Adv. Oper. Theory 2 (2017), no. 4, 428–434
http://doi.org/10.22034/aot.1703-1137
ISSN: 2538-225X (electronic)
http://aot-math.org

ON SYMMETRY OF BIRKHOFF-JAMES ORTHOGONALITY OF LINEAR OPERATORS

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Communicated by M. Omladič

Abstract. A bounded linear operator T on a normed linear space $\mathbb{X}$ is said to be right symmetric (left symmetric) if $A \perp_B T \Rightarrow T \perp_B A$ ($T \perp_B A \Rightarrow A \perp_B T$) for all $A \in B(\mathbb{X})$, the space of all bounded linear operators on $\mathbb{X}$. Turnšek [Linear Algebra Appl., 407 (2005), 189-195] proved that if $\mathbb{X}$ is a Hilbert space then T is right symmetric if and only if T is a scalar multiple of an isometry or coisometry. This result fails in general if the Hilbert space is replaced by a Banach space. The characterization of right and left symmetric operators on a Banach space is still open. In this paper we study the orthogonality in the sense of Birkhoff-James of bounded linear operators on $(\mathbb{R}^n, \|.\|_\infty)$ and characterize the right symmetric and left symmetric operators on $(\mathbb{R}^n, \|.\|_\infty)$.

1. Introduction

Let $(\mathbb{X}, \|.\|)$ be a real normed linear space and $B(\mathbb{X})$ be the space of all bounded linear operators on $\mathbb{X}$. For any two elements x, y in $\mathbb{X}$, x is said to be orthogonal to y in the sense of Birkhoff-James [1, 2, 3], written as $x \perp_B y$, if and only if $\|x\| \leq \|x + \lambda y\|$ for all $\lambda \in \mathbb{R}$. In [2, 3] James studied many important properties related to the notion of orthogonality in the sense of Birkhoff-James. Orthogonality is related to many important geometric properties of normed linear spaces, including strict convexity, uniform convexity and smoothness of the space. For any two elements x, y in $\mathbb{X}$, x is said to be strongly orthogonal to y in the sense of Birkhoff-James [5], written as $x \perp_{SB} y$, if and only if $\|x\| < \|x + \lambda y\|$ for

Date: Received: Mar. 15, 2017; Accepted: Jun 12, 2017.
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2010 *Mathematics Subject Classification.* Primary 46B20; Secondary 47A30.
Key words and phrases. Birkhoff-James Orthogonality; left syemmetric operator; right symmetric operator.

all $0 \neq \lambda \in \mathbb{R}$. In [5] Paul et al. characterized exposed point of the unit ball in terms of strong orthogonality. Following the notion introduced by Sain [6], left symmetric and right symmetric points in a normed space are defined as follows:
Left symmetric point: An element $x \in \mathbb{X}$ is called left symmetric if $x \perp_B y \Rightarrow y \perp_B x$ for all $y \in \mathbb{X}$.
Right symmetric point: An element $x \in \mathbb{X}$ is called right symmetric if $y \perp_B x \Rightarrow x \perp_B y$ for all $y \in \mathbb{X}$.
An element $x \in \mathbb{X}$ is said to be symmetric if it is both left and right symmetric, i.e., $x \perp_B y \Leftrightarrow y \perp_B x$ for all $y \in \mathbb{X}$. James [2] proved that Birkhoff-James orthogonality is symmetric in a normed linear space $\mathbb{X}$ of three or more dimensions if and only if a compatible inner product can be defined on $\mathbb{X}$. For any two elements $T, A \in B(\mathbb{X})$, T is said to be orthogonal to A, in the sense of Birkhoff-James, written as $T \perp_B A$, if and only if

$$\|T\| \leq \|T + \lambda A\|, \text{ for all } \lambda \in \mathbb{R}.$$

Since $B(\mathbb{X})$ is not an inner product space so it is interesting to study the symmetry of orthogonality of operators in $B(\mathbb{X})$.

