

ON SYMMETRY OF BIRKHOFF-JAMES ORTHOGONALITY OF LINEAR OPERATORS

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ABSTRACT. A bounded linear operator T on a normed linear space \mathbb{X} is said to be right symmetric (left symmetric) if $A \perp_B T \Rightarrow T \perp_B A$ ($T \perp_B A \Rightarrow A \perp_B T$) for all $A \in B(\mathbb{X})$, the space of all bounded linear operators on \mathbb{X} . Turnšek [Linear Algebra Appl., 407 (2005), 189-195] proved that if \mathbb{X} is a Hilbert space then T is right symmetric if and only if T is a scalar multiple of an isometry or coisometry. This result fails in general if the Hilbert space is replaced by a Banach space. The characterization of right and left symmetric operators on a Banach space is still open. In this paper we study the orthogonality in the sense of Birkhoff-James of bounded linear operators on $(\mathbb{R}^n, \|\cdot\|_\infty)$ and characterize the right symmetric and left symmetric operators on $(\mathbb{R}^n, \|\cdot\|_\infty)$.

1. INTRODUCTION

Let $(\mathbb{X}, \|\cdot\|)$ be a real normed linear space and $B(\mathbb{X})$ be the space of all bounded linear operators on \mathbb{X} . For any two elements x, y in \mathbb{X} , x is said to be orthogonal to y in the sense of Birkhoff-James [1, 2, 3], written as $x \perp_B y$, if and only if $\|x\| \leq \|x + \lambda y\|$ for all $\lambda \in \mathbb{R}$. In [2, 3] James studied many important properties related to the notion of orthogonality in the sense of Birkhoff-James. Orthogonality is related to many important geometric properties of normed linear spaces, including strict convexity, uniform convexity and smoothness of the space. For any two elements x, y in \mathbb{X} , x is said to be strongly orthogonal to y in the sense of Birkhoff-James [5], written as $x \perp_{SB} y$, if and only if $\|x\| < \|x + \lambda y\|$ for

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all $0 \neq \lambda \in \mathbb{R}$. In [5] Paul et al. characterized exposed point of the unit ball in terms of strong orthogonality. Following the notion introduced by Sain [6], left symmetric and right symmetric points in a normed space are defined as follows:

Left symmetric point: An element $x \in \mathbb{X}$ is called left symmetric if $x \perp_B y \Rightarrow y \perp_B x$ for all $y \in \mathbb{X}$.

Right symmetric point: An element $x \in \mathbb{X}$ is called right symmetric if $y \perp_B x \Rightarrow x \perp_B y$ for all $y \in \mathbb{X}$.

An element $x \in \mathbb{X}$ is said to be symmetric if it is both left and right symmetric, i.e., $x \perp_B y \Leftrightarrow y \perp_B x$ for all $y \in \mathbb{X}$. James [2] proved that Birkhoff-James orthogonality is symmetric in a normed linear space \mathbb{X} of three or more dimensions if and only if a compatible inner product can be defined on \mathbb{X} . For any two elements $T, A \in B(\mathbb{X})$, T is said to be orthogonal to A , in the sense of Birkhoff-James, written as $T \perp_B A$, if and only if

$$\|T\| \leq \|T + \lambda A\|, \text{ for all } \lambda \in \mathbb{R}.$$

Since $B(\mathbb{X})$ is not an inner product space so it is interesting to study the symmetry of orthogonality of operators in $B(\mathbb{X})$.

In [4] we proved that if T is a compact operator on a real Hilbert space H then T is left symmetric if and only if T is the zero operator, we also proved that if T is compact then T is right symmetric if and only if T is a scalar multiple of an isometry or a coisometry when H is finite dimensional and T is the zero operator when H is infinite dimensional. A more general characterisation of right symmetric operators was proved by Turnšek [8] in this connection, he proved that for a bounded linear operator T on a complex Hilbert space H , T is right symmetric if and only if T is a scalar multiple of an isometry or a coisometry. These results fail, in general, if the Hilbert space is replaced by a Banach space. The characterization of right and left symmetric operators on a Banach space, both in finite and infinite dimensional case, in general, is still open.

