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STRUCTURES ON THE WAY FROM CLASSICAL TO QUANTUM SPACES AND THEIR TENSOR PRODUCTS

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In memoriam: Professor Charles Read

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Abstract. We study tensor products of two structures situated, in a sense, between normed spaces and (abstract) operator spaces. We call them Lambert and proto-Lambert spaces and pay more attention to the latter ones. The considered two tensor products lead to essentially different norms in the respective spaces. Moreover, the proto-Lambert tensor product is especially nice for spaces with the maximal proto-Lambert norm and in particular, for L_1-spaces. At the same time the Lambert tensor product is nice for Hilbert spaces with the minimal Lambert norm.

1. Introduction

The subject of the present paper is a structure on a linear space that, in a reasonable sense, is situated between the classical structure of a normed space and the structure of an abstract operator, or quantum space. The latter structure was discovered about 35 years ago; nowadays the theory of operator spaces, sometimes called quantum functional analysis, is a well developed area of modern functional analysis, presented in widely known textbooks; see,e.,g., [5, 17, 15]. Leading idea of that area was to investigate not just norm on a given linear space, say E, but a sequence of norms $\|\cdot\|_n; n = 1, 2, \ldots$, each one on the space of $n \times n$ matrices

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with entries in E, mutually related by certain natural conditions, the so-called Ruan axioms.

The above-mentioned intermediate structure appeared in 2002 in the Ph.D thesis of A. Lambert [11]; his superviser was G.Wittstock, one of the founding fathers of operator space theory. It was Lambert who suggested to consider, for every n, not a norm on the matrix space $M_n(E)$ but a norm on the column space of length n, consisting of vectors from E. He called the resulting sequence of norms *operator-sequential norm on* E, if it satisfied two natural axioms. Lambert developed a beautiful and rich theory, in particular, clarifying (putting in proper perspective) some aspects of quantum as well as classical functional analysis. As one of the achievements of his theory, Lambert shows that for his spaces there exists a concept of tensor product with good properties. One can say that his tensor product is on the way from the projective tensor product of normed spaces to the operator-projective tensor product of quantum spaces.

In the present paper we study some properties of Lambert's tensor product. But we also pay much attention to a certain natural generalization of Lambert's "operator–sequential space". It is called a proto-Lambert space, and it arises when we assume that only the first of Lambert's axioms for his spaces is fulfilled. Our point is that one can obtain many good things, if he considers proto–Lambert, and not, generally speaking, Lambert spaces.

Note that these proto–Lambert spaces are, after translation into an equivalent language, an important particular case of the so–called p–multi–normed spaces. The latter were quite recently (just in time when this paper was under preparation) introduced and successfully studied by H.Dales, N.Laustsen, T.Oikhberg and V.Troitsky in the memoir [3]. Proto–Lambert spaces correspond to the case of $p = 2$. Thus, they share the general properties of p–multi–normed spaces, investigated in the cited memoir. However, the properties, considered in our paper, rely heavily on the specific advantages of that particular p, actually the same advantages that distinguish ℓ_2 among all ℓ_p.

Our presentation will be given in the frame–work of the so–called non–coordinate ("index–free") approach to the structures in question, similar to what was done in [9] for operator spaces. Thus, it is different from the original approach in [11]. Both ways of presentation have their own advantages (and drawbacks), but in questions, revolving around tensor products, the non–coordinate, "index–free" presentation is, in our subjective opinion, more elegant and transparent.

The contents of the paper are as follows.

The second section contains the definition of a proto–Lambert (still not Lambert) space and some examples, notably the space $L_p(X, E); 1 \leq p < \infty$ of relevant E–valued measurable functions on X. (Running ahead, we note that this space is a Lambert space only when $p \geq 2$).

In Section 3, we consider classes of maps that reflect in a proper way the structure of a proto–Lambert space: L–bounded and L–contractive linear and bilinear operators. We prove that some classes of (bi)linear operators have the relevant properties and, in particular, some bilinear operators, related to L_p–spaces and to "classical" projective tensor products of normed spaces, are completely contractive.

In Section 4, we show that proto–Lambert spaces have their own tensor product " $\otimes_{pl}$ ", possessing the universal property for the class of L–bounded bilinear operators between these spaces.

In Section 5, we concentrate on the case when one of the tensor factors is a space with the so–called maximal proto-Lambert norm, in particular, an $L_1(X)$–space, and give an explicit description of the resulting proto-Lambert tensor product. As a corollary, we obtain a version, for proto-Lambert spaces, of Grothendieck's theorem on tensoring by L_1–spaces in the "classical" context of Banach spaces: cf., e.g., [6, §2,n^{o}2].

In Section 6, we pass from proto–Lambert to Lambert spaces, adding in the relevant definition the non-coordinate analogue of the second of Lambert's axioms. We introduce the respective version " $\otimes_l$ "of Lambert's "maximal tensor product" of his Operatorfolgenräume and prove its existence.

We have seen in Section 5 that the proto-Lambert tensor product is especially nice for L_1–spaces. In Section 7 we show that the Lambert (without "proto–") tensor product is nice for Hilbert spaces. Namely, if we equip both of Hilbert spaces with the so–called minimal Lambert norm, then their completed Lambert tensor product is again a Hilbert space with the same structure.

In the last Section 8, we compare both tensor products," $\otimes_{pl}$ " and " $\otimes_l$ ", and show that the first one provides, generally speaking, essentially greater norms. In particular, for every n we display a certain element in the amplification of the tensor square of a certain Lambert space. It turns out that the norm of this element, provided by the proto–Lambert tensor product, is n, whereas the norm, provided by the Lambert tensor product, is $\sqrt{n}$.

2. Proto-Lambert spaces and their examples

To begin with, we choose an arbitrary, separable, infinite-dimensional Hilbert space, denote it by H and fix it throughout the whole paper. The identity operator on H will be denoted by $\mathbf{1}$.

As usual, by $\mathcal{B}(E,F)$ we denote the space of all bounded operators between the normed spaces E and F, endowed with the operator norm. We write $\mathcal{B}(E)$ instead of $\mathcal{B}(E,E)$, and also $\mathcal{B}$ instead of $\mathcal{B}(H)$.

If K, L are pre–Hilbert spaces, $x \in K, y \in L$, we denote by $x \circ y : L \to K$ the rank 1 operator, taking z to $\langle z, y\rangle x$. Note that we have $\|x \circ y\| = \|x\|\|y\|$.

The symbol $\otimes$ is used for the (algebraic) tensor product of linear spaces and for elementary tensors. The symbols $\otimes_p$ and $\otimes_i$ denote the non–completed projective and injective tensor product of normed spaces, respectively, and the symbol $\otimes_{hil}$ is used for the non-completed Hilbert tensor product of pre–Hilbert spaces. The symbols $\widehat{\otimes}_p$, $\widehat{\otimes}_i$ and $\widehat{\otimes}_{hil}$ are used for the respective completed tensor products. The complex–conjugate space of a linear space E is denoted by E^{cc}.

In what follows we need the triple notion of the so-called amplification. First, we amplify linear spaces, then linear operators and finally bilinear operators. Note that these amplifications differ from the amplifications, serving in the theory of quantum spaces (cf. [9]).

The *amplification* of a given linear space E is the tensor product $H\otimes E$. Usually we briefly denote it by HE, and an elementary tensor, say $\xi\otimes x; \xi\in H, x\in E$, by ξx. Note that HE is a left module over the algebra $\mathcal{B}$ with the outer multiplication "$\cdot$", well defined by $a\cdot(\xi x) := a(\xi)x$.

Remark 2.1. In the non-coordinate presentation of the operator space theory the amplification of E is $\mathcal{F}\otimes E$, where $\mathcal{F}$ is the space of finite rank bounded operators on H. But $\mathcal{F} = H\otimes H^{cc}$; so, passing to the "non-coordinate Lambert theory", we replace the whole tensor product by its first factor. (One can observe a similar transfer in the coordinate presentation: we replace $M_n(\mathbb{C}) = \mathbb{C}^n\otimes(\mathbb{C}^n)^{cc}$ by $\mathbb{C}^n$).

Note that the transfer from $\mathcal{F}\otimes E$ to $H\otimes E$ was actually used in the "non-coordinate" proof of the injective property of the Haagerup tensor product of operator spaces [9, Section 7.3] (such a property was discovered by Paulsen and Smith [16]).

Definition 2.2. A semi-norm on HE is called *proto-Lambert semi-norm* or briefly *PL-semi-norm* on E, if the left $\mathcal{B}$-module HE is contractive, that is we always have the estimate $\|a\cdot u\| \leq \|a\|\|u\|$. The space E, endowed by a PL-semi-norm, is called *semi-normed proto-Lambert space* or briefly *semi-normed PL-space*). If the semi-norm in question is actually a norm, we speak, naturally, about a *normed proto-Lambert* (= *normed PL–) space*, and in this case we often omit the word "normed".