In [4] we proved that if T is a compact operator on a real Hilbert space H then T is left symmetric if and only if T is the zero operator, we also proved that if T is compact then T is right symmetric if and only if T is a scalar multiple of an isometry or a coisometry when H is finite dimensional and T is the zero operator when H is infinite dimensional. A more general characterisation of right symmetric operators was proved by Turnšek [8] in this connection, he proved that for a bounded linear operator T on a complex Hilbert space H, T is right symmetric if and only if T is a scalar multiple of an isometry or a coisometry. These results fail, in general, if the Hilbert space is replaced by a Banach space. The characterization of right and left symmetric operators on a Banach space, both in finite and infinite dimensional case, in general, is still open.

In this paper we study the orthogonality of operators on $(\mathbb{R}^n, \|\cdot\|_\infty)$ in the sense of Birkhoff-James. We find a necessary and sufficient condition for an operator T to be right symmetric. Furthermore, we find a necessary and sufficient condition for an operator T to be left symmetric. We prove that $T = (t_{ij})$ is right symmetric if and only if for each $i \in \{1, 2, \ldots, n\}$, exactly one term of $t_{i1}, t_{i2}, \ldots, t_{in}$ is nonzero and of the same magnitude. We prove that T is left symmetric if and only if T is the zero operator when the dimension is more than 2. We also prove that if T is a linear operator on $(\mathbb{R}^2, \|\cdot\|_\infty)$, then T is left symmetric if and only if T attains norm at only one extreme point, say e, Te is a left symmetric point and image of the other extreme point is zero.

From now onwards, by $\mathbb{R}^n$ we will mean the normed linear space $\mathbb{R}^n$ equipped with the ℓ_∞ norm, which will be denoted by $\|.\|$. The following standard notation will be used, for $x \in \mathbb{R}$,

$$\begin{aligned} sgn(x) &= 1,\ x > 0 \\ sgn(x) &= -1,\ x < 0 \\ sgn(x) &= 0,\ x = 0 \end{aligned}$$

2. Main Results

We begin this section with a theorem that characterizes nonzero right symmetric linear operators on $\mathbb{R}^n$.

Theorem 2.1. *Suppose $T = (t_{ij})$ is a nonzero linear operator on $\mathbb{R}^n$. For any linear operator A on $\mathbb{R}^n$, $A \perp_B T \Rightarrow T \perp_B A$ if and only if for each $i \in \{1, 2, \ldots, n\}$, exactly one term of $t_{i1}, t_{i2}, \ldots, t_{in}$ is nonzero and of the same magnitude.*

Proof. Without any loss of generality we may assume that $\|T\| = 1$. We first prove the sufficient part. Assume that for each $i \in \{1, 2, \ldots, n\}$, there exists $k_i \in \{1, 2, \ldots, n\}$ such that $t_{ik_i} \neq 0$ and $t_{ij} = 0$ for all $j \neq k_i$ and $|t_{1k_1}| = |t_{2k_2}| = \ldots = |t_{nk_n}|$.
Let $A = (a_{mn})$ be a linear operator on $\mathbb{R}^n$ such that $A \perp_B T$. We show that $T \perp_B A$. For this we first claim that there exists $i, j \in \{1, 2, \ldots, n\}$ such that $sgn(a_{ik_i}) = sgn(t_{ik_i})$ and $sgn(a_{jk_j}) = -sgn(t_{jk_j})$. If possible, suppose that $sgn(a_{ik_i}) = sgn(t_{ik_i})$ for all $i \in \{1, 2, \ldots, n\}$. Choose $0 < \lambda < \max\{\frac{2|a_{ik_i}|}{|t_{ik_i}|}\ :\ i \in \{1, 2, \ldots, n\}\}$. Since for each $i \in \{1, 2, \ldots, n\}$, $|a_{i1} - \lambda t_{i1}| + \ldots + |a_{in} - \lambda t_{in}| = |a_{i1}| + |a_{i2}| + \ldots + |a_{ik_i} - \lambda t_{ik_i}| + \ldots + |a_{in}| < \|A\|$ it is easy to see that $\|A - \lambda T\| < \|A\|$ i.e., $A \not\perp_B T$.
Similarly, if $sgn(a_{ik_i}) = -sgn(t_{ik_i})$ for all $i \in \{1, 2, \ldots, n\}$, one can check that $A \not\perp_B T$.
So, there exist $i, j \in \{1, 2, \ldots, n\}$ such that $sgn(a_{ik_i}) = sgn(t_{ik_i})$ and $sgn(a_{jk_j}) = -sgn(t_{jk_j})$.
We next show that $T \perp_B A$. Let $\lambda > 0$ be fixed. Then