In this paper we study the orthogonality of operators on $(\mathbb{R}^n, \|\cdot\|_\infty)$ in the sense of Birkhoff-James. We find a necessary and sufficient condition for an operator T to be right symmetric. Furthermore, we find a necessary and sufficient condition for an operator T to be left symmetric. We prove that $T = (t_{ij})$ is right symmetric if and only if for each $i \in \{1, 2, \dots, n\}$, exactly one term of $t_{i1}, t_{i2}, \dots, t_{in}$ is nonzero and of the same magnitude. We prove that T is left symmetric if and only if T is the zero operator when the dimension is more than 2. We also prove that if T is a linear operator on $(\mathbb{R}^2, \|\cdot\|_\infty)$, then T is left symmetric if and only if T attains norm at only one extreme point, say e , Te is a left symmetric point and image of the other extreme point is zero.

From now onwards, by \mathbb{R}^n we will mean the normed linear space \mathbb{R}^n equipped with the ℓ_∞ norm, which will be denoted by $\|\cdot\|$. The following standard notation

will be used, for $x \in \mathbb{R}$,

$$\begin{aligned} \operatorname{sgn}(x) &= 1, \quad x > 0 \\ \operatorname{sgn}(x) &= -1, \quad x < 0 \\ \operatorname{sgn}(x) &= 0, \quad x = 0 \end{aligned}$$

2. MAIN RESULTS

We begin this section with a theorem that characterizes nonzero right symmetric linear operators on \mathbb{R}^n .

Theorem 2.1. *Suppose $T = (t_{ij})$ is a nonzero linear operator on \mathbb{R}^n . For any linear operator A on \mathbb{R}^n , $A \perp_B T \Rightarrow T \perp_B A$ if and only if for each $i \in \{1, 2, \dots, n\}$, exactly one term of $t_{i1}, t_{i2}, \dots, t_{in}$ is nonzero and of the same magnitude.*

Proof. Without any loss of generality we may assume that $\|T\| = 1$. We first prove the sufficient part. Assume that for each $i \in \{1, 2, \dots, n\}$, there exists $k_i \in \{1, 2, \dots, n\}$ such that $t_{ik_i} \neq 0$ and $t_{ij} = 0$ for all $j \neq k_i$ and $|t_{1k_1}| = |t_{2k_2}| = \dots = |t_{nk_n}|$.

Let $A = (a_{mn})$ be a linear operator on \mathbb{R}^n such that $A \perp_B T$. We show that $T \perp_B A$. For this we first claim that there exists $i, j \in \{1, 2, \dots, n\}$ such that $\operatorname{sgn}(a_{ik_i}) = \operatorname{sgn}(t_{ik_i})$ and $\operatorname{sgn}(a_{jk_j}) = -\operatorname{sgn}(t_{jk_j})$. If possible, suppose that $\operatorname{sgn}(a_{ik_i}) = \operatorname{sgn}(t_{ik_i})$ for all $i \in \{1, 2, \dots, n\}$. Choose $0 < \lambda < \max\{\frac{2|a_{ik_i}|}{|t_{ik_i}|} : i \in \{1, 2, \dots, n\}\}$. Since for each $i \in \{1, 2, \dots, n\}$, $|a_{i1} - \lambda t_{i1}| + \dots + |a_{in} - \lambda t_{in}| = |a_{i1}| + |a_{i2}| + \dots + |a_{ik_i} - \lambda t_{ik_i}| + \dots + |a_{in}| < \|A\|$ it is easy to see that $\|A - \lambda T\| < \|A\|$ i.e., $A \not\perp_B T$. Similarly, if $\operatorname{sgn}(a_{ik_i}) = -\operatorname{sgn}(t_{ik_i})$ for all $i \in \{1, 2, \dots, n\}$, one can check that $A \not\perp_B T$.