A semi-normed PL–space E becomes semi-normed space (in the usual sense), if for $x\in E$ we set $\|x\| := \|\xi x\|$, where $\xi\in H$ is an arbitrary vector with $\|\xi\| = 1$. Clearly, the result does not depend on a choice of ξ. The obtained semi-normed space is called *underlying space* of a given PL-space, and the latter is called a *PL–quantization* of a former. (We use such a term by analogy with quantizations in operator space theory; see, e.g., [4], [5] or [9]). Obviously, for all $\xi\in H$ and $x\in E$ we have $\|\xi x\| = \|\xi\|\|x\|$.

It is easy to verify that the space of scalars, $\mathbb{C}$, has the only PL–quantization, given by the identification $H\mathbb{C} = H$.

Proposition 2.3. *Let E be a semi-normed PL–space with a normed underlying space. Then the PL–semi-norm on HE is a norm.*

Proof. Take $u\in HE; u\neq 0$ and represent it as $\sum_{k=1}^n \xi_k x_k$, where ξ_k are linearly independent, $\|\xi_1\| = 1$ and $x_1\neq 0$. Further, take $\eta\in H$ with $\langle\xi_1,\eta\rangle = 1$ and $\langle\xi_k,\eta\rangle = 0$ for $k>1$. Then $(\xi_1\circ\eta)\cdot u = \xi_1 x_1$. Therefore we have $0 < \|\xi_1 x_1\| \leq \|\xi_1\circ\eta\|\|u\| = \|\xi_1\|\|\eta\|\|u\|$, hence $\|u\| > 0$. □

Example 2.4. Every normed space, say E, has, generally speaking, a lot of PL–quantizations. We distinguish two of them. The PL–space, denoted by $E_{\max}$, respectively $E_{\min}$, has the PL–norm, obtained by the endowing HE with the norm of $H\otimes_p E$, respectively of $H\otimes_i E$. We denote the norm on the former and on the latter space by $\|\cdot\|_{\max}$ and $\|\cdot\|_{\min}$, respectively; accordingly, the corresponding PL-quantizations of E will be called maximal and minimal. Clearly, the PL-norm of $E_{\max}$ is the greatest of all PL-norms of PL-quantizations of E. The adjective "minimal" will be justified a little bit later.

Example 2.5. Let (X,μ) be a measure space, $L_p(X); 1 \leq p < \infty$ the relevant Banach space, F a normed PL-space (say, $F := \mathbb{C}$ in the simplest case). We want to endow the "classical" space $L_p(X,F)$ (of relevant F-valued functions) with a PL-norm.

As a preliminary step, consider the normed space $L_p(X,HF)$ of relevant HF-valued measurable functions on X and observe that it is a left $\mathcal{B}$-module with the outer multiplication defined by $[a\cdot\bar{x}](t) := a\cdot[\bar{x}(t)]; a \in \mathcal{B}, \bar{x} \in L_p(X,HF), t \in X$. A routine calculation shows that this module is contractive.

Now consider the operator $\alpha : H(L_p(X,F)) \to L_p(X,HF)$, well defined on elementary tensors by taking $\xi x; x \in L_p(X,F), \xi \in H$ to the HF-valued function $\bar{x}(t) := \xi(x(t))$, and introduce the semi-norm on $H(L_p(X,F))$, setting $\|u\| := \|\alpha(u)\|$. It is easy to veryfy that α is a $\mathcal{B}$-module morphism. Thus there is an isometric morphism of the module $H(L_p(X,F))$ into a contractive module. It follows immediately that the former module is itself contractive, that is the introduced semi-norm on $H(L_p(X,F))$ is a PL-semi-norm on $L_p(X,F)$. Further, for $\xi \in H; \|\xi\| = 1$ and $x \in L_p(X,F)$ we have $\|\xi[x(t)]\| = \|x(t)\|$ for all $t \in X$. Therefore for $\xi x \in H(L_p(X,F))$ we have

$$\|\xi x\| = \left(\int_X \|[\xi x](t)\|^p d\mu(t)\right)^{\frac{1}{p}} = \left(\int_X \|x(t)\|^p d\mu(t)\right)^{\frac{1}{p}}.$$

We see that the underlying semi-normed space of the constructed PL-space is $L_p(X,F)$. Therefore Proposition 2.3 guarantees that the introduced PL-seminorm on $L_p(X,F)$ is actually a norm.

Example 2.6. We want to introduce a PL–quantization of the "classical" tensor product $E\otimes_p F$ of normed spaces, when one of tensor factors, say, to be definite, F, is a PL–space.

Consider the linear isomorphism $\beta : H(E\otimes F) \to E\otimes_p(HF) : \xi(x\otimes y) \mapsto x\otimes\xi y$ and introduce a norm on $H(E\otimes F)$ by setting $\|U\| := \|\beta(U)\|$. The space $E\otimes_p(HF)$, as a projective tensor product of a normed space and a contractive $\mathcal{B}$-module, has itself a standard structure of a contractive $\mathcal{B}$-module. Since β is a $\mathcal{B}$-module morphism, the same is true with $H(E\otimes F)$. Thus $E\otimes F$ becomes a PL–space, and we must show that its underlying normed space $(E\otimes F, \|\cdot\|)$ is exactly $E\otimes_p F$.

Denote the norm on $E\otimes_p F$ and on $E\otimes_p(HF)$ by $\|\cdot\|_p$. Take arbitrary $u \in E\otimes F$. It is easy to check that the norm $\|\cdot\|$ on $E\otimes F$ is a cross-norm, so we have $\|\xi u\| \leq \|u\|_p$. Therefore our task is to show that, for $\xi \in H, \|\xi\| = 1$, we have $\|\xi u\| \geq \|u\|_p$.

Identifying $\mathcal{B}$-modules $H(E\otimes F)$ and $E\otimes_p(HF)$ by means of β, represent ξu as $\sum_{k=1}^n x_k\otimes w_k; x_k \in E, w_k \in HF$. Set $p := \xi \circ \xi$. Obviously, $p\cdot w_k = \xi y_k$ for some $y_k \in F; k = 1,\cdots,n$. Therefore $\sum_{k=1}^n \|x_k\|\|w_k\| \geq \sum_{k=1}^n \|x_k\|\|p\cdot w_k\| = \sum_{k=1}^n \|x_k\|\|y_k\|$. But we have $\xi u = p\cdot(\xi u) = \sum_{k=1}^n x_k\otimes p\cdot w_k = \xi(\sum_{k=1}^n x_k\otimes y_k)$. It follows that $u = \sum_{k=1}^n x_k\otimes y_k$. Consequently, $\sum_{k=1}^n \|x_k\|\|w_k\| \geq \|u\|_p$, and we are done.

From now on we denote the constructed PL–quantization of $E\otimes_p F$ again by $E\otimes_p F$.

3. L-bounded linear and bilinear operators

Suppose we are given an operator $\varphi : E \to F$ between linear spaces. Denote, for brevity, the operator $\mathbf{1}\otimes\varphi : HE \to HF$ (taking ξx to $\xi\varphi(x)$) by φ_∞ and call it *amplification* of φ. Obviously, φ_∞ is a morphism of left $\mathcal{B}$-modules.

Definition 3.1. An operator $\varphi : E \to F$ between seminormed PL–spaces is called L*–bounded*, L*-contractive*, L*–isometric,* L*–isometric isomorphism*, if φ_∞ is bounded, contractive, isometric, isometric isomorphism, respectively. We set $\|\varphi\|_{lb} := \|\varphi_\infty\|$.

If φ is bounded, being considered between the respective underlying seminormed spaces, we say that it is (just) *bounded*, and denote its operator seminorm, as usual, by $\|\varphi\|$. Clearly, every L–bounded operator $\varphi : E \to F$ is bounded, and $\|\varphi\| \leq \|\varphi\|_{lb}$.

Some operators between PL–spaces, bounded as operators between underlying spaces, are "automatically" L–bounded. Here is the first phenomenon of that kind.

Proposition 3.2. *Let E be a PL–space. Then every bounded functional $f : E \to \mathbb{C}$ is L–bounded, and $\|f\|_{lb} := \|f\|$.*

Proof. Clearly, it is sufficient to show that for every $u \in HE$ we have $\|f_\infty(u)\| \leq \|f\|\|u\|$. Recall that $\|f_\infty(u)\| = \max\{|\langle f_\infty(u), \xi\rangle|; \xi \in H, \|\xi\| = 1\}$. Presenting u as a sum of elementary tensors, we see that, for every $\eta \in H; \|\eta\| = 1$ we have $\langle f_\infty(u), \xi\rangle\eta = f_\infty[(\eta \circ \xi)\cdot u]$ and also $(\eta \circ \xi)\cdot u = \eta x_\xi$ for some $x_\xi \in E$. It follows that $\|x_\xi\| = \|(\eta \circ \xi)\cdot u\| \leq \|u\|$, hence $|\langle f_\infty(u), \xi\rangle| = \|f_\infty[(\eta \circ \xi)\cdot u]\| = \|f_\infty(\eta x_\xi)\| = |f(x_\xi)| \leq \|f\|\|u\|$. □

Thus for every PL–space E and $u \in HE$ we have $\|u\| \geq \sup\{\|f_\infty(u)\|\}$, where supremum is taken over all $f \in E^*; \|f\| \leq 1$. But such a supremum is exactly $\|u\|_{\min}$. This justifies the word "minimal" in Example 2.4.

