ and let $C \geq 1$. We say that T has the C-BAP (C-bounded approximation property), if for every compact subset K of E and for each $\epsilon > 0$, there exists a finite rank operator $R : E \to \ F$ such that $\|R\| \leq C\|T\|$ and $\sup_{f \in K} \|Rf - Tf\| \leq \epsilon$.

The operator T is said to have the BAP, if it has the C-BAP for some constant $C \in [1, \infty)$.

O. Reinov gave the following characterization of O-frames in terms of BAP.

Theorem 2.6. [11] *Let E and F be Banach spaces and let $T \in B(E, \ F)$. Then the following statements are equivalent:*

(1) *T has an O-frame.*
(2) *T has BAP.*
(3) *The operator T can be factored through a Banach space with a Schauder basis.*

Recall that an operator $T \in B(E, F)$ is said to factor through a Banach space G if there exist operators $R \in B(E, G)$ and $S \in B(G, F)$ such that $T = SR$.

Definition 2.7. A sequence $\{f_k\} \subset E$ is said to be ω-linearly independent if $\{c_k\} \subset \mathbb{K}$, $\sum_{k=1}^{\infty} c_k f_k = 0$ imply $c_k = 0$, for all $k \in \mathbb{N}$.

Next, we state a result in the form of a lemma that will be used in the subsequent work.

Lemma 2.8. [6] *Let $\{f_k\} \subset E$ and let $\sum_{k=1}^{\infty} f_k$ be a series of vectors in E. Then the following statements are equivalent:*

(1) *If $\sigma(.)$ is any permutation of $\mathbb{N}$, then $\sum_{k=1}^{\infty} f_{\sigma(k)} = f, for \ all \ f \in E$.*

(2) *For each* $\epsilon > 0$*, there is a finite set* $\Omega \subset \mathbb{N}$ *such that*

$$\Big\|f - \sum_{j \in \Omega_0} f_j\Big\| < \epsilon,$$

whenever $\Omega_0 \subset \mathbb{N}$ *is a finite set satisfying* $\Omega \subset \Omega_0$.

Definition 2.9. An O-frame $(\{f_k^*\}, \{g_k\}) \subset E^* \times F$ for an operator T is said to be boundedly complete if for each $\phi^{**} \in E^{**}$, the series $\sum_{k=1}^{\infty} \phi^{**}(f_k^*)g_k$ converges in F.

In the following result, we give a sufficient condition under which an O-frame is boundedly complete:

Theorem 2.10. *Let* $(\{f_k^*\}, \{g_k\}) \subset E^* \times F$ *be an O-frame for* T *such that*

$$\sup_n \Big\|\sum_{k=1}^{n} \alpha_k f_k^*(f) g_k\Big\| < \infty \Rightarrow \sum_{k=1}^{\infty} \alpha_k f_k^*(f) g_k \quad \text{converges in } F,$$

where $\{\alpha_k\}$ *is any sequence of scalars and* $f \in E$*. Then,* $(\{f_k^*\}, \{g_k\})$ *is a boundedly complete O-frame for* T*.*

Proof. Let $\phi^{**} \in E^{**}$. If $0 \neq \phi^{**} \in [f_k^*]^{\perp}$, then $\phi^{**}(f_k^*) = 0$, for all $k \in \mathbb{N}$. So, the series $\sum_{k=1}^{\infty} \phi^{**}(f_k^*)g_k$ converges in F. Suppose that $\phi^{**} \notin [f_k^*]^{\perp}$. Define $T_n : E \to F$ by

$$T_n f = \sum_{k=1}^{n} f_k^*(f) g_k, \quad f \in E.$$

Let T_n^* be the adjoint operator of T_n. Then

$$(T_n^*(g^*))(f) = \Big(\sum_{k=1}^{n} g^*(g_k) f_k^*\Big)(f), \quad g^* \in F^*, f \in E.$$