So, there exist $i, j \in \{1, 2, \dots, n\}$ such that $\operatorname{sgn}(a_{ik_i}) = \operatorname{sgn}(t_{ik_i})$ and $\operatorname{sgn}(a_{jk_j}) = -\operatorname{sgn}(t_{jk_j})$.

We next show that $T \perp_B A$. Let $\lambda > 0$ be fixed. Then

$$\|T + \lambda A\| \geq |t_{i1} + \lambda a_{i1}| + |t_{i2} + \lambda a_{i2}| + \dots + |t_{in} + \lambda a_{in}| \geq |t_{ik_i} + \lambda a_{ik_i}| > |t_{ik_i}| = \|T\|$$

Also

$$\|T - \lambda A\| \geq |t_{j1} - \lambda a_{j1}| + |t_{j2} - \lambda a_{j2}| + \dots + |t_{jn} - \lambda a_{jn}| \geq |t_{jk_j} - \lambda a_{jk_j}| > |t_{jk_j}| = \|T\|$$

This proves that $T \perp_B A$. This completes the proof of the sufficient part.

Conversely, let T be a linear operator on \mathbb{R}^n such that $A \perp_B T \Rightarrow T \perp_B A$ for any linear operator A on \mathbb{R}^n . We show that for each $i \in \{1, 2, \dots, n\}$, exactly one term of $t_{i1}, t_{i2}, \dots, t_{in}$ is nonzero and are of the same magnitude. We complete the proof in the following two steps.

Step 1: We prove that $|t_{i1}| + |t_{i2}| + |t_{i3}| + \dots + |t_{in}| = 1$ for each $i \in \{1, 2, \dots, n\}$.

Case 1. If possible, suppose $|t_{11}| + |t_{12}| + |t_{13}| + \dots + |t_{1n}| = 0$. Then $|t_{1j}| = 0$ for all $j \in \{1, 2, \dots, n\}$. Take $t = \min\{|t_{ij}| : t_{ij} \neq 0\}$. Now there exists a natural number p such that $\frac{1}{n^p} < t$.

Take

$$A = \begin{bmatrix} -n & -n & \dots & -n \\ -t\text{sgn}(t_{21}) & -t\text{sgn}(t_{22}) & \dots & -t\text{sgn}(t_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ -t\text{sgn}(t_{n1}) & -t\text{sgn}(t_{n2}) & \dots & -t\text{sgn}(t_{nn}) \end{bmatrix}$$

It is easy to see that $\|A\| = n^2$. For any scalar λ , we have $\|A + \lambda T\| \geq n + n + \dots + n = n^2 = \|A\|$ which shows that $A \perp_B T$. Take $\lambda_0 = \frac{1}{n^{p+2}}$. For any $i \neq 1$, $|t_{i1} + \lambda_0 a_{i1}| + \dots + |t_{in} + \lambda_0 a_{in}| = |t_{i1} - \frac{1}{n^{p+2}} t\text{sgn}(t_{i1})| + \dots + |t_{in} - \frac{1}{n^{p+2}} t\text{sgn}(t_{in})| < |t_{i1}| + \dots + |t_{in}| \leq 1 = \|T\|$ and so $|t_{11} + \lambda_0 a_{11}| + \dots + |t_{1n} + \lambda_0 a_{1n}| = \frac{n}{n^{p+1}} = \frac{1}{n^p} < t \leq \|T\|$. Then $\|T - \lambda_0 A\| < \|T\|$ i.e., $T \not\perp_B A$. Thus we get $|t_{i1}| + |t_{i2}| + |t_{i3}| + \dots + |t_{in}| \neq 0$ for each $i \in \{1, 2, \dots, n\}$.