A PL–space is called *complete* (or Banach), if its underlying normed space is complete. As in the "classical" context, for every PL–space E there exists its *completion*, which is defined as a pair $(\overline{E}, i : E \to \overline{E})$, consisting of a complete PL–space and an L–isometric operator, such that the same pair, considered for respective underlying spaces and operators, is the "classical" completion of E as of a normed space. The proof of the respective existence theorem repeats, with obvious modifications, the simple argument given in [9, Chapter 4] for quantum spaces. We only recall that the norm on $H\overline{E}$ is introduced with the help of the natural embedding of $H\overline{E}$ into $\overline{HE}$, the "classical" completion of HE. This embedding is well defined by taking an elementary tensor $\xi x; \xi \in H, x \in \overline{E}$ to $\lim_{n\to\infty} \xi x_n$, where $x_n \in E$ converges to x; hence ξx_n can be considered as a converging sequence in $\overline{HE}$. (Here, of course, we identify E with a subspace of $\overline{E}$ and HE with a subspace of $\overline{HE}$.)

It is easy to observe that the characteristic universal property of the "classical" completion has its proto–Lambert version. Namely, if $(\overline{E}, i)$ is the completion of a PL–space E, F a PL–space and $\varphi : E \to F$ is an L–bounded operator, then there exists a unique L–bounded operator $\overline{\varphi} : \overline{E} \to \overline{F}$, which is, in obvious sense, the *continuous extension* of φ. Moreover, we have $\|\overline{\varphi}\|_{lb} = \|\varphi\|_{lb}$.

Distinguish the following useful fact. Its proof is the same, up to obvious modifications, as of Proposition 4.8 in [9].

Proposition 3.3. *Let $\varphi : E \to F$ be an L–isometric isomorhism between PL–spaces. Then its continuous extension $\overline{\varphi} : \overline{E} \to \overline{F}$ is also an L–isometric isomorhism.*

We pass to bilinear operators. By virtue of Riesz/Fisher Theorem, we can arbitrarily choose a unitary isomorphism $\iota : H\widehat{\otimes}_{hil}H \to H$ and fix it throughout the whole paper. Following [10], for $\xi, \eta \in H$ we denote the vector $\iota(\xi \otimes \eta) \in H$ by $\xi \diamondsuit \eta$, and for $a, b \in \mathcal{B}$ we denote the operator $\iota(a \overset{\cdot}{\otimes} b)\iota^{-1}$ on H by $a \diamondsuit b$; obviously, the latter is well defined by the equality $(a \diamondsuit b)(\xi \diamondsuit \eta) = a(\xi) \diamondsuit b(\eta)$. Evidently, we have

$$\|\xi \diamondsuit \eta\| = \|\xi\|\|\eta\| \qquad \text{and} \qquad \|a \diamondsuit b\| = \|a\|\|b\|. \tag{3.1}$$

If E is a linear space, $\xi \in H$ and $u \in HE$, we set $\xi \diamondsuit u := T_\xi \cdot u$, where $T_\xi \in \mathcal{B}$ sends η to $\xi \diamondsuit \eta$. Thus, this version of the operation ‘ $\diamondsuit$ ’ is well defined on elementary tensors by $\xi \diamondsuit \eta x := (\xi \diamondsuit \eta)x$. Similarly, we introduce $u \diamondsuit \eta \in HE$ by $\xi x \diamondsuit \eta := (\xi \diamondsuit \eta)x$. By (3.1), $T_\xi = \|\xi\|S$, where S is an isometry. Therefore, if E is a PL–space, we have

$$\|\xi \diamondsuit u\| = \|\xi\|\|u\| \qquad \text{and similarly} \qquad \|u \diamondsuit \eta\| = \|\eta\|\|u\|. \tag{3.2}$$

Now let $\mathcal{R} : E \times F \to G$ be a bilinear operator between linear spaces. Its *amplification* is the bilinear operator $\mathcal{R}_\infty : HE \times HF \to HG$, associated with the 4-linear operator $H \times E \times H \times F \to HG : (\xi, x, \eta, y) \mapsto (\xi \diamondsuit \eta)\mathcal{R}(x, y)$. In other words, $\mathcal{R}_\infty$ is well defined on elementary tensors by $\mathcal{R}_\infty(\xi x, \eta y) = (\xi \diamondsuit \eta)\mathcal{R}(x, y)$.

Definition 3.4. A bilinear operator $\mathcal{R}$ between PL-spaces is called *L-bounded*, respectively, *L-contractive*, if its amplification is (just) bounded, respectively, contractive. We put $\|\mathcal{R}\|_{lb} := \|\mathcal{R}_\infty\|$.

It is easy to see that an L-bounded bilinear operator, being considered between respective underlying (semi-)normed spaces is just bounded, and $\|\mathcal{R}\| \le \|\mathcal{R}\|_{lb}$. On the other hand, similarly to linear operators, sometimes the “classical” boundedness automatically implies the L–boundedness.

Proposition 3.5. *Let E, F be PL-spaces, $f : E \to \mathbb{C}$, and $g : F \to \mathbb{C}$ bounded functionals. Then the bilinear functional $f \times g : E \times F \to \mathbb{C} : (x, y) \mapsto f(x)g(y)$ is L–bounded and $\|f \times g\|_{lb} = \|f\|\|g\|$.*

Proof. Since $\|f \times g\| = \|f\|\|g\|$, it suffices to show that $\|f \times g\|_{lb} \le \|f\|\|g\|$. Indeed, combining the obvious formula $(f \times g)_\infty(u, v) = f_\infty(u) \diamondsuit g_\infty(v)$, Proposition 3.2 and (3.1), we have $\|(f \times g)_\infty(u, v)\| \le \|f\|\|g\|\|u\|\|v\|$. ☐

Proposition 3.6. *Let $L_p(X) := L_p(X, \mathbb{C})$ and $L_p(X, E)$ be the PL-spaces from Example 2.5. Then the bilinear operator $\mathcal{R} : L_p(X) \times E \to L_p(X, E)$, taking (z, x) to the E-valued function $t \mapsto z(t)x; t \in X$, is L-contractive.*

Proof. Recall the isometric operator $\alpha : H(L_p(X,E)) \to L_p(X,HE)$ and distinguish its particular case $\alpha_0 : H(L_p(X)) \to L_p(X,H)$. Also consider the bilinear operator $\mathcal{S} : L_p(X,H) \times HE \to L_p(X,HE)$, taking a pair (ω,u) to the HE–valued function $t \mapsto \omega(t)\diamondsuit u; t \in X$. With the help of (3.2), a routine calculation gives $\|\mathcal{S}(\omega,u)\| = \|\omega\|\|u\|$.

Now consider the diagram

$$\begin{array}{ccc} H(L_p(X)) \times HE & \xrightarrow{\mathcal{R}_\infty} & H(L_p(X,E)) \\ \downarrow{\scriptstyle \alpha_0 \times \mathbf{1}_{HE}} & & \downarrow{\scriptstyle \alpha} \\ L_p(X,H) \times HE & \xrightarrow{\mathcal{S}} & L_p(X,HE) \end{array},$$

It is easy to check on elementary tensors in the respective amplifications that it is commutative. Therefore, for $w \in H(L_p(X))$ and $u \in HE$ we have

$$\|\mathcal{R}_\infty(w,u)\| = \|\alpha(\mathcal{R}_\infty(w,u))\| = \|\mathcal{S}(\alpha_0(w),u)\| = \|\alpha_0(w)\|\|u\| = \|w\|\|u\|.$$

□

We shall denote the completion of the PL-space $E\otimes_p F$ from Example 2.6 by $E\widehat{\otimes}_p F$. Clearly, it is the PL–quantization of the "classical" Banach space $E\widehat{\otimes}_p F$.

Proposition 3.7. *Let E be a normed space, F a PL–space, $E\otimes_p F$ the resulting PL–space. Then the canonical bilinear operator $\vartheta : E_{\max} \times F \to E\otimes_p F$, considered between the respective PL–spaces, is L–contractive. Moreover, $\widehat{\vartheta} : E_{\max} \times F \to E\widehat{\otimes}_p F$, that is ϑ, considered with $E\widehat{\otimes}_p F$ as its range, is also L–contractive.*

Proof. Consider the trilinear operator $\mathcal{T} : E \times H \times HF \to E\otimes_p(HF) : (x,\xi,v) \mapsto x\otimes(\xi\diamondsuit v)$. It follows from (3.2) that $\mathcal{T}$ is contractive. Therefore the bilinear operator $\mathcal{S} : (E\otimes_p H)\times HF \to E\otimes_p HF : (x\otimes\xi, v) \mapsto x\otimes(\xi\diamondsuit v)$ is also contractive.

