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COMPLETELY POSITIVE CONTRACTIVE MAPS AND PARTIAL ISOMETRIES

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Dedicated to the memory of Uffe Haagerup

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Abstract. Associated with a completely positive contractive map φ of a C^*-algebra A is a universal C^*-algebra generated by the C^*-algebra A along with a contraction implementing φ. We prove a dilation theorem: the map φ may be extended to a completely positive contractive map of an augmentation of A. The associated C^*-algebra of the augmented system contains the original universal C^*-algebra as a corner, and the extended completely positive contractive map is implemented by a partial isometry.

Introduction

The Cuntz–Pimsner C^*-algebras naturally associated with a completely positive contractive (cpc) map of a C^*-algebra are considered. A C^*-algebra and cpc map may be viewed as a dynamical system, and the Cuntz–Pimsner C^*-algebra may be viewed as a crossed product C^*-algebra of this system; it is a universal C^*-algebra that is generated by the given C^*-algebra along with a contraction implementing the action of the completely positive map. Such a crossed product for this general setting is introduced and explored in [11]. We show below that up to a Morita equivalent crossed product C^*-algebra the implementing contraction may be replaced by a partial isometry. The strategy may be summarized as follows. The given cpc map of a C^*-algebra may be extended to a cpc map of

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an 'augmentation' of the given C^*-algebra. This (extended) cpc map is implemented by a partial isometry in the Cuntz–Pimsner C^*-algebra associated with the augmented dynamical system. The Cuntz–Pimsner C^*-algebra of the original dynamical system is a corner in the C^*-algebra of the augmented dynamical system, and the two Cuntz–Pimsner C^*-algebras are Morita equivalent. The approach involving the augmented C^*-algebra extends to a general setting a process found in [3]. Various examples involving these results will be collected in a separate paper, however we note now that the Morita equivalence result of [4] follows from the considerations below. Also note ([11] Subsections 3.4, 3.5, Section 4) that Cuntz–Pimsner algebras arising from systems defined by completely positive maps include crossed products by endomorphisms ([12]), Exel's crossed products ([6]), and when A is commutative, correspond to C^*-algebras of topological relations ([2]). It is established in [11] that C^*-algebras of many discrete graphs are included in this context.

The paper is organized as follows. The first section includes preliminaries on the C^*-correspondence and the resultant universal Cuntz–Pimsner C^*-algebras associated with a completely positive contraction φ on a C^*-algebra A, namely a dynamical system (A, φ).

The second section introduces an augmented C^*-algebra A_q, obtained by adjoining a projection to A, and a completely positive contraction $\widetilde{\varphi}$ defined on A_q extending a given cpc map φ on A. From here on, for technical simplicity, the C^*-algebra A is assumed to be unital, although the augmented C^*-algebra is non-unital. Any representation of the correspondence for this augmented system implements $\widetilde{\varphi}$ through a partial isometry with initial projection given by the unit of A.

The following sections, Section 3 and Section 4, first involve a restriction, and then an inducing process, that are used to establish an isomorphism result. A natural restriction process is described in the Section 3; namely representations of the correspondence for the augmented dynamical system $(A_q, \widetilde{\varphi})$ restrict to representations of the correspondence for the original system (A, φ). Relationships between the ideals of compact adjointable operators for the two correspondences and the various coisometry ideals of A and A_q defining the relative Cuntz–Pimsner C^*-algebras are explored. This is accomplished by introducing a technically useful intermediate correspondence. Section 4 proceeds by forming an induced representation of the C^*-correspondence associated with the augmented dynamical system from a representation of the original system. The section concludes with the isomorphism of a Cuntz–Pimsner C^*-algebra of the given system (A, φ) with a corner of the Cuntz–Pimsner C^*-algebra of the augmented system.

Section 5 briefly considers some special cases of a cpc map φ on a C^*-algebra A, namely those that map the unit of A to a projection, those that are unital, and specializing further, those that are $*$-endomorphisms of A.

In Section 6, given an ideal of coisometry in A, a quotient system (A_1, φ_1) of the augmented system is formed. The main result here is that the universal Cuntz–Pimsner C^*-algebra of this quotient system is isomorphic to the Cuntz–Pimsner C^*-algebra of the augmented dynamical system.