Case 2. If possible suppose that $0 < |t_{11}| + |t_{12}| + |t_{13}| + \dots + |t_{1n}| < 1$. Then there exists at least one j such that $t_{1j} \neq 0$. Without any loss of generality we assume that $t_{11} \neq 0$. Let

$$A = \begin{bmatrix} -\text{sgn}(t_{11}) & 0 & \dots & 0 \\ t_{21} & t_{22} & \dots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n1} & t_{n2} & \dots & t_{nn} \end{bmatrix}$$

Clearly, $\|A\| = 1$. Let $\lambda > 0$, then $\|A + \lambda T\| \geq |a_{n1} + \lambda t_{n1}| + |a_{n2} + \lambda t_{n2}| + \dots + |a_{nn} + \lambda t_{nn}| = |t_{n1} + \lambda t_{n1}| + |t_{n2} + \lambda t_{n2}| + \dots + |t_{nn} + \lambda t_{nn}| = (|t_{n1}| + |t_{n2}| + \dots + |t_{nn}|)|1 + \lambda| = |1 + \lambda| > 1$. For $\lambda < 0$, $\|A + \lambda T\| \geq |a_{11} + \lambda t_{11}| + |a_{12} + \lambda t_{12}| + \dots + |a_{1n} + \lambda t_{1n}| = |-\text{sgn}(t_{11}) + \lambda t_{11}| + |\lambda t_{n2}| + |\lambda t_{n3}| + \dots + |\lambda t_{nn}| \geq |-\text{sgn}(t_{11}) + \lambda t_{11}| \geq 1$. So $A \perp_B T$.

Now take $0 < \lambda_0 < 1 - (|t_{11}| + |t_{12}| + \dots + |t_{1n}|)$. Then we get

$$(T - \lambda_0 A) = \begin{bmatrix} t_{11} - \lambda_0 a_{11} & \dots & t_{1n} - \lambda_0 a_{1n} \\ t_{21} - \lambda_0 a_{21} & \dots & t_{2n} - \lambda_0 a_{2n} \\ \vdots & \ddots & \vdots \\ t_{n1} - \lambda_0 a_{n1} & \dots & t_{nn} - \lambda_0 a_{nn} \end{bmatrix}$$

For any $i \neq 1$, $|t_{i1} - \lambda_0 a_{i1}| + \dots + |t_{in} - \lambda_0 a_{in}| = |t_{i1} - \lambda_0 t_{i1}| + \dots + |t_{in} - \lambda_0 t_{in}| = (|t_{i1}| + \dots + |t_{in}|)|1 - \lambda_0| < |t_{i1}| + \dots + |t_{in}| \leq 1 = \|T\|$ and $|t_{11} - \lambda_0 a_{11}| + \dots + |t_{1n} - \lambda_0 a_{1n}| = |t_{11} + \lambda_0 \text{sgn}(t_{11})| + |t_{12}| + \dots + |t_{1n}| = \||t_{11}| + \lambda_0| + |t_{12}| + \dots + |t_{1n}| \leq |t_{11}| + \dots + |t_{1n}| + |\lambda_0| < 1 = \|T\|$. So $\|T - \lambda_0 A\| < \|T\|$ i.e., $T \not\perp_B A$. This contradiction leads to $|t_{11}| + |t_{12}| + |t_{13}| + \dots + |t_{1n}| = 1$. This completes the proof of Step 1 i.e., $|t_{i1}| + |t_{i2}| + |t_{i3}| + \dots + |t_{in}| = 1$ for each $i \in \{1, 2, \dots, n\}$.

Step 2. We prove that for each $i \in \{1, 2, \dots, n\}$ exactly one of $t_{i1}, t_{i2}, \dots, t_{in}$ is nonzero. Fix $i \in \{1, 2, \dots, n\}$. Since T is nonzero, using Step 1 it is easy to see that at least one of $t_{i1}, t_{i2}, \dots, t_{in}$ is nonzero. If possible suppose that, there exists $k, l \in \{1, 2, \dots, n\}$ ($k < l$) such that $t_{ik}, t_{il} \neq 0$.