This gives

$$T_n^*(g^*) = \sum_{k=1}^{n} g^*(g_k) f_k^*, \quad g^* \in F^*, n = 1, 2, 3, ...$$

Further, for every $g^* \in F^*$, we have

$$(T_n^{**}(\phi^{**}))(g^*) = \phi^{**}(T_n^*(g^*)) = g^*\Big(\sum_{k=1}^{n} \phi^{**}(f_k^*) g_k\Big).$$

Therefore, we obtain

$$T_n^{**}(\phi^{**}) = \pi\Big(\sum_{k=1}^{n} \phi^{**}(f_k^*) g_k\Big),$$

where π is the canonical mapping of F into F^{**}. Since π is an isometry, it follows that

$$\begin{aligned}\Big\|\sum_{k=1}^{n}\phi^{**}(f_k^*)g_k\Big\| &= \Big\|\pi\Big(\sum_{k=1}^{n}\phi^{**}(f_k^*)g_k\Big)\Big\|\\ &= \|T_n^{**}(\phi^{**})\|\\ &\leq \|T_n\|\|\phi^{**}\|.\end{aligned}$$

This gives, $\sup_n \Big\|\sum_{k=1}^{n}\phi^{**}(f_k^*)g_k\Big\| < \infty$. Without loss of generality we may assume that $f_k^* \neq 0$, for all $k \in \mathbb{N}$. Then, there exists a non-zero $f \in E$ such that $f_k^*(f) \neq 0$, for all $k \in \mathbb{N}$. Choose $\{\alpha_k\} \subset \mathbb{K}$ (where $\mathbb{K}$ is the scalar field) such that $\phi^{**}(f_k^*) = \alpha_k f_k^*(f),\ k = 1, 2, 3, \ldots$. Then, $\sup_n \Big\|\sum_{k=1}^{n}\alpha_k f_k^*(f)g_k\Big\| < \infty$. Therefore, by hypotheses, $\sum_{k=1}^{\infty}\phi^{**}(f_k^*)g_k$ converges in F. Hence $(\{f_k^*\},\{g_k\})$ is a boundedly complete O-frame for T. □

Now, we discuss O-frames in finite dimensional Banach spaces.

Theorem 2.11. *If E and F are finite dimensional Banach spaces, then every operator $T \in B(E,F)$ has an O-frame.*

Proof. Let E and F be finite dimensional Banach spaces. Then, there exist sequences $\{h_k^*\}_{k=1}^{n} \subset E^*$ and $\{h_k\}_{k=1}^{n} \subset E$ such that

$$f = \sum_{k=1}^{n} h_k^*(f)h_k, \quad \text{for all} \quad f \in E.$$

Let $T : E \to F$ be a bounded linear operator. Define sequences $\{g_n\} \subset F$ and $\{f_n^*\} \subset E^*$ as follows:

$$\left.\begin{aligned} g_{tn^2+ln+\xi} &= \frac{1}{2^{t+1}n}Th_\xi \\ f^*_{tn^2+ln+\xi} &= h_\xi^* \end{aligned}\right\} \Big(t = 0, 1, 2, \ldots; l = 0, 1, \ldots, n-1; \xi = 1, 2, \ldots, n\Big).$$

Then, for each $f \in E$ we have

$$\begin{aligned}\sum_{k=1}^{\infty} f_k^*(f)g_k &= \sum_{t=0}^{\infty}\sum_{l=0}^{n-1}\sum_{\xi=1}^{n} f^*_{tn^2+ln+\xi}(f)g_{tn^2+ln+\xi}\\ &= \sum_{t=0}^{\infty} n\sum_{\xi=1}^{n}\frac{1}{2^{t+1}n}h_\xi^*(f)Th_\xi\\ &= T\Big(\sum_{t=0}^{\infty} n\sum_{\xi=1}^{n}\frac{1}{2^{t+1}n}h_\xi^*(f)h_\xi\Big)\\ &= T\Big(\sum_{\xi=1}^{n}h_\xi^*(f)h_\xi\Big)\\ &= Tf.\end{aligned}$$

Hence $(\{f_k^*\},\{g_k\})$ is an O-frame for T. □

Next, we discuss a special type of perturbation of an O-frame for $T \in B(E,F)$ and obtained a sufficient condition for the perturbed system to be an O-frame for T.