$$\|T+\lambda A\| \geq |t_{i1}+\lambda a_{i1}|+|t_{i2}+\lambda a_{i2}|+\ldots+|t_{in}+\lambda a_{in}| \geq |t_{ik_i}+\lambda a_{ik_i}| > |t_{ik_i}| = \|T\|$$

Also

$$\|T-\lambda A\| \geq |t_{j1}-\lambda a_{j1}|+|t_{j2}-\lambda a_{j2}|+\ldots+|t_{jn}-\lambda a_{jn}| \geq |t_{jk_j}-\lambda a_{jk_j}| > |t_{jk_j}| = \|T\|$$

This proves that $T \perp_B A$. This completes the proof of the sufficient part.

Conversely, let T be a linear operator on $\mathbb{R}^n$ such that $A \perp_B T \Rightarrow T \perp_B A$ for any linear operator A on $\mathbb{R}^n$. We show that for each $i \in \{1, 2, \ldots, n\}$, exactly one term of $t_{i1}, t_{i2}, \ldots, t_{in}$ is nonzero and are of the same magnitude. We complete the proof in the following two steps.

Step 1: We prove that $|t_{i1}|+|t_{i2}|+|t_{i3}|+\ldots+|t_{in}| = 1$ for each $i \in \{1, 2, \ldots, n\}$.

Case 1. If possible, suppose $|t_{11}| + |t_{12}| + |t_{13}| + \ldots + |t_{1n}| = 0$. Then $|t_{1j}| = 0$ for all $j \in \{1, 2, \ldots, n\}$. Take $t = \min\{|t_{ij}| : t_{ij} \neq 0\}$. Now there exists a natural number p such that $\frac{1}{n^p} < t$.

Take

$$A = \begin{bmatrix} -n & -n & . & . & -n \\ -tsgn(t_{21}) & -tsgn(t_{22}) & . & . & -tsgn(t_{2n}) \\ . & . & . & . & . \\ . & . & . & . & . \\ -tsgn(t_{n1}) & -tsgn(t_{n2}) & . & . & -tsgn(t_{nn}) \end{bmatrix}$$

It is easy to see that $\|A\| = n^2$. For any scalar λ, we have $\|A + \lambda T\| \geq n + n + \ldots + n = n^2 = \|A\|$ which shows that $A \perp_B T$. Take $\lambda_0 = \frac{1}{n^{p+2}}$. For any $i \neq 1$, $|t_{i1} + \lambda_0 a_{i1}| + \ldots + |t_{in} + \lambda_0 a_{in}| = |t_{i1} - \frac{1}{n^{p+2}} tsgn(t_{i1})| + \ldots + |t_{in} - \frac{1}{n^{p+2}} tsgn(t_{in})| < |t_{i1}| + \ldots + |t_{in}| \leq 1 = \|T\|$ and so $|t_{11} + \lambda_0 a_{11}| + \ldots + |t_{1n} + \lambda_0 a_{1n}| = \frac{n}{n^{p+1}} = \frac{1}{n^p} < t \leq \|T\|$. Then $\|T - \lambda_0 A\| < \|T\|$ i.e., $T \not\perp_B A$. Thus we get $|t_{i1}| + |t_{i2}| + |t_{i3}| + \ldots + |t_{in}| \neq 0$ for each $i \in \{1, 2, \ldots, n\}$.