Case 1: $t_{ik}t_{il} > 0$. Without any loss of generality we may assume that $t_{ik}, t_{il} > 0$ and $t_{ik} \geq t_{il}$. Let $c = \frac{1}{(\frac{|t_{il}|}{2} + |t_{ik}|)}$. Take

$$A = \begin{bmatrix} \frac{t_{11}}{c} & \frac{t_{12}}{c} & \dots & \dots & \frac{t_{1k}}{c} & \dots & \dots & \dots & \frac{t_{1l}}{c} & \dots & \dots & \frac{t_{1n}}{c} \\ \frac{t_{21}}{c} & \frac{t_{22}}{c} & \dots & \dots & \frac{t_{2k}}{c} & \dots & \dots & \dots & \frac{t_{2l}}{c} & \dots & \dots & \frac{t_{2n}}{c} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 0 & -\frac{t_{il}}{2} & 0 & \dots & 0 & t_{ik} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{t_{n1}}{c} & \frac{t_{n2}}{c} & \dots & \dots & \frac{t_{nk}}{c} & \dots & \dots & \dots & \frac{t_{nl}}{c} & \dots & \dots & \frac{t_{nn}}{c} \end{bmatrix}$$

Clearly, $\|A\| = \frac{1}{c} = \frac{t_{il}}{2} + t_{ik}$. Let $\lambda > 0$ be fixed. Then $\|A + \lambda T\| \geq |\frac{t_{11}}{c} + \lambda t_{11}| + |\frac{t_{12}}{c} + \lambda t_{12}| + \dots + |\frac{t_{1n}}{c} + \lambda t_{1n}| > \frac{|t_{11}|}{c} + \frac{|t_{12}|}{c} + \dots + \frac{|t_{1n}|}{c} = \frac{1}{c} = \|A\|$ and $\|A - \lambda T\| \geq |-\frac{t_{il}}{2} - \lambda t_{ik}| + |t_{ik} - \lambda t_{il}| \geq |(\frac{t_{il}}{2} + t_{ik}) + \lambda(t_{ik} - t_{il})| \geq |\frac{t_{il}}{2} + t_{ik}| = \|A\|$. So $A \perp_B T$.

Now take $\lambda_0 = \frac{t_{il}}{t_{ik}}$. We have

$$(T - \lambda_0 A) = \begin{bmatrix} t_{11} - \lambda_0 a_{11} & \dots & \dots & t_{1n} - \lambda_0 a_{1n} \\ t_{21} - \lambda_0 a_{21} & \dots & \dots & t_{2n} - \lambda_0 a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ t_{n1} - \lambda_0 a_{n1} & \dots & \dots & t_{nn} - \lambda_0 a_{nn} \end{bmatrix}$$

For any $j \neq i$, $|t_{j1} - \lambda_0 a_{j1}| + \dots + |t_{jn} - \lambda_0 a_{jn}| = |t_{j1} - \frac{t_{il}}{t_{ik}} \frac{t_{j1}}{c}| + \dots + |t_{jn} - \frac{t_{il}}{t_{ik}} \frac{t_{jn}}{c}| < |t_{j1}| + \dots + |t_{jn}| = \|T\|$. Also, $|t_{i1} - \lambda_0 a_{i1}| + \dots + |t_{in} - \lambda_0 a_{in}| = |t_{i1} - \frac{t_{il}}{t_{ik}} \cdot 0| + \dots + |t_{ik} + \frac{t_{il}}{t_{ik}} \frac{t_{il}}{2}| + \dots + |t_{il} - \frac{t_{il}}{t_{ik}} t_{ik}| + \dots + |t_{in} - \frac{t_{il}}{t_{ik}} \cdot 0| = |t_{i1}| + |t_{i2}| + \dots + |t_{ik} + \frac{t_{il}^2}{2t_{ik}}| + \dots + |t_{in}| \leq |t_{i1}| + \dots + |t_{ik}| + |\frac{t_{il}^2}{2t_{ik}}| + \dots + |t_{in}| < |t_{i1}| + \dots + |t_{ik}| + |t_{il}| + \dots + |t_{in}| = \|T\|$. So $\|T - \lambda_0 A\| < \|T\|$ i.e., $T \not\perp_B A$. This is a contradiction.