Theorem 2.12. *Let $(\{f_k^*\},\{g_k\}) \subset E^* \times F$ be an O-frame for $T \in B(E,F)$. For a given $\epsilon > 0$ and a fixed $f_0 \in E$, let $\{h_k^*\} \subset E^*$ and $\{d_k\} \subset F$ be given by*

$$h_k^* = \frac{1}{|f_k^*(f_0)|+\epsilon} f_k^* - \frac{1}{|f_{k+1}^*(f_0)|+\epsilon} f_{k+1}^*, \ \textit{for all } k \in \mathbb{N}$$

and

$$d_k = \sum_{n=1}^{k} (|f_n^*(f_0)|+\epsilon) g_n, \ \textit{for all } k \in \mathbb{N}.$$

If $\lim_{n\to\infty} \frac{f_{n+1}^*(f)}{|f_{n+1}^*(f_0)|+\epsilon} d_n = 0$, then $(\{h_k^*\},\{d_k\})$ is an O-frame for T.

Proof. By hypotheses, we have

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=1}^{n} h_k^*(f) d_k &= \lim_{n\to\infty} [h_1^*(f)d_1 + h_2^*(f)d_2 + ...h_{n-1}^*(f)d_{n-1} + h_n^*(f)d_n] \\
&= \lim_{n\to\infty} [\frac{f_1^*(f)}{|f_1^*(f_0)|+\epsilon}(|f_1^*(f_0)|+\epsilon)g_1 - \frac{f_2^*(f)}{|f_2^*(f_0)|+\epsilon}(|f_1^*(f_0)|+\epsilon)g_1 \\
&+ \frac{f_2^*(f)}{|f_2^*(f_0)|+\epsilon}\{(|f_1^*(f_0)|+\epsilon)g_1 + (|f_2^*(f_0)|+\epsilon)g_2\} \\
&- \frac{f_3^*(f)}{|f_3^*(f_0)|+\epsilon}\{(|f_1^*(f_0)|+\epsilon)g_1 + (|f_2^*(f_0)|+\epsilon)g_2\} \\
&.... + \frac{f_n^*(f)}{|f_n^*(f_0)|+\epsilon} d_n - \frac{f_{n+1}^*(f)}{|f_{n+1}^*(f_0)|+\epsilon} d_n] \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} f_k^*(f) g_k - \lim_{n\to\infty} \frac{f_{n+1}^*(f)}{|f_{n+1}^*(f_0)|+\epsilon} d_n \\
&= Tf.
\end{aligned}$$

Hence $(\{h_k^*\},\{d_k\})$ is an O-frame for T. □

3. Unconditional convergence associated with O-frames

In this section, we study the notion of an unconditional O-frame defined by Reinov [11]. We begin with the following definition:

Definition 3.1. [11] Let E and F be infinite dimensional separable Banach spaces over the scalar field ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Let $(\{f_k^*\},\{g_k\}) \subset E^* \times F$ and $T \in B(E,F)$. We say that the pair $(\{f_k^*\},\{g_k\}$ is an UO-frame (unconditional O-frame) for T if

$$Tf = \sum_{k=1}^{\infty} f_k^*(f) g_k, \quad \text{for all} \quad f \in E, \tag{3.1}$$

where the series in (3.1) converges unconditionally for each $f \in E$ in the norm topology of F.