Recall the isometric operator $\beta : H(E\otimes_p F) \to E\otimes_p HF$ from Example 2.6 and distinguish its particular case, the "flip" $\beta_0 : HE_{\max} \to E\otimes_p H$. Consider the diagram

$$\begin{array}{ccc} HE_{\max} \times HF & \xrightarrow{\vartheta_\infty} & H(E\otimes_p F) \\ \downarrow{\scriptstyle \beta_0 \times \mathbf{1}_{HF}} & & \downarrow{\scriptstyle \beta} \\ (E\otimes_p H) \times HF & \xrightarrow{\mathcal{S}} & E\otimes_p HF \end{array},$$

which is obviously commutative. Therefore a routine calculation shows that for $w \in HE_{\max}$ and $v \in HF$ we have $\|\vartheta_\infty(w,v)\| \leq \|w\|\|u\|$, and we are done. □

4. Proto-Lambert tensor product

We proceed to show that L-bounded bilinear operators between PL-spaces can be linearized with the help of a specific tensor product " $\otimes_{pl}$ ", which seems to be new. But before, since in this paper we shall come across several varieties of a tensor product, it is convenient to give a general definition, embracing all particular cases.

Let us fix, throughout this section, two *arbitrary chosen* PL–spaces E and F. Further, let $\mho$ be a subclass of the class of all normed PL-spaces.

Definition 4.1. A pair (Θ, θ) that consists of $\Theta \in \mho$ and an L-contractive bilinear operator $\theta : E \times F \to \Theta$ is called *tensor product of* E *and* F *relative to* $\mho$ if, for every $G \in \mho$ and every L–bounded bilinear operator $\mathcal{R} : E \times F \to G$, there exists a unique L–bounded operator $R : \Theta \to G$ such that the diagram

$$\begin{array}{ccc} E \times F & & \\ \downarrow \theta & \searrow^{\mathcal{R}} & \\ \Theta & \xrightarrow{R} & G \end{array}$$

is commutative, and moreover $\|R\|_{lb} = \|\mathcal{R}\|_{lb}$.

Such a pair is unique in the following sense: if $(\Theta_k, \theta_k); k = 1, 2$ are two pairs, satisfying the given definition for a certain $\mho$, then there is a L–isometric isomorphism $I : \Theta_1 \to \Theta_2$, such that $I\theta_1 = \theta_2$. This fact is a particular case of a general–categorical observation concerning the uniqueness of an initial object in a category; cf., e.g., [13], [7, Theorem 2.73]. However, the question about the existence of such a pair depends on our luck with the choice of the class $\mho$.

Definition 4.2. The tensor product of E and F relative to the class of all normed PL–spaces is called *non-completed* PL*–tensor product* of our spaces.

We shall prove the existence of such a pair, displaying its explicit construction.

First, we need a sort of "extended" version of the diamond multiplication, this time between elements of amplifications of linear spaces. Namely, for $u \in HE, v \in HF$ we consider the element $u\diamondsuit v := \vartheta_\infty(u, v) \in H(E\otimes F)$, where $\vartheta : E \times F \to E\otimes F$ is the canonical bilinear operator. In other words, this "diamond operation" is well defined by $\xi x\diamondsuit \eta y := (\xi\diamondsuit\eta)(x\otimes y)$.

Proposition 4.3. *Every* $U \in H(E\otimes F)$ *can be represented as* $\sum_{k=1}^{n} a_k\cdot(u_k\diamondsuit v_k)$ *for some natural* n *and* $a_k \in \mathcal{B}, u_k \in HE, v_k \in HF, k = 1, \cdots, n$.

Proof. Evidently, it suffices to consider the simplest case, when $U = \xi(x\otimes y); \xi \in H, x \in E, y \in F$. Take arbitrary non-zero $\eta, \zeta \in H$; then we have $\xi = a(\eta\diamondsuit\zeta)$, for some $a \in \mathcal{B}$. Consequently, $U = a\cdot(\eta x\diamondsuit\zeta y)$. ☐

As a corollary, the operator $\mathcal{B}\otimes HE\otimes HF \to H(E\otimes F)$, associated with the 3-linear operator $(a, u, v) \mapsto a\cdot(u\diamondsuit v)$, is surjective. Thus $H(E\otimes F)$ can be endowed with the seminorm of the respective quotient space of $\mathcal{B}\otimes_p HE\otimes_p HF$, denoted by $\|\cdot\|_{pl}$. In other words, we have

$$\|U\|_{pl} := \inf\{\sum_{k=1}^{n} \|a_k\|\|u_k\|\|v_k\|\}, \tag{4.1}$$

where the infimum is taken over all possible representations of U as indicated in Proposition 4.3.

Proposition 4.4. *The seminormed* $\mathcal{B}$*–module* $(H(E\otimes F), \|\cdot\|_{pl})$ *is contractive.*

Proof. Clearly, $\mathcal{B}\otimes_p HE\otimes_p HF$ is a contractive left $\mathcal{B}$-module as a tensor product of the left $\mathcal{B}$-module $\mathcal{B}$ and the linear space $HE\otimes HF$. Therefore $H(E\otimes F)$ is the image of a contractive left $\mathcal{B}$-module with respect to a quotient map of seminormed spaces. Since the latter map is a module morphism, we easily obtain the desired property. □

Thus, $\|\cdot\|_{pl}$ is a PL–seminorm on $E\otimes F$. Denote the respective PL–space by $E\otimes_{pl}F$.

Observe the obvious estimate

$$\|u\diamondsuit v\|_{pl} \le \|u\|\|v\|; \quad u \in HE, v \in HF. \tag{4.2}$$

Since $u\diamondsuit v = \vartheta_\infty(u, v)$, we see that ϑ, *considered with range* $E\otimes_{pl}F$, *is L-contractive.*

Looking at the underlying spaces and using (3.1), we easily obtain that

$$\|x\otimes y\| \le \|x\|\|y\|; \quad x \in E, y \in F. \tag{4.3}$$

(In fact, in (4.2) and (4.3) we have the equality, but we shall not discuss it now).

Proposition 4.5. *Let G be a PL-space, $\mathcal{R} : E \times F \to G$ an L–bounded bilinear operator, $R : E\otimes_{pl}F \to G$ the associated linear operator. Then R is L–bounded, and $\|\mathcal{R}\|_{lb} = \|R\|_{lb}$.*

Proof. Take $U \in H(E\otimes_{pl}F))$ and represent it according to Proposition 4.3. We remember that R_∞ is a $\mathcal{B}$-module morphism. Therefore, using the obvious equality $R_\infty(u\diamondsuit v) = \mathcal{R}_\infty(u, v)$, we have that $R_\infty(U) = \sum_{k=1}^n a_k\cdot\mathcal{R}_\infty(u_k, v_k)$, hence

$$\|R_\infty(U)\| \le \sum_{k=1}^n \|a_k\|\|\mathcal{R}_\infty(u_k, v_k)\| \le \|\mathcal{R}\|_{lb} \sum_{k=1}^n \|a_k\|\|u_k\|\|v_k\|.$$

From this, using 4.1, we obtain that $\|R_\infty(U)\| \le \|\mathcal{R}\|_{lb}\|U\|_{pl}$. Thus our R is L–bounded, and $\|R\|_{lb} \le \|\mathcal{R}\|_{lb}$. The converse inequality easily follows from (4.2). □

Proposition 4.6. (As a matter of fact), $\|\cdot\|_{pl}$ *is a norm.*

Proof. By Proposition 2.3, it is sufficient to show that, for a non-zero elementary tensor $\xi w; w \in E\otimes_{pl}F, \xi \in H; \|\xi\| = 1, w \ne 0$ we have $\|\xi w\|_{pl} \ne 0$. Since E and F are normed spaces, then, as it is known, there exist bounded functionals $f : E \to \mathbb{C}, g : F \to \mathbb{C}$ such that $(f\otimes g)w \ne 0$. Now consider in the previous proposition $\mathcal{R} := f\times g : E\times F \to \mathbb{C}$. By virtue of Proposition 3.5, $\mathcal{R}$ is L–bounded, hence the operator $(f\otimes g)_\infty$ is bounded. At the same time $(f\otimes g)_\infty(\xi w) = [(f\otimes g)(w)]\xi \ne 0$, and we are done. □

Combining Propositions 4.5 and 4.6, we immediately obtain

Theorem 4.7. (Existence theorem). *The pair $(E\otimes_{pl}F, \vartheta)$ is a non-completed PL-tensor product of E and F.*

We can also speak about the "completed" version of Definition 4.2.

Definition 4.8. The tensor product of E and F relative to the class of all complete PL–spaces is called *completed, or Banach PL–tensor product* of our spaces.