Case 2: $t_{ik}t_{il} < 0$. Assume that $t_{ik} < 0, t_{il} > 0$ and $|t_{ik}| \geq |t_{il}|$. Let $c = \frac{1}{(\frac{|t_{il}|}{2} + |t_{ik}|)}$.

$$A = \begin{bmatrix} -\frac{t_{11}}{c} & -\frac{t_{12}}{c} & \dots & \dots & -\frac{t_{1k}}{c} & \dots & \dots & \dots & -\frac{t_{1l}}{c} & \dots & \dots & -\frac{t_{1n}}{c} \\ -\frac{t_{21}}{c} & -\frac{t_{22}}{c} & \dots & \dots & -\frac{t_{2k}}{c} & \dots & \dots & \dots & -\frac{t_{2l}}{c} & \dots & \dots & -\frac{t_{2n}}{c} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 0 & -\frac{t_{il}}{2} & 0 & \dots & 0 & t_{ik} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -\frac{t_{n1}}{c} & -\frac{t_{n2}}{c} & \dots & \dots & -\frac{t_{nk}}{c} & \dots & \dots & \dots & -\frac{t_{nl}}{c} & \dots & \dots & -\frac{t_{nn}}{c} \end{bmatrix}$$

As before, we can show that $A \perp_B T$ but, $T \not\perp_B A$.

We next assume that $t_{ik} > 0, t_{il} < 0$ and $|t_{ik}| \geq |t_{il}|$. In this case take $U = -T$. As before we can show that there exists a linear operator A such that $A \perp_B U$ but $U \not\perp_B A$. By the homogeneity of Birkhoff-James orthogonality it follows that $A \perp_B T$ but $T \not\perp_B A$. Therefore, for each $i \in \{1, 2, \dots, n\}$ exactly one term of $t_{i1}, t_{i2}, \dots, t_{in}$ is nonzero. This completes the proof of our Step 2. The proof of the necessary part now follows from Step 1 and Step 2. \square

Remark 2.2. The right symmetric linear operators on \mathbb{R}^n attains norm at all extreme points and images of the extreme points are also extreme points.

We next characterize the left symmetric linear operators on \mathbb{R}^2 . Note that the unit ball of \mathbb{R}^2 has only two pair of extreme points which are denoted as $\pm e_1, \pm e_2$.

Theorem 2.3. *Suppose T is a linear operator on \mathbb{R}^2 . Then for any linear operator A on \mathbb{R}^2 , $T \perp_B A \Rightarrow A \perp_B T$ if and only if T attains norm at only one extreme point, say e_1 , Te_1 is a left symmetric point and image of the other extreme point is zero.*

Proof. Let the four extreme points of the unit ball of \mathbb{R}^2 be $\pm e_1, \pm e_2$. Suppose T attains norm at e_1 and $Te_2 = 0$. Let A be a linear operator such that $T \perp_B A$. Then by Theorem 2.1 of Sain and Paul [7] $Te_1 \perp_B Ae_1$. As Te_1 is a left symmetric point, it follows that $Ae_1 \perp_B Te_1$. Also $Ae_2 \perp_B Te_2 = 0$. Clearly, A attains norm at either e_1 or e_2 and $Ae_j \perp_B Te_j$ for $j = 1, 2$. So we get $A \perp_B T$.