Case 2. If possible suppose that $0 < |t_{11}| + |t_{12}| + |t_{13}| + \ldots + |t_{1n}| < 1$. Then there exists at least one j such that $t_{1j} \neq 0$. Without any loss of generality we assume that $t_{11} \neq 0$. Let

$$A = \begin{bmatrix} -sgn(t_{11}) & 0 & . & . & 0 \\ t_{21} & t_{22} & . & . & t_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ t_{n1} & t_{n2} & . & . & t_{nn} \end{bmatrix}$$

Clearly, $\|A\| = 1$. Let $\lambda > 0$, then $\|A + \lambda T\| \geq |a_{n1} + \lambda t_{n1}| + |a_{n2} + \lambda t_{n2}| + \ldots + |a_{nn} + \lambda t_{nn}| = |t_{n1} + \lambda t_{n1}| + |t_{n2} + \lambda t_{n2}| + \ldots + |t_{nn} + \lambda t_{nn}| = (|t_{n1}| + |t_{n2}| + \ldots + |t_{nn}|)|1 + \lambda| = |1 + \lambda| > 1$. For $\lambda < 0$, $\|A + \lambda T\| \geq |a_{11} + \lambda t_{11}| + |a_{12} + \lambda t_{12}| + \ldots + |a_{1n} + \lambda t_{1n}| = |-sgn(t_{11}) + \lambda t_{11}| + |\lambda t_{n2}| + |\lambda t_{n3}| + \ldots + |\lambda t_{nn}| \geq |-sgn(t_{11}) + \lambda t_{11}| \geq 1$. So $A \perp_B T$.

Now take $0 < \lambda_0 < 1 - (|t_{11}| + |t_{12}| + \ldots + |t_{1n}|)$. Then we get

$$(T - \lambda_0 A) = \begin{bmatrix} t_{11} - \lambda_0 a_{11} & . & . & t_{1n} - \lambda_0 a_{1n} \\ t_{21} - \lambda_0 a_{21} & . & . & t_{2n} - \lambda_0 a_{2n} \\ . & . & . & . \\ . & . & . & . \\ t_{n1} - \lambda_0 a_{n1} & . & . & t_{nn} - \lambda_0 a_{nn} \end{bmatrix}$$

For any $i \neq 1$, $|t_{i1} - \lambda_0 a_{i1}| + \ldots + |t_{in} - \lambda_0 a_{in}| = |t_{i1} - \lambda_0 t_{i1}| + \ldots + |t_{in} - \lambda_0 t_{in}| = (|t_{i1}| + \ldots + |t_{in}|)|1 - \lambda_0| < |t_{i1}| + \ldots + |t_{in}| \leq 1 = \|T\|$ and $|t_{11} - \lambda_0 a_{11}| + \ldots + |t_{1n} - \lambda_0 a_{1n}| = |t_{11} + \lambda_0 sgn(t_{11})| + |t_{12}| + \ldots + |t_{1n}| = ||t_{11}| + \lambda_0| + |t_{12}| + \ldots + |t_{1n}| \leq |t_{11}| + \ldots + |t_{1n}| + |\lambda_0| < 1 = \|T\|$. So $\|T - \lambda_0 A\| < \|T\|$ i.e., $T \not\perp_B A$. This contradiction leads to $|t_{11}| + |t_{12}| + |t_{13}| + \ldots + |t_{1n}| = 1$. This completes the proof of Step 1 i.e., $|t_{i1}| + |t_{i2}| + |t_{i3}| + \ldots + |t_{in}| = 1$ for each $i \in \{1, 2, \ldots, n\}$.

Step 2. We prove that for each $i \in \{1, 2, \ldots, n\}$ exactly one of $t_{i1}, t_{i2}, \ldots, t_{in}$ is nonzero. Fix $i \in \{1, 2, \ldots, n\}$. Since T is nonzero, using Step 1 it is easy to see that at least one of $t_{i1}, t_{i2}, \ldots, t_{in}$ is nonzero. If possible suppose that, there exists $k, l \in \{1, 2, \ldots, n\}$ $(k < l)$ such that $t_{ik}, t_{il} \neq 0$.