Regarding the existence of an unconditional O-frame for T, we have the following example:

Example 3.2. Let $E = F = L^2(\mathbb{N}, \mu)$ be discrete signal spaces, where μ is counting measure. Let $\{\chi_k\}$ be the sequence of standard unit vectors in E. Define sequences $\{f_k^*\} \subset E^*$ and $\{g_k\} \subset E$ by

$$f_k^*(f) = \frac{\xi_k}{k}, \quad f = \{\xi_k\} \in E \ (k \in \mathbb{N})$$

and

$$g_k = \chi_k, \quad (k \in \mathbb{N}).$$

Consider the operator $T : E \to E$ given by

$$T(f) = \{\xi_1, \frac{\xi_2}{2}, \frac{\xi_3}{3}, ..., \}, \ f = \{\xi_j\} \in \ E.$$

Then, $T \in B(E)$ and for each $f \in E$ we have

$$Tf = \sum_{k=1}^{\infty} f_k^*(f) g_k.$$

Hence the pair $(\{f_k^*\}, \{g_k\})$ is an O-frame for T. Also, the O-frame $(\{f_k^*\}, \{g_k\})$ is unconditional. Indeed, let $f = \{\xi_k\} \subset E$. Then, for $n, p \in \mathbb{N}$, we have

$$\Big|\Big| \sum_{k=n}^{n+p} f_k^*(f) g_k \Big|\Big|_2^2 = \sum_{k=n}^{n+p} \Big| \frac{\xi_k}{k} \Big|^2.$$

Since the series $\sum_{k=1}^{\infty} |\frac{\xi_k}{k}|^2$ converges in $\mathbb{K}$, the series $\sum_{k=1}^{\infty} f_k^*(f) g_k$ converges unconditionally. Hence $(\{f_k^*\}, \{g_k\})$ is an UO-frame for T.

Next, we give an example of an O-frame which is not an unconditional O-frame.

Example 3.3. Let $E = F = (c_0, \|.\|_\infty)$, where $c_0 = \{\{\alpha_n\} \subset \mathbb{C} : \lim_{n\to\infty} \alpha_n \to 0\}$. Define $\{f_k^*\} \subset E$ by

$$f_k^*(f) = (0, 0, ..., \xi_k - \xi_{k+1}, 0, 0..., 0), \quad f = \{\xi_k\} \ \ (k \in \mathbb{N}).$$

Take $g_k = \sum_{i=1}^{k} \mathcal{X}_{i+1}$, where $\{\mathcal{X}_i\}$ is the sequence of canonical unit vectors. Consider the operator $T : E \to E$ given by

$$T(f) = \{0, \xi_1, \xi_2, \xi_3, ..., \underbrace{\xi_n}_{(n+1)^{\text{th place}}}, 0, 0, 0, ...\}, \ f = \{\xi_n\} \in \ E.$$

Then, $T \in B(E)$ and for each $f \in E$ we have

$$Tf = \sum_{k=1}^{\infty} f_k^*(f) g_k.$$

Thus, the pair $(\{f_k^*\}, \{g_k\})$ is an O-frame for T. In order to show that $(\{f_k^*\}, \{g_k\})$ is not unconditional, let $f \in E$ and $n, p \in \mathbb{N}$. Then

$$\Big|\Big| \sum_{k=n}^{n+p} f_k^*(f) g_k \Big|\Big|_\infty = \sup_{n \le l \le n+p} \Big| \sum_{k=l}^{n+p} f_k^*(f) \Big|.$$

Take $f_0 = \{0, \frac{1}{2}, 0, \frac{1}{3}, 0, ...\}$. Then, for this f_0, the series $\sum_{k=1}^{\infty} f_k^*(f_0)$ is conditionally convergent. Therefore $(\{f_k^*\}, \{g_k\})$ is not an UO-frame for T.

Next, we give a necessary condition for an unconditional O-frame for T.