1. Preliminaries

1.1. **Cuntz–Pimsner C^*-algebras.** The construction of Cuntz–Pimsner C^*-algebras is based on C^*-correspondences. We include some notation and background for Cuntz–Pimsner C^*-algebras associated with a correspondence over a C^*-algebra, and refer to [17], [15], [7], [9] and the references therein for further details. A C^*-correspondence from A to B, denoted ${}_A\mathcal{E}_B$, is a Hilbert B-module $\mathcal{E}_B$ along with a specified $*$-homomorphism $\phi : A \to \mathcal{L}(\mathcal{E}_B)$. A $B - B$ correspondence ${}_B\mathcal{E}_B$ is referred to as a 'C^*-correspondence over B'. A C^*-algebra C may be viewed as a correspondence over itself; the Hilbert C-module structure is given by $\langle a, b\rangle = a^*b$ for $a, b \in C$. If ${}_B\mathcal{E}_B$ is a C^*-correspondence over a C^*-algebra B then a representation $(T, \pi) : \mathcal{E} \to C$ of ${}_B\mathcal{E}_B$ in a C^*-algebra C is a $*$-homomorphism $\pi : B \to C$ along with a linear map $T : \mathcal{E} \to C$ which is a bimodule map and (when viewing C as correspondence over itself) intertwines the inner products: so the pair (T, π) satisfies the covariance conditions $T(\phi(b)x) = \pi(b)T(x)$, $T(xb) = T(x)\,\pi(b)$, and $T^*(x)T(y)$ (which equals $\langle T(x), T(y)\rangle_C) = \pi(\langle x, y\rangle_B)$ for $b \in B$, $x, y \in \mathcal{E}$. The C^*-subalgebra of C generated by $T(\mathcal{E}) \cup \pi(B)$ is denoted $C^*(T, \pi)$. The representation (T, π) is called injective if π is injective.

Given a C^*-correspondence $\mathcal{E}$ over B, $\mathcal{L}(\mathcal{E})$ denotes the C^*-algebra of adjointable linear operators and $\mathcal{K}(\mathcal{E})$ denotes its closed two-sided ideal of 'compact' operators generated by $\{\theta_{x,y} | x, y \in \mathcal{E}\}$, where $\theta^{\mathcal{E}}_{x,y}(z) = x\,\langle y, z\rangle$, $(z \in \mathcal{E})$. A representation $(T, \pi) : \mathcal{E} \to C$ in a C^*-algebra C yields a $*$-homomorphism $\Psi_T : \mathcal{K}(\mathcal{E}) \to C$ determined by $\theta_{x,y} \to T(x)T^*(y)$. Denote the ideal $\phi^{-1}(\mathcal{K}(\mathcal{E}))$ of B by $J(\mathcal{E})$. Given an ideal K contained in $J(\mathcal{E})$ we say that a representation $(T, \pi) : \mathcal{E} \to C$ is coisometric on K if $\Psi_T(\phi(b)) = \pi(b)$ for all $b \in K$. Given a C^*-correspondence $\mathcal{E}$ over B there is a representation $(T_{\mathcal{E}}, \pi_{\mathcal{E}})$ of $\mathcal{E}$ in a C^*-algebra which is coisometric on K and universal among all such representations of $\mathcal{E}$; namely if (T, π) is a representation of $\mathcal{E}$ in a C^*-algebra C coisometric on K then there is a $*$-homomorphism $\rho : C^*(T_{\mathcal{E}}, \pi_{\mathcal{E}}) \to C$ with $(T, \pi) = \rho \circ (T_{\mathcal{E}}, \pi_{\mathcal{E}})$, where $\rho \circ (T_{\mathcal{E}}, \pi_{\mathcal{E}})$ denotes the representation $(\rho \circ T_{\mathcal{E}}, \rho \circ \pi_{\mathcal{E}})$ of $\mathcal{E}$. We remark that if K is an ideal contained in $J(\mathcal{E})$ and (T, π) a representation of $\mathcal{E}$ coisometric on K then ideal $K \cap (\ker\phi) \subseteq \ker\pi$. The universal C^*-algebra $C^*(T_{\mathcal{E}}, \pi_{\mathcal{E}})$, called the relative Cuntz–Pimsner algebra of $\mathcal{E}$ (determined by K), is denoted $\mathcal{O}(K, \mathcal{E})$ ([15], and [7] Proposition 1.3).

For the ideal $K = 0$ the universal Cuntz–Pimsner algebra of $\mathcal{E}$ is often referred to as the Toeplitz algebra of the correspondence, and is denoted $\mathcal{T}_{\mathcal{E}}$. For the ideal $J_{\mathcal{E}} = \phi^{-1}(\mathcal{K}(\mathcal{E})) \cap (\ker\phi)^{\perp}$ of B (where for an ideal J of a C^*-algebra A, $J^{\perp}$ denotes the ideal $\{a \in A \mid ab = 0, (b \in J)\}$) the universal Cuntz–Pimsner C^*-algebra $\mathcal{O}(J_{\mathcal{E}}, \mathcal{E})$ determined by this ideal is denoted $\mathcal{O}_{\mathcal{E}}$. The universal representation $(T_{\mathcal{E}}, \pi_{\mathcal{E}}) : \mathcal{E} \to \mathcal{O}(K, \mathcal{E})$ coisometric on the ideal K is injective if and only if $K \subseteq J_{\mathcal{E}}$ ([15] Proposition 2.21 and [9] Proposition 3.3).