Case 1: $t_{ik}t_{il} > 0$. Without any loss of generality we may assume that $t_{ik}, t_{il} > 0$ and $t_{ik} \geq t_{il}$. Let $c = \frac{1}{(\frac{|t_{il}|}{2}+|t_{ik}|)}$. Take

$$A = \begin{bmatrix} \frac{t_{11}}{c} & \frac{t_{12}}{c} & . & . & . & . & \frac{t_{1k}}{c} & . & . & . & . & . & \frac{t_{1l}}{c} & . & . & . & . & \frac{t_{1n}}{c} \\ \frac{t_{21}}{c} & \frac{t_{22}}{c} & . & . & . & . & \frac{t_{2k}}{c} & . & . & . & . & . & \frac{t_{2l}}{c} & . & . & . & . & \frac{t_{2n}}{c} \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & -\frac{t_{il}}{2} & 0 & . & . & . & 0 & t_{ik} & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ \frac{t_{n1}}{c} & \frac{t_{n2}}{c} & . & . & . & . & \frac{t_{nk}}{c} & . & . & . & . & . & \frac{t_{nl}}{c} & . & . & . & . & \frac{t_{nn}}{c} \end{bmatrix}$$

Clearly, $\|A\| = \frac{1}{c} = \frac{t_{il}}{2} + t_{ik}$. Let $\lambda > 0$ be fixed. Then $\|A + \lambda T\| \geq |\frac{t_{11}}{c} + \lambda t_{11}| + |\frac{t_{12}}{c} + \lambda t_{12}| + \ldots + |\frac{t_{1n}}{c} + \lambda t_{1n}| > \frac{|t_{11}|}{c} + \frac{|t_{12}|}{c} + \ldots + \frac{|t_{1n}|}{c} = \frac{1}{c} = \|A\|$ and $\|A-\lambda T\| \geq |-\frac{t_{il}}{2} - \lambda t_{ik}| + |t_{ik} - \lambda t_{il}| \geq |(\frac{t_{il}}{2} + t_{ik}) + \lambda(t_{ik} - t_{il})| \geq |\frac{t_{il}}{2} + t_{ik}| = \|A\|$. So $A \perp_B T$.

Now take $\lambda_0 = \frac{t_{il}}{t_{ik}}$. We have

$$(T - \lambda_0 A) = \begin{bmatrix} t_{11} - \lambda_0 a_{11} & . & . & t_{1n} - \lambda_0 a_{1n} \\ t_{21} - \lambda_0 a_{21} & . & . & t_{2n} - \lambda_0 a_{2n} \\ . & . & . & . \\ . & . & . & . \\ t_{n1} - \lambda_0 a_{n1} & . & . & t_{nn} - \lambda_0 a_{nn} \end{bmatrix}$$

For any $j \neq i$, $|t_{j1} - \lambda_0 a_{j1}| + \ldots + |t_{jn} - \lambda_0 a_{jn}| = |t_{j1} - \frac{t_{il}}{t_{ik}} \frac{t_{j1}}{c}| + \ldots + |t_{jn} - \frac{t_{il}}{t_{ik}} t_{jn}| < |t_{j1}| + \ldots + |t_{jn}| = \|T\|$. Also, $|t_{i1} - \lambda_0 a_{i1}| + \ldots + |t_{in} - \lambda_0 a_{in}| = |t_{i1} - \frac{t_{il}}{t_{ik}}.0| + \ldots + |t_{ik} + \frac{t_{il}}{t_{ik}} \frac{t_{il}}{2}| + \ldots + |t_{il} - \frac{t_{il}}{t_{ik}} t_{ik}| + \ldots + |t_{in} - \frac{t_{il}}{t_{ik}}.0| = |t_{i1}| + |t_{i2}| + \ldots + |t_{ik} + \frac{t_{il}^2}{2t_{ik}}| + \ldots + |t_{in}| \leq |t_{i1}| + \ldots + |t_{ik}| + |\frac{t_{il}^2}{2t_{ik}}| + \ldots + |t_{in}| < |t_{i1}| + \ldots + |t_{ik}| + |t_{il}| + \ldots + |t_{in}| = \|T\|$. So $\|T - \lambda_0 A\| < \|T\|$ i.e., $T \not\perp_B A$. This is a contradiction.