Theorem 3.4. *Let $(\{f_k^*\}, \{g_k\}) \subset E^* \times F$ be an UO-frame for T. Then, for every $f \in E$*

$$\lim_{n\to\infty} \sup_{g\in F^*, \|f\|\leq 1} \sum_{i=n+1}^{\infty} |f_k^*(f)||g(g_k)| = 0.$$

Proof. Let $\epsilon > 0$ be given. Since $(\{f_k^*\}, \{g_k\})$ is an UO-frame for T, by Lemma 2.8, there exists a finite subset d of $\mathbb{N}$ such that

$$\|Tf - \sum_{i\in d'} f_k^*(f)g_i\| < \frac{\epsilon}{4}, \quad \text{for all finite subsets } d' \text{ of } \mathbb{N} \text{ with } d' \subset d. \tag{3.2}$$

Define sets

$$d_1(f) = \{i \in \{n+1, n+2, ..., n+m\} : \text{Real } g^*(g_i)f_i^*(f) \geq 0\}$$

and

$$d_2(f) = \{i \in \{n+1, n+2, ..., n+m\} : \text{Real } g^*(g_i)f_i^*(f) < 0\},$$

where $n \geq n_0 = \max_{i\in d'} i, m \geq 1$ and $g^* \in F^*$ is such that $\|g^*\| \leq 1$. Then, by using (3.2), we have

$$\begin{aligned}
\sum_{i=n+1}^{n+m} |\text{Real } g^*(g_i)f_i^*(f)| &= \sum_{j=1}^{2} \sum_{i\in d_j(f)} |\text{Real } g^*(g_i)f_i^*(f)| \\
&= \sum_{j=1}^{2} \Big|\text{Real } g^*\Big(\sum_{i\in d_j(f)} f_i^*(f)g_i\Big)\Big| \\
&\leq \sum_{j=1}^{2} \Big| g^*\Big(\sum_{i\in d_j(f)} f_i^*(f)g_i\Big)\Big| \\
&\leq \sum_{j=1}^{2} \|g^*\|\Big\|\sum_{i\in d_j(f)} f_i^*(f)g_i\Big\| \\
&\leq \sum_{j=1}^{2} \Big(\Big\|Tf - \sum_{i\in d_j(f)\cup d} f_i^*(f)g_i\Big\| + \Big\|Tf - \sum_{i\in d_j(f)\cup d} f_i^*(f)g_i\Big\|\Big) \\
&< \frac{\epsilon}{2}, \quad \text{for all } f \in E.
\end{aligned}$$

Similarly, we can show that

$$\sum_{i=n+1}^{n+m} |\text{Im } g^*(g_i)f_i^*(f)| < \frac{\epsilon}{2}, \quad \text{for all } f \in E.$$

Hence

$$\lim_{n\to\infty} \sup_{g\in F^*, \|f\|\leq 1} \sum_{i=n+1}^{\infty} |f_k^*(f)||g(g_k)| = 0, \quad f \in E.$$

□

Next, we obtain a condition on $T \in B(E,F)$ under which an O-frame for T is a Schauder frame for F.

Proposition 3.5. *Let E and F be separable Banach spaces and let $(\{f_k^*\},\{g_k\}) \subset E^* \times F$ be an O-frame for $T \in B(E,F)$. If T is invertible, then $(\{T^{-1^*} f_k^*\},\{g_k\})$ is a Schauder frame for F. Moreover, if $(\{f_k^*\},\{g_k\})$ is an unconditional O-frame for $T \in B(E,F)$, then $(\{T^{-1^*} f_k^*\},\{g_k\})$ is an unconditional Schauder frame for F.*

Proof. For $g \in F$, we have

$$\begin{aligned} g &= \sum_{k=1}^{\infty} f_k^*(T^{-1}g)g_k \\ &= \sum_{k=1}^{\infty} (T^{-1})^* f_k^*(g)g_k. \end{aligned}$$

Hence $(\{T^{-1^*} f_k^*\},\{g_k\})$ is a Schauder frame for F. Moreover, the series $\sum_{k=1}^{\infty}(T^{-1})^* f_k^*(f)g_k$ converges unconditionally as $(\{f_k^*\},\{g_k\})$ is an unconditional O-frame for $T \in B(E,F)$. Thus, $(\{T^{-1^*} f_k^*\},\{g_k\})$ is an unconditional Schauder frame for F. □