Proposition 4.9. *The Banach PL–tensor product of PL–spaces E and F exists, and it is the pair $(E\widehat{\otimes}_{pl}F, \widehat{\vartheta})$, where $E\widehat{\otimes}_{pl}F$ is the completion of the PL–space $E\otimes_{pl}F$, and $\widehat{\vartheta}$ acts as ϑ, but with range $E\widehat{\otimes}_{pl}F$.*

Proof. This is an immediate corollary of the universal property of the completion. □

5. Tensoring by maximal PL–spaces and by $L_1(\cdot)$

In this section we show that for certain concrete tensor factors their PL–tensor product also becomes something concrete and transparent.

Theorem 5.1. *Let E be a normed space, F a PL–space, $E\otimes_p F$ the PL–space from Example 2.6. Then there exists an L–isometric isomorphism $I : E_{\max}\otimes_{pl}F \to E\otimes_p F$, acting as the identity operator on the common underlying linear space of our PL–spaces. As a corollary* (see Proposition 3.3), *there exists an L–isometric isomorphism $\widehat{I} : E_{\max}\widehat{\otimes}_{pl}F \to E\widehat{\otimes}_p F$, which is the extension by continuity of I.*

Proof. Consider ϑ from Proposition 3.7. By Proposition 4.7, ϑ gives rise to the L–contractive operator I, acting as in the formulation. Therefore it is sufficient to show that the operator I_∞ does not decrease norms of elements.

Take $U \in H(E\otimes F)$. Identifying the latter space with $E\otimes HF$, we can represent U as $\sum_{k=1}^n x_k\otimes v_k; x_k \in E, v_k \in HF$. Choose $e \in H; \|e\| = 1$ and denote by $S \in \mathcal{B}$ the isometric operator $\xi \mapsto e\diamondsuit\xi; \xi \in H$. We easily see that

$$U = S^*\cdot \left[\sum_{k=1}^n ex_k\diamondsuit v_k\right].$$

From this, by (4.1), we obtain the estimate $\|U\|_{pl} \le \sum_{k=1}^n \|x_k\|\|v_k\|$. Hence, we have

$$\|U\|_{pl} \le \inf\{\sum_{k=1}^n \|x_k\|\|v_k\|\}. \tag{5.1}$$

where the infimum is taken over all representations of U in the indicated form.

Now look at $I_\infty(U)$. It is the same $\sum_{k=1}^n x_k\otimes v_k$, only considered in the normed space $E\otimes_p HF$. It follows that $\|I_\infty(U)\|$ is exactly the infimum, indicated in (5.1). Thus, $\|I_\infty(U)\| \ge \|U\|_{pl}$ and we are done. □

Remark 5.2. As an easy corollary of this theorem, we have, up to an L–isometric isomorphism, that $E_{\max}\widehat{\otimes}_{pl}F_{\max} = [E\widehat{\otimes}_p F]_{\max}$ for all normed spaces E and F. In particular, we have $H_{\max}\widehat{\otimes}_{pl}H_{\max} = \mathcal{N}(H)_{\max}$, where $\mathcal{N}(H)$ is the Banach space of trace class operators on H.

Now we want to apply this theorem to the description of PL–tensor products in the situation, when one of tensor factors is $L_1(X)$ from Example 2.5. As a "classical" prototype of that description, we recall the following theorem, due to Grothendieck.

Let (X,μ) be a measure space, F a Banach space. Then there exists an isometric isomorphism $\mathcal{G}_F : L_1(X)\widehat{\otimes}_p F \to L_1(X,F)$, well defined by taking an elementary tensor $z\otimes x; z\in L_1(X), x\in F$ to the F-valued function $t\mapsto z(t)x; t\in X$.

Theorem 5.3. *Let (X,μ) be a measure space, F a complete PL–space. Then there exists an L–isometric isomorphism $\mathcal{I} : L_1(X)\widehat{\otimes}_{pl} F \to L_1(X,F)$, well defined in the same way as $\mathcal{G}_F$ in the Grothendieck theorem.*

Proof. First we note that *the PL-norm on $L_1(X)$, introduced in Example 2.5, coincides with the maximal PL-norm, introduced in Example 2.4*. This is because the identity operator $I : H\otimes_p L_1(X) \to H(L_1(X))$ participates in the commutative diagram

$$\begin{array}{ccc} H\otimes_p L_1(X) & \xrightarrow{\ I\ } & H(L_1(X)) \ , \\ {\scriptstyle flip}\downarrow & & \downarrow{\scriptstyle \alpha_1} \\ L_1(X)\otimes_p H & \xrightarrow{\ \mathcal{G}_0\ } & L_1(X,H) \end{array}$$

where $\mathcal{G}_0$ is the restriction of $\mathcal{G}_H$, α_1 is a particular case of α from Example 2.5, and these operators, as well as "flip", are isometric.

Combining this with Theorem 5.1, we come to the L–isometric isomorphism $\widehat{I} : L_1(X)\widehat{\otimes}_{pl} F \to L_1(X)\widehat{\otimes}_p F$.

Now we show that *the isometric isomorphism $\mathcal{G}_F : L_1(X)\widehat{\otimes}_p F \to L_1(X,F)$ is L–isometric with respect to the PL–norm in the latter space, taken from Example 2.5.* We see that $\mathcal{G}_F$ is the extension by continuity (cf. Section 3) of its restriction $\mathcal{G}_F^0$ to $L_1(X)\otimes_p F$, and this restriction maps the latter space onto a dense subspace of $L_1(X,F)$. Therefore, by virtue of Proposition 3.3, it is sufficient to show that the operator $\mathcal{G}_F^0$ is L–isometric, or, equivalently, that $(\mathcal{G}_F^0)_\infty$ is isometric. But the latter participates in the commutative diagram

$$\begin{array}{ccc} H(L_1(X)\otimes_p F) & \xrightarrow{\ (\mathcal{G}_F^0)_\infty\ } & H(L_1(X,F)) \ , \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \alpha} \\ L_1(X)\otimes_p HF & \xrightarrow{\ \mathcal{G}_{HF}^0\ } & L_1(X,HF) \end{array}$$

where α and (with $L_1(X)$ as E) β are operators from the Examples 2.5 and 2.6, respectively, and $\mathcal{G}_{HF}^0$ is the restriction to the respective subspaces of the isometric isomorphism, provided by the Grothendieck theorem (this time with the completion of HF in the role of F). Since α,β and $\mathcal{G}_{HF}^0$ are isometric, then $(\mathcal{G}_0)_\infty$ is also isometric.

After this, to end the proof, it remains to set $\mathcal{I} := \mathcal{G}_F\widehat{I}$. □

Like in the "classical" context of Grothendieck theorem, we can distinguish a transparent particular case, concerning integrable functions of two variables.

Proposition 5.4. *Let $(X,\mu),(Y,\nu)$ be two measure spaces, $(X\times Y,\mu\times\nu)$ their product measure space. Then $L_1(X)\widehat{\otimes}_{pl} L_1(Y) = L_1(X\times Y)$ up to an L–isometric isomorphism. More precisely, there exists an L–isometric isomorphism between* *the indicated PL–spaces, well defined by taking $x \otimes y; x \in L_1(X), y \in L_1(Y)$ to the function $(s,t) \mapsto x(s)y(t); (s,t) \in X \times Y$.*

Proof. By virtue of Theorem 5.3, it is sufficient to show that the "classical" isometric isomorphism $I : L_1(X, L_1(Y)) \to L_1(X \times Y)$, taking the $L_1(Y)$-valued integrable function $\bar{x}$ to the function $I(\bar{x}) : (s,t) \mapsto [\bar{x}(s)](t)$, is L–isometric. Since every $u \in H(L_1(X, L_1(Y))$ is (finite) sum of elementary tensors, a routine calculation, using the construction of the PL–norm on relevant L_p–spaces and, of course, Fubini Theorem, shows that indeed $\|u\| = \|I_\infty(u)\|$. □

Remark 5.5. The results of this section can lead to the conjecture that something similar, at least in formulations, can be said in the context of the so-called proto-quantum spaces in the operator space theory (cf. [9, Ch. 2]). It happened that to some extent it is indeed so, but proofs of crucial facts become more complicated. However, we do not discuss it in the present paper.

6. Lambert spaces and the Lambert tensor product

From now on we concentrate on a special type of PL-spaces, which is a non-coordinate form of *Operatorfolgenräume* of Lambert.

If X is a left $\mathcal{B}$-module and $x \in X$, we say that a projection $P \in \mathcal{B}$ is a *support* of x, if $P \cdot x = x$. A *contractive* seminormed left $\mathcal{B}$-module Y is called *semi–Ruan module*, if it has the following "property (sR)": for $x, y \in Y$ with orthogonal supports we have

$$\|x+y\|^2 \leq \|x\|^2 + \|y\|^2,$$

hence for every $x_k \in x; k = 1, \cdots, n$ with pairwise orthogonal supports we have $\| \sum_{k=1}^n x_k\|^2 \leq \sum_{k=1}^n \|x_k\|^2$.

(Semi-Ruan modules were introduced and studied in [10]. Then, in more general context and with more advanced results, they were investigated in [18]. However, earlier the same class of modules actually cropped up in [14]).