1.2. **Crossed products by completely positive maps.** A linear map of C^*-algebras $\varphi : A \to B$ is positive if it maps the positive cone A_+ of A to the positive cone B_+ of B, so $\varphi(a^*a) \in B_+$ for all $a \in A$. Such a map is necessarily Hermitian ($*$-preserving) and bounded. The map is completely positive, abbreviated cp, if its amplification to n by n matrices, $\varphi_n : \mathrm{M}_n(A) \to \mathrm{M}_n(B)$ defined by $\varphi_n([a_{i,j}]) = [\varphi(a_{i,j})]$, is a positive map for all $n \in \mathbb{N}$, in which case it is bounded. If the map has bound 1, in other words a contraction, then φ is a completely positive contraction, abbreviated cpc (see for example, [1], [13], [16]).

Associated with a completely positive map $\varphi : A \to B$, a specialization of the KSGNS construction ([13]) yields a C^*-correspondence ${}_A\mathcal{E}_B$ from A to B. For example, set $\mathcal{E}_B$ to be the Hilbert B-module $A \otimes_\varphi B$ obtained by completing the quotient inner-product right B-module $(A \otimes_{\mathrm{alg}} B)/N$ where the B-valued inner product is given on simple tensors in the algebraic tensor product $A \otimes_{\mathrm{alg}} B$ by

$$\langle r \otimes u, s \otimes v \rangle = \langle u, \varphi(\langle r, s \rangle) v \rangle_B \text{ for } r, s \in A \text{ and } u, v \in B,$$

and where N is the subspace of elements $z \in A \otimes_{\mathrm{alg}} B$ with $\langle z, z \rangle = 0$. There is a unital C^*-homomorphism φ_* from the multiplier algebra $\mathcal{M}(A)$ to the adjointable operators $\mathcal{L}(A \otimes_\varphi B)$ determined by $\varphi_*(a)(r \otimes u) = ar \otimes u$ ([13]). Denoting the restriction of φ_* to A by ϕ_φ, or ϕ if the context is clear, describes a left action of A on the Hilbert module $\mathcal{E}_B$, yielding a C^*-correspondence ${}_A\mathcal{E}_B$ from A to B. Note that this correspondence does not require assumptions on A being unital, or additional continuity properties for φ.

Definition 1.1. A cp system is a pair (A, φ) where $\varphi : A \to A$ is a cp map of a C^*-algebra A. For $\varphi : A \to A$ a cp map the C^*-correspondence $A \otimes_\varphi A$ over A is denoted $\mathcal{E}_\varphi$. To make clear the underlying system denote the universal C^*-algebras $\mathcal{T}_{\mathcal{E}_\varphi}$ by $\mathcal{T}_{(A,\varphi)}$ and $\mathcal{O}_{\mathcal{E}_\varphi}$ by $\mathcal{O}_{(A,\varphi)}$.

We note that Theorem 3.13 of [11] provides an alternative description of $\mathcal{O}_{(A,\varphi)}$ without the use of correspondences.

We show in Proposition 1.4 below, that for a given ideal $K \subseteq J(\mathcal{E}_\varphi)$ the universal relative C^*-algebra $\mathcal{O}(K, \mathcal{E}_\varphi)$ of the A-A correspondence $\mathcal{E}_\varphi$ is an isomorphism invariant for a basic equivalence relation on cp systems described by intertwining $*$-isomorphisms.

Definition 1.2. Two cp systems (A, φ) and (B, ψ) are equivalent if there is a $*$-isomorphism $\gamma : A \to B$ with $\gamma \circ \varphi = \psi \circ \gamma$.

Lemma 1.3. *Assume cp systems (A, φ) and (B, ψ) are equivalent via $\gamma : A \to B$. Then $J(\mathcal{E}_\varphi) = \gamma^{-1}(J(\mathcal{E}_\psi))$ and $J_{\mathcal{E}_\varphi} = \gamma^{-1}(J_{\mathcal{E}_\psi})$.*

Proof. Assume $\gamma \circ \varphi = \psi \circ \gamma$. We have

$$\gamma(\langle r \otimes_\varphi u, s \otimes_\varphi v \rangle) = \langle \gamma(r) \otimes_\psi \gamma(u), \gamma(s) \otimes_\psi \gamma(v) \rangle\,,$$

so since a $*$-isomorphism is norm preserving, there is a $\mathbb{C}$-linear isometric isomorphism $\xi : \mathcal{E}_\varphi \to \mathcal{E}_\psi$ determined by mapping $r \otimes_\varphi u$ in $\mathcal{E}_\varphi$ to $\gamma(r) \otimes_\psi \gamma(u)$ in $\mathcal{E}_\psi$. We have $\xi(ar \otimes_\varphi u) = \phi_\psi(\gamma(a))\xi(r \otimes_\varphi u)$, so

$$\xi \circ \phi_\varphi(a) = \phi_\psi(\gamma(a)) \circ \xi \text{ for } a \in A,$$

and therefore $\ker(\phi_\varphi) = \gamma^{-1}(\ker(\phi_\psi))$.