4. Absolute O-frames

In this section, we define and study absolute O-frames. We begin with the following definition:

Definition 4.1. Let $(\{f_k^*\},\{g_k\}) \subset E^* \times F$ be an O-frame for $T \in B(E,F)$. We say that the pair $(\{f_k^*\},\{g_k\}$ is an absolute O-frame for T if the series

$$\sum_{k=1}^{\infty} f_k^*(f)g_k,$$

converges absolutely for each $f \in E$. That is, $\sum_{k=1}^{\infty} \|f_k^*(f)g_k\|$ converges in $\mathbb{R}$, for all $f \in E$.

Existence of an absolute O-frame is ensured by the following example:

Example 4.2. Let $E = F = L^1(\mathbb{N},\mu)$ be discrete signal spaces, where μ is counting measure. Let $\{\chi_k\}$ be the sequence of standard unit vectors in E.
Define sequences $\{f_k^*\} \subset E^*$ and $\{g_k\} \subset E$ by

$$\begin{cases} f_1^*(f) = \xi_1, \\ f_2^*(f) = f_3^*(f) = \xi_2, \\ f_4^*(f) = f_5^*(f) = f_6^*(f) = \xi_3, \\ \ldots \end{cases}$$

and

$$\begin{cases} g_1 = 0, \\ g_2 = g_3 = \dfrac{\chi_2}{2}, \\ g_4 = g_5 = g_6 = \dfrac{\chi_3}{3}, \\ \dots . \end{cases}$$

Consider the operator $T : E \to E$ given by

$$T(f) = \{0, \xi_2, \xi_3, ..., \}, \ f = \{\xi_j\} \in \ E.$$

Then, $T \in B(E)$ and for each $f \in E$, we have

$$Tf = \sum_{k=1}^{\infty} f_k^*(f) g_k.$$

Thus, the pair $(\{f_k^*\}, \{g_k\})$ is an O-frame for T. Also, the O-frame $(\{f_k^*\}, \{g_k\})$ is absolute. Indeed, let $f = \{\xi_k\} \subset E$. Then

$$\sum_{k=1}^{\infty} \Big|\Big| f_k^*(f) g_k \Big|\Big| = \sum_{k=2}^{\infty} |\xi_k|.$$

Since the series $\sum_{k=2}^{\infty} |\xi_k|$ is convergent, the series $\sum_{k=1}^{\infty} f_k^*(f) g_k$ converges absolutely.

Next, we define a positively confined O-frame for T as follows:

Definition 4.3. Let $(\{f_k^*\}, \{g_k\}) \subset E^* \times F$ be an O-frame for $T \in B(E, F)$. Then, the pair $(\{f_k^*\}, \{g_k\})$ is said to be

(1) pre-positively confined, if there exist strictly positive constants α and β such that

$$\alpha \le \|g_k\| \le \beta, \quad \text{for all} \quad k \in \mathbb{N},$$

(2) post-positively confined, if there exist strictly positive constants α^0 and β^0 such that

$$\alpha^0 \le \|f_k^*\| \le \beta^0, \ \text{for all} \quad k \in \mathbb{N},$$

(3) positively confined, if it is both pre and post-positively confined.

The following result provides a necessary and sufficient condition for a pre-positively confined O-frame for T to be absolute.