Definition 6.1. For a linear space E, a seminorm on HE is called *Lambert seminorm,* or briefly, *L–seminorm*, if the left $\mathcal{B}$-module HE is a semi–Ruan module. In other words, L–seminorm is a PL–seminorm, satisfying the property (sR). The linear space, endowed with an L–seminorm, is called *seminormed Lambert space*, or briefly *seminormed L–space.* In a similar way, we use the term *normed Lambert space (= normed L–space)*, but in this case we usually omit the word "normed".

As to examples of PL–spaces, it is easy to see that $E_{\min}$ is actually an L–space for all normed spaces E, whereas $E_{\max}$ is, generally speaking, not an L–space. The PL–space $L_p(X)$ is an L–space if, and only if $2 \leq p$. (Of course, we suppose here that our measure space is not a single atom). Finally, if K is a pre–Hilbert space, we can endow it with a so-called *Hilbert L-norm*, obtained after identifying HK with $H \otimes_{hil} K$.

The following example will not be used in this paper. However, we mention it because of its importance in the theory of L–spaces.

Example 6.2. ("Concrete L–space"). Suppose that E is given as a subspace of $\mathcal{B}(K,L)$ for some Hilbert spaces K,L. Consider the operator $\gamma : HE \to \mathcal{B}(K, H \dot{\otimes} L)$, well defined by taking $\xi T : \xi \in H, T \in E$ to the operator $x \mapsto \xi \otimes T(x); x \in K$. Introduce a seminorm on HE, setting $\|u\| := \|\gamma(u)\|$. Then it is not difficult to show that $\|\cdot\|$ is actually a norm, making E an L–space.

As a matter of fact, this example is, in a sense, universal: every L–space is L–isometrically isomorphic to some concrete, i.e. operator space. This assertion can be rather quickly derived from the non–coordinate version of the result of Lambert [11, Folgerung 1.3.6] about the embedding of his spaces into products of copies of ℓ_2. However, details are outside the scope of this paper.

From now on we proceed to a special tensor product within the class of L–spaces. As we shall see, its definition is parallel to that of the PL–tensor product, but the resulting object turns out to be a quite different thing.

Let us fix, for a time, two PL-spaces E and F.

Definition 6.3. The tensor product of E and F relative to the class of all normed L–spaces is called *non-completed L–tensor product of our spaces.*

Remark 6.4. This tensor product is a "non–coordinate' version of what Lambert calls maximal tensor product. Indeed, in a sense, it is maximal within a reasonable class of tensor products, and it plays in Lambert's theory a role similar to the role of the operator-projective tensor product in the theory of quantum (= abstract operator) spaces. See details in [11, 3.1.1].

We shall prove the existence of this kind of tensor product, displaying its explicit construction. Such a construction and the crucial Proposition 6.7 can be considered as the "non–coordinate" version of what was done by Lambert.

Recall the diamond product in all its varieties. The following proposition concerns all linear spaces E,F without any additional structure. First, note two identities

$$a\cdot u \diamondsuit b\cdot v = (a\diamondsuit b)(u\diamondsuit v); \quad a\cdot[(b\cdot u)\diamondsuit v] = a(b\diamondsuit \mathbf{1})\cdot(u\diamondsuit v); a,b\in\mathcal{B}, u\in HE, v\in HF \tag{6.1}$$

that can be easily checked on elementary tensors.

Proposition 6.5. *Every $U \in H(E\otimes F)$ can be represented as $a\cdot(\sum_{k=1}^n u_k\diamondsuit v_k)$, $a \in \mathcal{B}, u_k \in HE, v_k \in HF, k = 1,\cdots,n$, where u_k have pairwise orthogonal supports.*

Proof. Represent U as in Proposition 4.3. Choose isometric operators $S_1,\cdots,S_n \in \mathcal{B}$ with pairwise orthogonal final projections. Set

$$a := \sum_{k=1}^n a_k S_k^*\diamondsuit \mathbf{1} \qquad \text{and} \qquad u_k' := S_k\cdot u_k; k = 1,\cdots,n.$$

Since $S_k^* S_l = \delta_k^l \mathbf{1}$, the identities (6.1) imply that

$$a\cdot\left[\sum_{k=1}^n u_k'\diamondsuit v_k\right] = \sum_{k,l=1}^n [a_k(S_k^*\diamondsuit \mathbf{1})]\cdot[(S_l\cdot u_l)\diamondsuit v_l] =$$

$$\sum_{k,l=1}^{n}[a_k(S_k^*\diamondsuit\mathbf{1})(S_l\diamondsuit\mathbf{1})]\cdot[u_l\diamondsuit v_l]=\sum_{k=1}^{n}a_k\cdot(u_k\diamondsuit v_k)=U.$$

Finally, the elements u'_k have orthogonal supports $S_kS_k^*; k=1,\cdots,n$. □

Now, for a given $U\in H(E\otimes F)$, we introduce the number

$$\|U\|_l=\inf\left\{\|a\|\left(\sum_{k=1}^{n}\|u\|^2\|v\|^2\right)^{\frac{1}{2}}\right\},\tag{6.2}$$

where the infimum is taken over all possible representations of U in the form indicated by Proposition 6.5. We distinguish the obvious

Proposition 6.6. *For every $U\in H(E\otimes F)$ and $a\in\mathcal{B}$ we have $\|a\cdot U\|_l\leq\|a\|\|U\|_l$.*

Proposition 6.7. *The function $U\mapsto\|U\|_l$ is a seminorm on $H(E\otimes F)$.*

Proof. Let $U=a\cdot(\sum_{k=1}^{n}u_k^1\diamondsuit v_k^1)$, $V=b\cdot(\sum_{l=1}^{m}u_l^2\diamondsuit v_l^2)$, where the elements $u_k^1\in HE$, respectively $u_l^2\in HE$, have pairwise orthogonal supports P_k, respectively Q_l. Take arbitrary isometric operators $S,T\in\mathcal{B}$ with orthogonal final projections and observe that

$$U+V=(a(S^*\diamondsuit\mathbf{1})+b(T^*\diamondsuit\mathbf{1}))\cdot\left(\sum_{k=1}^{n}(S\cdot u_k^1)\diamondsuit v_k^1+\sum_{l=1}^{m}(T\cdot u_l^2)\diamondsuit v_l^2\right).$$

The elements $S\cdot u_k^1$ have supports SP_kS^*, whereas the elements $T\cdot u_l^2$ have supports TQ_lT^*; hence, taking together, these elements have pairwise orthogonal supports. Therefore, by virtue of Proposition 6.6 and of (6.2), we have that

$$\|U+V\|_l\leq\|a(S^*\diamondsuit\mathbf{1})+b(T^*\diamondsuit\mathbf{1})\|\left(\sum_{k=1}^{n}\|(S\cdot u_k^1)\|^2\|v_k^1\|^2+\sum_{l=1}^{m}\|(T\cdot u_l^2\|^2\|v_l^2\|^2\right)^{\frac{1}{2}}.$$

Combining this with the operator C^*-property, we obtain that

$$\|U+V\|_l\leq(\|a\|^2+\|b\|^2)^{\frac{1}{2}}\left(\sum_{k=1}^{n}\|u_k^1)\|^2\|v_k^1\|^2+\sum_{l=1}^{m}\|u_l^2\|^2\|v_l^2\|^2\right)^{\frac{1}{2}}.$$

Further, obviously we can assume that

$$\|a\|=\left(\sum_{k=1}^{n}\|u_k^1\|^2\|v_k^2\|^2\right)^{\frac{1}{2}}\quad\text{and}\quad\|b\|=\left(\sum_{l=1}^{m}\|u_l^2\|^2\|v_l^2\|^2\right)^{\frac{1}{2}}.$$

Therefore we have

$$\|U+V\|_l\leq\|a\|^2+\|b\|^2=\|a\|\left(\sum_{k=1}^{n}\|u_k^1\|^2\|v_k^1\|^2\right)^{\frac{1}{2}}+\|b\|\left(\sum_{l=1}^{m}\|u_l^2\|^2\|v_l^2\|^2\right)^{\frac{1}{2}}.$$

From this with the help of (6.2) we obtain that $\|U+V\|_l\leq\|U\|_l+\|V\|_l$.

The property of seminorms, concerning the scalar multiplication, is immediate. □

Proposition 6.8. *The module* $(H(E \otimes F), \|\cdot\|_l)$ *has the property (sR).*

Proof. Let $U, V \in H(E \otimes F)$ have orthogonal supports P and Q. Choose their arbitrary suitable representation, say $U = a \cdot (\sum_{k=1}^n u_k^1 \diamondsuit v_k^1)$, $V = b \cdot (\sum_{l=1}^m u_l^2 \diamondsuit v_l^2)$. Take S, T as in Proposition 6.7; then, by a similar argument, we have

$$\|U + V\|_l \leq \|a(S^* \diamondsuit \mathbf{1}) + b(T^* \diamondsuit \mathbf{1})\| \left(\sum_{k=1}^n \|u_k^1\|^2 \|v_k^1\|^2 + \sum_{l=1}^m \|u_l^2\|^2 \|v_l^2\|^2 \right)^{\frac{1}{2}}.$$

Evidently, we can assume that $a = Pa, b = Qb$ and $\|a\| = \|b\| = 1$. Therefore, by the operator C^*-property, we have

$$\|a(S^* \diamondsuit \mathbf{1}) + b(T^* \diamondsuit \mathbf{1})\| = \|Paa^*P + Qbb^*Q\|^{\frac{1}{2}} = \max\{\|a\|, \|b\|\} = 1.$$

Consequently, by (6.2), we have that $\|U + V\|_l^2 \leq \|U\|_l^2 + \|V\|_l^2$. □

Combining the last three propositions, we see that $\|\cdot\|_l$ *is an L–seminorm on* $E \otimes F$. We denote the resulting semi-normed L-space by $E \otimes_l F$.