Case 2: $t_{ik}t_{il} < 0$. Assume that $t_{ik} < 0, t_{il} > 0$ and $|t_{ik}| \geq |t_{il}|$. Let $c = \frac{1}{(\frac{|t_{il}|}{2}+|t_{ik}|)}$.

$$A = \begin{bmatrix} -\frac{t_{11}}{c} & -\frac{t_{12}}{c} & . & . & . & . & -\frac{t_{1k}}{c} & . & . & . & . & . & -\frac{t_{1l}}{c} & . & . & . & . & -\frac{t_{1n}}{c} \\ -\frac{t_{21}}{c} & -\frac{t_{22}}{c} & . & . & . & . & -\frac{t_{2k}}{c} & . & . & . & . & . & -\frac{t_{2l}}{c} & . & . & . & . & -\frac{t_{2n}}{c} \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & -\frac{t_{il}}{2} & 0 & . & . & . & 0 & t_{ik} & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ -\frac{t_{n1}}{c} & -\frac{t_{n2}}{c} & . & . & . & . & -\frac{t_{nk}}{c} & . & . & . & . & . & -\frac{t_{nl}}{c} & . & . & . & . & -\frac{t_{nn}}{c} \end{bmatrix}$$

As before, we can show that $A \perp_B T$ but, $T \not\perp_B A$.

We next assume that $t_{ik} > 0, t_{il} < 0$ and $|t_{ik}| \geq |t_{il}|$. In this case take $U = -T$. As before we can show that there exists a linear operator A such that $A \perp_B U$ but $U \not\perp_B A$. By the homogeneity of Birkhoff-James orthogonality it follows that $A \perp_B T$ but $T \not\perp_B A$. Therefore, for each $i \in \{1, 2, \ldots, n\}$ exactly one term of $t_{i1}, t_{i2}, \ldots, t_{in}$ is nonzero. This completes the proof of our Step 2. The proof of the necessary part now follows from Step 1 and Step 2. $\square$

Remark 2.2. The right symmetric linear operators on $\mathbb{R}^n$ attains norm at all extreme points and images of the extreme points are also extreme points.

We next characterize the left symmetric linear operators on $\mathbb{R}^2$. Note that the unit ball of $\mathbb{R}^2$ has only two pair of extreme points which are denoted as $\pm e_1, \pm e_2$.

Theorem 2.3. *Suppose T is a linear operator on $\mathbb{R}^2$. Then for any linear operator A on $\mathbb{R}^2$, $T \perp_B A \Rightarrow A \perp_B T$ if and only if T attains norm at only one extreme point, say e_1, Te_1 is a left symmetric point and image of the other extreme point is zero.*

Proof. Let the four extreme points of the unit ball of $\mathbb{R}^2$ be $\pm e_1, \pm e_2$. Suppose T attains norm at e_1 and $Te_2 = 0$. Let A be a linear operator such that $T \perp_B A$. Then by Theorem 2.1 of Sain and Paul [7] $Te_1 \perp_B Ae_1$. As Te_1 is a left symmetric point, it follows that $Ae_1 \perp_B Te_1$. Also $Ae_2 \perp_B Te_2 = 0$. Clearly, A attains norm at either e_1 or e_2 and $Ae_j \perp_B Te_j$ for $j = 1, 2$. So we get $A \perp_B T$.