Theorem 4.4. *Let $(\{f_k^*\}, \{g_k\}) \subset E^* \times F$ be a pre-positively confined O-frame for T. Then, the O-frame $(\{f_k^*\}, \{g_k\})$ is absolute if and only if the series $\sum_{k=1}^{\infty} |f_k^*(f)|$ converges for all $f \in E$.*

Proof. Since the O-frame $(\{f_k^*\}, \{g_k\})$ is pre-positively confined, there exist positive constants α and β such that $\alpha \le \|g_k\| \le \beta$, for all $k \in \mathbb{N}$. Suppose that $(\{f_k^*\}, \{g_k\})$ is absolute. Then, for all $f \in E$ we have

$$\begin{aligned}\sum_{k=1}^{\infty} |f_k^*(f)| &= \sum_{k=1}^{\infty} \Big|\Big| \frac{f_k^*(f) g_k}{\|g_k\|} \Big|\Big| \\ &\leq \frac{1}{\alpha} \sum_{k=1}^{\infty} \Big|\Big| f_k^*(f) g_k \Big|\Big| < \infty.\end{aligned}$$

Conversely, suppose that $\sum_{k=1}^{\infty} |f_k^*(f)|$ converges for all $f \in E$. Then

$$\begin{aligned}\sum_{k=n}^{m} \Big|\Big| f_k^*(f) g_k \Big|\Big| &= \sum_{k=n}^{m} |f_k^*(f)| \|g_k\| \\ &\leq \beta \sum_{k=n}^{m} |f_k^*(f)| \to 0 \quad \text{as} \quad m, n \to \infty.\end{aligned}$$

Therefore $\sum_{k=1}^{\infty} \Big|\Big| f_k^*(f) g_k \Big|\Big|$ converges in $\mathbb{R}$. Hence the O-frame $(\{f_k^*\}, \{g_k\})$ is absolute. □

Next, we give a necessary and sufficient condition for a post-positively confined O-frame for T^* to be absolute.

Theorem 4.5. *Let $(\{g_k\}, \{f_k^*\}) \subset E^* \times F$ be a post-positively confined O-frame for T^*. Then, the O-frame $(\{g_k\}, \{f_k^*\})$ is absolute if and only if the series $\sum_{k=1}^{\infty} |g^*(g_k)|$ converges for all $g^* \in F^*$.*

Proof. It can be worked out on the lines of Theorem 4.4. □

Next, we prove the following result related to an absolute O-frame satisfying certain conditions.

Theorem 4.6. *Let $(\{f_k^*\}, \{g_k\}) \subset E^* \times F$ be an absolute O-frame for $T \in B(E, F)$. If $\{g_k\}$ is ω-linearly independent and T is surjective, then there exists a topological isomorphism of $\ell^1(\mathbb{N})$ onto F.*

Proof. Define $\Psi : \ell^1(\mathbb{N}) \to F$ by

$$\Psi(\{\xi_k\}) = \sum_{k=1}^{\infty} \frac{\xi_k g_k}{\|g_k\|}, \quad \{\xi_k\} \in \ell^1(\mathbb{N}).$$

Then, for all $\{\xi_k\} \in \ell^1(\mathbb{N})$ we have

$$\begin{aligned}\|\Psi(\{\xi_k\})\| &= \Big|\Big| \sum_{k=1}^{\infty} \frac{\xi_k g_k}{\|g_k\|} \Big|\Big| \\ &\leq \sum_{k=1}^{\infty} |\xi_k| < \infty.\end{aligned}$$

Therefore, Ψ is a bounded linear operator such that $\mathrm{Ker}\Psi = \{0\}$ (where $\mathrm{Ker}\Psi$ denotes the kernel of Ψ). This follows from the fact that $\{g_k\}$ is ω-linearly independent. To show that ψ is onto, let $g \in F$ be any arbitrary element. Since

T is onto, there is an $f \in E$ such that $Tf = g$. Choose $\alpha_k = f_k^*(f)\|g_k\|$, for all $k \in \mathbb{N}$. Since $(\{f_k^*\}, \{g_k\})$ is absolute, $\{\alpha_k\} \in \ell^1(\mathbb{N})$. Also, we have

$$\begin{aligned}\Psi(\{\alpha_k\}) &= \sum_{k=1}^{\infty} \frac{\alpha_k g_k}{\|g_k\|} \\ &= \sum_{k=1}^{\infty} \frac{f_k^*(f)\|g_k\| g_k}{\|g_k\|} \\ &= g.\end{aligned}$$

Thus Ψ is onto. Therefore, using Open Mapping Theorem, we conclude that ψ is a topological isomorphism of $\ell^1(\mathbb{N})$ onto F. □

If T_1 and T_2 are bounded linear operators, then it is easy to verify that their product $T_1 \times T_2$ is also a bounded linear operator. The following result shows that if T_1 and T_2 are bounded linear operators having an absolute O-frame, then their product $T_1 \times T_2$ with a suitable norm also has an absolute O-frame.