Conversely, let $T \perp_B A \Rightarrow A \perp_B T$ for all linear operator A on \mathbb{R}^2 . Clearly, T attains norm at an extreme point, say e_1 . We claim that $Te_2 = 0$. Suppose $Te_2 \neq 0$. Define a linear operator A on \mathbb{R}^2 as $Ae_1 = 0, Ae_2 = Te_2$. It is easy to verify that A attains norm only at $\pm e_2$. Also $T \perp_B A$, as $Te_1 \perp_B Ae_1$ and $\|Te_1\| = \|T\|$. But $A \not\perp_B T$ as $Ae_2 \not\perp_B Te_2$. So $Te_2 = 0$.

Our next claim is that Te_1 is a left symmetric point. Suppose Te_1 is not a left symmetric point, i.e., there exists w such that $Te_1 \perp_B w$ but $w \not\perp_B Te_1$. Define a linear operator A on \mathbb{R}^2 as $Ae_1 = w, Ae_2 = 0$. It is easy to verify that A attains norm only at $\pm e_1$. Also $T \perp_B A$, as $Te_1 \perp_B Ae_1$ and $\|Te_1\| = \|T\|$. But $A \not\perp_B T$ as $Ae_1 \not\perp_B Te_1$. Thus we get $T \perp_B A$ but $A \not\perp_B T$, a contradiction to our hypothesis. This completes the proof of the theorem. \square

Example 2.4. Note that, $(1, 0), (0, 1)$ are nonzero left symmetric points of \mathbb{R}^2 . So, there are nonzero left symmetric linear operators on \mathbb{R}^2 . One such linear operator can be given in the following way:

$$\begin{aligned} T(1, 1) &= (1, 0) \\ T(1, -1) &= (0, 0) \end{aligned}$$

It is easy to verify that T attains norm only at $\pm(1, 1)$, image of which is a nonzero left symmetric point of \mathbb{R}^2 and image of the other extreme point is zero.

The next theorem characterizes the left symmetric linear operators on $\mathbb{R}^n, n \geq 3$.

Theorem 2.5. *Suppose T is a linear operators on $\mathbb{R}^n, n \geq 3$. Then T is left symmetric if and only if T is the zero operator.*

Proof. One part of the proof is obvious. For the other part, suppose that T is a nonzero linear operator on \mathbb{R}^n such that for any linear operator A on \mathbb{R}^n , $T \perp_B A \Rightarrow A \perp_B T$. Now T attains norm at an extreme point, say e_1 .

We claim that $Te = 0$ for all extreme point $e \neq \pm e_1$. If possible, suppose that, there exists an extreme point $e_2 \neq \pm e_1$ such that $Te_2 \neq 0$. As $e_2 \perp_{SB}$

e_1 , there exists a hyperplane H such that $e_1 \in H$ and $e_2 \perp_{SB} H - \{0\}$. Let $\{e_1, e_3, e_4, \dots, e_n\}$ be a basis of H so that $\{e_2, e_1, e_3, e_4, \dots, e_n\}$ is a basis of \mathbb{R}^n . Define a linear operator A on \mathbb{R}^n such that

$$\begin{aligned} Ae_2 &= Te_2 \\ Ae_i &= 0, \quad i \neq 2 \end{aligned}$$

It is easy to verify that A attains norm only at $\pm e_2$. Also $T \perp_B A$, as $Te_1 \perp_B Ae_1$ and $\|Te_1\| = \|T\|$. But $A \not\perp_B T$ as $Ae_2 \not\perp_B Te_2$. So $Te = 0$ for all extreme point $e \neq \pm e_1$. Let S denote the set of extreme points e different from $\pm e_1$. Then S contains a basis B and $Te = 0$ for all $e \in B$, which forces T to be the zero operator on the whole space. This completes the proof. \square

Remark 2.6. The question that still remains to be answered is the characterization of right and left symmetric operators on $\ell_p(1 < p < \infty)$ spaces and more generally on a normed linear space.

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