Conversely, let $T \perp_B A \Rightarrow A \perp_B T$ for all linear operator A on $\mathbb{R}^2$. Clearly, T attains norm at an extreme point, say e_1. We claim that $Te_2 = 0$. Suppose $Te_2 \neq 0$. Define a linear operator A on $\mathbb{R}^2$ as $Ae_1 = 0,\ Ae_2 = Te_2$. It is easy to verify that A attains norm only at $\pm e_2$. Also $T \perp_B A$, as $Te_1 \perp_B Ae_1$ and $\|Te_1\| = \|T\|$. But $A \not\perp_B T$ as $Ae_2 \not\perp_B Te_2$. So $Te_2 = 0$.

Our next claim is that Te_1 is a left symmetric point. Suppose Te_1 is not a left symmetric point, i.e., there exists w such that $Te_1 \perp_B w$ but $w \not\perp_B Te_1$. Define a linear operator A on $\mathbb{R}^2$ as $Ae_1 = w,\ Ae_2 = 0$. It is easy to verify that A attains norm only at $\pm e_1$. Also $T \perp_B A$, as $Te_1 \perp_B Ae_1$ and $\|Te_1\| = \|T\|$. But $A \not\perp_B T$ as $Ae_1 \not\perp_B Te_1$. Thus we get $T \perp_B A$ but $A \not\perp_B T$, a contradiction to our hypothesis. This completes the proof of the theorem. □

Example 2.4. Note that, $(1,0),\ (0,1)$ are nonzero left symmetric points of $\mathbb{R}^2$. So, there are nonzero left symmetric linear operators on $\mathbb{R}^2$. One such linear operator can be given in the following way:

$$\begin{aligned} T(1,1) &= (1,0) \\ T(1,-1) &= (0,0) \end{aligned}$$

It is easy to verify that T attains norm only at $\pm(1,1)$, image of which is a nonzero left symmetric point of $\mathbb{R}^2$ and image of the other extreme point is zero.

The next theorem characterizes the left symmetric linear operators on $\mathbb{R}^n, n \geq 3$.

Theorem 2.5. *Suppose T is a linear operators on $\mathbb{R}^n, n \geq 3$. Then T is left symmetric if and only if T is the zero operator.*

Proof. One part of the proof is obvious. For the other part, suppose that T is a nonzero linear operator on $\mathbb{R}^n$ such that for any linear operator A on $\mathbb{R}^n$, $T \perp_B A \Rightarrow A \perp_B T$. Now T attains norm at an extreme point, say e_1.
We claim that $Te = 0$ for all extreme point $e \neq \pm e_1$. If possible, suppose that, there exists an extreme point $e_2 \neq \pm e_1$ such that $Te_2 \neq 0$. As $e_2 \perp_{SB}$

e_1, there exists a hyperplane H such that $e_1 \in H$ and $e_2 \perp_{SB} H - \{0\}$. Let $\{e_1, e_3, e_4, \ldots, e_n\}$ be a basis of H so that $\{e_2, e_1, e_3, e_4, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$. Define a linear operator A on $\mathbb{R}^n$ such that

$$\begin{aligned} Ae_2 &= Te_2 \\ Ae_i &= 0, \; i \neq 2 \end{aligned}$$

It is easy to verify that A attains norm only at $\pm e_2$. Also $T \perp_B A$, as $Te_1 \perp_B Ae_1$ and $\|Te_1\| = \|T\|$. But $A \not\perp_B T$ as $Ae_2 \not\perp_B Te_2$. So $Te = 0$ for all extreme point $e \neq \pm e_1$. Let S denote the set of extreme points e different from $\pm e_1$. Then S contains a basis B and $Te = 0$ for all $e \in B$, which forces T to be the zero operator on the whole space. This completes the proof. $\square$

Remark 2.6. The question that still remains to be answered is the characterization of right and left symmetric operators on $\ell_p (1 < p < \infty)$ spaces and more generally on a normed linear space.

Acknowledgments. Dr. Debmalya Sain lovingly acknowledges the blissful presence of his childhood friend, the distinguished physician Dr. Chandan Das, in every sphere of his life!

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