Theorem 4.7. *Let E_1, E_2, F_1 and F_2 be Banach spaces. Let $(\{f_k^*\}, \{g_k\}) \subset E_1^* \times F_1$ and $(\{p_k^*\}, \{q_k\}) \subset E_2^* \times F_2$ be absolute O-frames for operators $T_1 \in B(E_1, F_1)$ and $T_2 \in B(E_2, F_2)$, respectively. Then, $T_1 \times T_2$ also has an absolute O-frame.*

Proof. Let $h = (f, g) \in E_1 \times E_2$, where $f \in E_1$ and $g \in E_2$. Define $\{h_k\} \subset F_1 \times F_2$ and $\{h_k^*\} \subset (E_1 \times E_2)^*$ by

$$\begin{cases} h_{2k} = (g_k, 0) \\ h_{2k-1} = (0, q_k) \end{cases}$$

and

$$\begin{cases} h_{2k}^*(f, g) = f_k^*(f) \\ h_{2k-1}^*(f, g) = p_k^*(g). \end{cases}$$

Also, define $T_1 \times T_2 : E_1 \times E_2 \to F_1 \times F_2$ by

$$(T_1 \times T_2)(f, g) = (T_1 f, T_2 g).$$

Then, for each $h \in E_1 \times E_2$ we have

$$\begin{aligned}\sum_{k=1}^{\infty} h_k^*(f, g) h_k &= \sum_{k=1}^{\infty} h_{2k}^*(f, g) h_{2k} + \sum_{k=1}^{\infty} h_{2k-1}^*(f, g) h_{2k-1} \\ &= \left(\sum_{k=1}^{\infty} f_k^*(f) g_k, \sum_{k=1}^{\infty} p_k^*(g) q_k \right) \\ &= (T_1 f, T_2 g) \\ &= (T_1 \times T_2)(h).\end{aligned}$$

Thus $(\{h_k^*\}, \{h_k\})$ is an O-frame for $T_1 \times T_2$. Since $(\{f_k^*\}, \{g_k\}) \subset E_1^* \times F_1$ and $(\{p_k^*\}, \{q_k\}) \subset E_2^* \times F_2$ are absolute O-frames for operators $T_1 \in B(E_1, F_1)$ and $T_2 \in B(E_2, F_2)$, respectively, the series $\sum_{k=1}^{\infty} \|f_k^*(f) g_k\|$ converges for each $f \in E_1$ and the series $\sum_{k=1}^{\infty} \|p_k^*(f) q_k\|$ converges for each $f \in E_2$. Thus, by the definition of the system $(\{h_k^*\}, \{h_k\})$, the series $\sum_{k=1}^{\infty} \|h_k^*(f) h_k\|$ converges for all $h \in E_1 \times E_2$. Hence $(\{h_k^*\}, \{h_k\})$ is an absolute O-frame for $T_1 \times T_2$. □

Finally, as an application, we give the following result.

Corollary 4.8. *If T_1 and T_2 are bounded linear operators having BAP, then the product $T_1 \times T_2$ with a suitable norm on the underlying space also has BAP.*

Proof. If T_1 and T_2 have BAP, then by Theorem 2.6, T_1 and T_2 both have an O-frame. Therefore, by Theorem 4.7, $T_1 \times T_2$ has an O-frame. Hence by Theorem 2.6 again, $T_1 \times T_2$ has bounded approximation property. □

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