Like in the "PL–case" (cf. (4.2)), we have the obvious estimation

$$\|u \diamondsuit v\|_l \leq \|u\| \|v\|; \quad u \in HE, v \in HF. \tag{6.3}$$

Consequently, *the canonical bilinear operator* $\vartheta : E \times F \to E \otimes_l F$ *is L-contractive.*

Proposition 6.9. *Let G be an L-space, $\mathcal{R} : E \times F \to G$ an L–bounded bilinear operator, $R : E \otimes_{pl} F \to G$ the associated linear operator. Then R is L-bounded, and* $\|\mathcal{R}\|_{lb} = \|R\|_{lb}$.

Proof. Take $U \in H(E \otimes_l F))$ and represent it as in Proposition 6.5. Since R_∞ is a $\mathcal{B}$-module morphism, and $R_\infty(u_k \diamondsuit v_k) = \mathcal{R}_\infty(u_k, v_k)$ for all k, we have that $R_\infty(U) = a \cdot (\sum_{k=1}^n \mathcal{R}_\infty(u_k, v_k))$.

Now look at $\mathcal{R}_\infty(u_k, v_k) \in HG$ for some k. Obviously we have

$$(P_k \diamondsuit \mathbf{1}) \cdot \mathcal{R}_\infty(u_k, v_k) = \mathcal{R}_\infty(P_k \cdot u_k, v_k).$$

This implies that the elements $\mathcal{R}_\infty(u_k, v_k) \in HG$ have pairwise orthogonal supports, namely $P_k \diamondsuit \mathbf{1}$. Therefore, since G is an L-space, we have

$$\|R_\infty(U)\| \leq \|a\| \left(\sum_{k=1}^n \|\mathcal{R}_\infty(u_k, v_k)\|^2 \right)^{\frac{1}{2}} \leq$$

$$\|a\| \left(\sum_{k=1}^n \|\mathcal{R}\|_{lb}^2 \|u_k\|^2 \|v_k\|^2 \right)^{\frac{1}{2}} = \|\mathcal{R}\|_{lb} \|a\| \left(\sum_{k=1}^n \|u_k\|^2 \|v_k\|^2 \right)^{\frac{1}{2}}.$$

From this, using (6.2), we obtain the estimate $\|R_\infty(U)\| \leq \|\mathcal{R}\|_{lb} \|U\|_l\|$. Consequently, $\|R\|_{lb} \leq \|\mathcal{R}\|_{lb}$. The converse inequality easily follows from (6.3). □

Proposition 6.10. (As a matter of fact), $\|\cdot\|_l$ *is a norm.*

Proof. Since $\mathbb{C}$ is an L–space, the argument in Proposition 4.6 works with obvious modifications. □

Combining Propositions 6.6–6.10, we immediately obtain

Theorem 6.11. (Existence theorem) *The pair* $(E \otimes_l F, \vartheta)$ *is a non-completed* L–*tensor product of* E *and* F*.*

The non-completed L–tensor product has an obvious "completed" version. The definition of the *completed* L–*tensor product of two* PL–*spaces* and the relevant existence theorem repeat what was said about completed PL–tensor product, only we replace "PL" by "L" and the subscript "*pl*" by "l". Thus, *the completed* L–*tensor product of two* PL–*spaces* E *and* F *exists, and it is the pair* $(E \widehat{\otimes}_l F, \widehat{\vartheta})$, *where* $E \widehat{\otimes}_p F$ *is the completion of the* L–*space* $E \otimes_l F$, *and* $\widehat{\vartheta}$ *acts as* ϑ, *but with range* $E \widehat{\otimes}_l F$.

Remark 6.12. We do not discuss here the non-coordinate version of another tensor product, the so–called minimal, introduced in [11, 3.1.3]. Unlike the maximal tensor product (cf. Remark 6.4), it corresponds, in a sense, to the operator–injective tensor product in the operator space theory.

7. The Lambert tensor product of Hilbert spaces

As we have seen before, the PL–tensor product is especially good for maximal PL–spaces and L_1-spaces with their specific PL–norm. Here we shall show that, in the same sense, the L–tensor product is good for Hilbert spaces with the minimal L–norm, that is $\|\cdot\|_{\min}$ of Example 2.4. *Throughout this section, all Hilbert and pre–Hilbert spaces are supposed to be endowed with that* L–*norm.*

We shall use one of equivalent definitions of the minimal L–norm. It is a particular case of the definition of the norm in the injective tensor product $E \otimes_i F$ of two normed spaces, expressed by means of an injective operator $E \otimes_i F \to \mathcal{B}(E^*, F)$ (see, e.g.,[2, pp. 62-63]). In the particular case of a pre-Hilbert space K we obtain the following observation that we distinguish for the convenience of references.

Proposition 7.1. *There is an isometric operator* $\mathcal{I} : HK \to \mathcal{B}(K^{cc}, H)$, *well defined by* $\xi x \mapsto \xi \circ x$.

This, in its turn, implies

Proposition 7.2. (i) *For* $u = \sum_{k=1}^n \lambda_k \xi_k x_k \in HK$, *where* $\lambda_k \in \mathbb{C}$ *and* ξ_k, x_k *are orthonormal systems in* H *and* K *respectively, we have* $\|u\| = \max\{|\lambda_k|; k = 1, \cdots, n\}$.

(ii) *Every* $u \in HK$ *can be represented as* $\sum_{k=1}^n s_k \xi_k x_k$, *where* ξ_k *and* x_k *are orthonormal systems in* H *and* K *respectively,* $s_1 \geq s_2 \geq \cdots \geq s_n > 0$.

Proof. (i) is immediate. To prove (ii), we recall that $\mathcal{I}(u)$, being a *finite rank* operator between pre–Hilbert spaces, has the form $\sum_{k=1}^n s_k \xi_k \circ x_k$, where ξ_k, x_k and s_k have the indicated properties. (E.g., the argument in [7, Section 3.4] works with obvious modifications). It follows that u have the desired representation. □

Proposition 7.3. *Let* K *and* L *be pre-Hilbert spaces. Then the canonical bilinear operator* $\vartheta : K \times L \to K \otimes_{hil} L : (x, y) \mapsto x \otimes y$ *is* L*-contractive.*

Proof. As we know, $\vartheta_\infty : HK \times HL \to H(K\otimes_{hil}L)$ takes a pair (u,v) to $u\diamondsuit v$. By Proposition 7.2(ii), u has the form $\sum_{k=1}^n s_k\xi_k x_k$ with the mentioned properties, and v has the form $\sum_{l=1}^m s'_l\eta_l y_l$ with similar properties. Consequently,

$$u\diamondsuit v = \sum_{k=1}^{n}\sum_{l=1}^{m} s_k s'_l(\xi_k\diamondsuit\eta_l)(x_k\otimes y_l),$$

where the systems $\xi_k\diamondsuit\eta_l$ and $x_k\otimes y_l$ are orthonormal in H and $K\otimes_{hil}L$, respectively. Therefore, by Proposition 7.2(i), $\|u\diamondsuit v\| = s_1 s'_1 = \|u\|\|v\|$. □

Theorem 7.4. *Let K and L be pre–Hilbert spaces. Then we have $K\otimes_l L = K\otimes_{hil}L$ and $K\widehat{\otimes}_l L = K\widehat{\otimes}_{hil}L$. Both equalities are up to an L–isometric isomorphism, well defined by taking an elementary tensor $x\otimes y$ to the same $x\otimes y$, but considered in $K\otimes_{hil}L$ and $K\widehat{\otimes}_{hil}L$, respectively.*

Proof. Since ϑ from the previous proposition has values in an L–space, it gives rise to the L–contractive operator $R : K\otimes_l L \to K\otimes_{hil}L$, which is the identity map of the underlying linear spaces. Our task is to show that it is an L–isometric isomorphism.

Take $U \in H(K\otimes_l L)$. Since it is the sum of several elementary tensors of the form $\xi(x\otimes y)$, it easily follows that U can be represented as $\sum_{k=1}^n\sum_{l=1}^m \xi_{kl}(x_k\otimes y_l)$, where x_k and y_l are orthonormal systems in K and L, respectively, and $\xi_{kl} \in H$. Applying to $R_\infty(U) \in H(K\otimes_{hil}L)$ Proposition 7.1, we see that $\|R_\infty(U)\|$ is the norm of the operator $S := \sum_{k=1}^n\sum_{l=1}^m \xi_{kl}\circ(x_k\otimes y_l) : (K\otimes_{hil}L)^{cc} \to H$. Set $M := span\{x_k\otimes y_l\} \subset (K\otimes_{hil}L)^{cc}$. Since $\dim M < \infty$, the pre–Hilbert space $(K \dot{\otimes} L)^{cc}$ decomposes as $M\oplus M^\perp$, and S takes $M^\perp$ to 0. Therefore $\|S\| = \|S_0\|$, where S_0 is the restriction of S to M. Thus, we have

$$\|R_\infty(U)\| = \|S_0\|. \tag{7.1}$$

Now return to our initial U. Choose arbitrary orthonormal systems $\eta_k; k = 1,\cdots,n$ and $\zeta_l; l = 1,\cdots,m$ in H. Set $u := \sum_{k=1}^n \eta_k x_k \in HK$ and $v := \sum_{l=1}^m \zeta_l y_l \in HL$. We see that $u\diamondsuit v = \sum_{k=1}^n\sum_{l=1}^m(\eta_k\diamondsuit\zeta_l)(x_k\otimes y_l)$. Consider the finite rank operator

$$T := \sum_{k=1}^{n}\sum_{l=1}^{m}\xi_{kl}\circ(\eta_k\diamondsuit\zeta_l) : H \to H.$$

An easy calculation shows that $T\cdot(u\diamondsuit v) = U$. Further, as a particular case of Proposition 7.1, $\|u\| = \|v\| = 1$. Finally, if we set $N := span\{\eta_k\diamondsuit\zeta_l\}$ and denote by T_0 the restriction of T to N, we obviously have $\|T\| = \|T_0\|$. Combining this with (6.2), we obtain that

$$\|U\|_l \le \|T_0\|. \tag{7.2}$$

The systems $\{x_k\otimes y_l\}$ and $\{\eta_k\diamondsuit\zeta_l\}$ are orthonormal bases in M and N, respectively, and $S_0(x_k\otimes y_l) = \xi_{kl} = T_0(\eta_k\diamondsuit\zeta_l)$. It follows that $\|T_0\| = \|S_0\|$. Combining this with (7.1) and (7.2), and remembering that R_∞ is contractive, we obtain that $\|U\|_l = \|R_\infty(U)\|$.

This gives the first of L–isometric isomorphisms, claimed in the theorem. The extension of the latter by continuity provides the second L–isometric isomorphism. □

8. Comparison of both tensor products

In conclusion, we want to compare PL– and L–tensor products. Since the class of PL-spaces is larger than that of L-spaces, it immediately follows from the definition of both tensor products in terms of their universal properties, that $\|\cdot\|_{pl} \geq \|\cdot\|_l$. We shall show that the first number is sometimes essentially greater than the second number.

Endow the space ℓ_2 with the Hilbert L–norm (see above), and the space $\ell_1 = L_1(\mathbb{N})$ with the PL-norm from Example 2.5.

Proposition 8.1. *The bilinear operator $\mathcal{M} : \ell_2 \times \ell_2 \to \ell_1$, acting as the coordinate-wise multiplication, is L-contractive.*

Proof. Our task is to show that the bilinear operator $\mathcal{M}_\infty : H\ell_2 \times H\ell_2 \to H\ell_1$ is contractive. Consider the isometric operators $I : H\ell_2 = H \otimes_{hil} \ell_2 \to \ell_2(H)$ and $\alpha : H\ell_1 \to \ell_1(H)$; both are well defined by taking an elementary tensor $\xi\widetilde{\lambda}; \widetilde{\lambda} = (\dots, \lambda_n, \dots)$ to $(\dots, \lambda_n \xi, \dots)$. Consider the diagram

$$\begin{array}{ccc} H\ell_2 \times H\ell_2 & \xrightarrow{I \times I} & \ell_2(H) \times \ell_2(H) \\ \downarrow{\scriptstyle \mathcal{M}_\infty} & & \downarrow{\scriptstyle \mathcal{S}} \\ H\ell_1 & \xrightarrow{\alpha} & \ell_1(H) \end{array} ,$$

where $\mathcal{S}$ takes a pair $(\widetilde{\xi} := (\dots, \xi_n, \dots), \widetilde{\eta} := (\dots, \eta_n, \dots))$ to $(\dots, \xi_n \diamondsuit \eta_n, \dots)$. Since the Cauchy-Schwarz inequality implies that $\|\mathcal{S}(\widetilde{\xi}, \widetilde{\eta})\| \leq \|\widetilde{\xi}\| \|\widetilde{\eta}\|$, the latter sequence indeed belongs to $\ell_1(H)$, and, moreover, $\mathcal{S}$ is contractive.

Now observe that our diagram, as one can easily verify on elementary tensors, is commutative. Consequently, for $u, v \in H\ell_2$ we have $\|\mathcal{M}_\infty(u, v)\| \leq \|u\| \|v\|$. □

Have a look at the PL– and L–tensor square of the same Hilbert L–space ℓ_2. Fix $n \in \mathbb{N}$, denote by $\mathbf{p}^m; m = 1, 2, \cdots$ sequences $(\dots, 0, 1, 0, \dots) \in \ell_2$ and choose an arbitrary orthonormal system, say $e_1, e_2, \dots$, in H. In what follows, we set

$$V := \sum_{k=1}^{n} e_k (\mathbf{p}^k \otimes \mathbf{p}^k) \in H(\ell_2 \otimes \ell_2).$$

It obviously can be presented as

$$V = S \cdot \sum_{k=1}^{n} u_k \diamondsuit v_k, \tag{8.1}$$

where $u_k = v_k = e_k \mathbf{p}^k$, and $S = \sum_{k=1}^{n} e_k \circ (e_k \diamondsuit e_k)$.

Proposition 8.2. *We have $\|V\|_{pl} = n$.*

Proof. Consider the operator $M : \ell_2 \otimes_{pl} \ell_2 \to \ell_1$, associated with $\mathcal{M}$ from the previous proposition. Then it is L–contractive together with the latter: in particular, $\|M_\infty(V)\| \leq \|V\|_{pl}$. But we obviously have $M_\infty(V) = \sum_{k=1}^n e_k(\mathbf{p}^k)$; therefore, since we are in $L\ell_1$, we have $\|M_\infty(V)\| = n$. Consequently, $\|V\|_{pl} \geq n$. On the other hand, it follows from (8.1) that $\|V\|_{pl} \leq \sum_{k=1}^n \|S\|\|u_k\|\|v_k\| = n$. □

Proposition 8.3. (At the same time) *we have* $\|V\|_l = \sqrt{n}$.

Proof. Consider the bilinear operator $\mathcal{N} : \ell_2 \times \ell_2 \to \ell_2$, acting as $\mathcal{M}$, but with the other range. Since the norm of an element in $H\ell_1$ can only decrease, if we shall consider this element in $H\ell_2$, our $\mathcal{N}$ is L–contractive together with $\mathcal{M}$. But ℓ_2 (contrary to ℓ_1!) is an L–space; therefore the operator $N : \ell_2 \otimes_l \ell_2 \to \ell_2$, associated with $\mathcal{N}$, is also L–contractive. In particular, $\|N_\infty(V)\| \leq \|V\|_l$. Of course, $N_\infty(V)$ is the same $\sum_{k=1}^n e_k(\mathbf{p}^k)$ as in the previous proposition. However, since now we are in $H\ell_2 = H \otimes_{hil} \ell_2$, we have $\|N_\infty(V)\| = \sqrt{n}$, and hence $\|V\|_l \geq \sqrt{n}$.

On the other hand, since the elements $u_k \in H\ell_2$ form an orthonormal system, we obtain, by (8.1), that $\|V\|_l \leq \|S\|(\sum_{k=1}^n \|u_k\|^2\|v_k\|^2)^{\frac{1}{2}} = \sqrt{n}$. □

Nevertheless, despite PL– and L–tensor products of the same PL-spaces usually have essentially different norms, their underlying spaces coincide:

Proposition 8.4. *Let E and F be PL–spaces. Then the identity operator on the linear space $E \otimes F$ is an isometric isomorphism, being considered as an operator between underlying spaces of PL–spaces $E \otimes_{pl} F$ and $E \otimes_l F$.*

Proof. Since the latter operator is obviously contractive, our task is to show that its inverse operator is also contractive. Denote by G be the underlying normed space of $E \otimes_{pl} F$, endowed with the minimal PL-norm (see Example 2.4). Since $\vartheta : E \times F \to E \otimes_{pl} F$ is L–contractive, the same is true, if we consider ϑ with range G. But, as we know, G is an L–space. Therefore ϑ gives rise to the L–contractive operator between $E \otimes_l F$ and G, which is contractive as an operator between the underlying normed spaces. But the latter is, of course, the desired inverse operator. □

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