Computations show that $\xi(x \cdot a) = \xi(x)\gamma(a)$ and $\xi \circ \theta_{x,y} = \theta_{\xi x, \xi y} \circ \xi$ for $x, y \in \mathcal{E}_\varphi$ and $a \in A$, yielding $\xi \circ \mathcal{K}(\mathcal{E}_\varphi) \circ \xi^{-1} = \mathcal{K}(\mathcal{E}_\psi)$. The conclusions follow by combining this with the preceding displayed expression. ∎

Proposition 1.4. *Assume cp systems* (A, φ) *and* (B, ψ) *are equivalent via a* $*$-*isomorphism* $\gamma : A \to B$. *If* $K \trianglelefteq J(\mathcal{E}_\varphi)$ *then* $\mathcal{O}(K, \mathcal{E}_\varphi) \cong \mathcal{O}(\gamma^{-1}(K), \mathcal{E}_\psi)$. *In particular the universal* C^*-*algebras* $\mathcal{T}_{(A,\varphi)} \cong \mathcal{T}_{(B,\psi)}$ *and* $\mathcal{O}_{(A,\varphi)} \cong \mathcal{O}_{(B,\psi)}$.

Proof. For a representation (T, π) of $\mathcal{E}_\varphi$ in a C^*-algebra C coisometric on K define $S = T \circ \xi^{-1}$ and $\sigma = \pi \circ \gamma^{-1}$, where $\xi : \mathcal{E}_\varphi \to \mathcal{E}_\psi$ is the isometric isomorphism of the previous Lemma. Using the identities in the previous Lemma it is routine to check that (S, σ) is a representation of $\mathcal{E}_\psi$ in C coisometric on $\gamma^{-1}(K)$. □

If φ is such that the homomorphism ϕ implementing the left action of A on $\mathcal{E}_\varphi$ is injective then the ideal $J_{\mathcal{E}_\varphi} = J(\mathcal{E}_\varphi)$. We include the following lemma for completeness although it is generally known (cf. Remark 3.7 [11]).

Lemma 1.5. *Let* $\varphi : A \to A$ *be a cp map and consider the associated* C^*-*correspondence* $\mathcal{E}_\varphi$ *over* A *with its left action homomorphism* $\phi : A \to \mathcal{L}(\mathcal{E}_\varphi)$. *Then*

$$\ker \phi = \{a \in A \mid \varphi(b^* a^* ab) = 0, (b \in A)\},$$

and this ideal is contained in the subspace $\ker \varphi$.

Proof. The element $\phi(a) \in \mathcal{L}(\mathcal{E}_\varphi)$ is zero if and only if $\phi(a)(b \otimes c) \in N$, the subspace of elements $z \in A \otimes_{\text{alg}} A$ with $\langle z, z\rangle = 0$, for all simple tensors $b \otimes c$ in $A \otimes_{\text{alg}} A$. Therefore $\phi(a) = 0$ if and only if $c^* \varphi(b^* a^* ab)c = 0$ for all $b, c \in A$. Now let c run through an approximate unit of A to obtain $\varphi(b^* a^* ab) = 0$ for all $b \in A$. To obtain the second statement, let b run through an approximate identity of A. It follows that $\ker \phi$ is contained in the left ideal $\{a \in A \mid \varphi(a^* a) = 0\}$, which is in turn contained in $\ker \varphi$ by Kadison's inequality ([1] p. 129). □

For our purposes we restrict attention to cp maps that are contractive, so cpc maps φ on A.

Definition 1.6. A cpc system is a pair (A, φ) where $\varphi : A \to A$ is a cp contractive map of a C^*-algebra A.

If A is not unital one can consider the multiplier algebra $\mathcal{M}(A)$ of A and assume, for example, that φ is a strict cpc map from A to $\mathcal{M}(A)$, which is our standing assumption from now on. Recall that strict means that $\varphi(e_\lambda)$ is strictly Cauchy in $\mathcal{M}(A)$ for some approximate unit $\{e_\lambda \mid \lambda \in \Lambda\}$ of A. As φ is contractive, and positive, and the unit ball of $\mathcal{M}(A)$ is strictly complete, this means $\varphi(e_i)$ converges strictly to a positive element in the unit ball of $\mathcal{M}(A)$. If φ is strict then φ extends to a cpc map of $\mathcal{M}(A)$ ([13]) or, if something less encompassing is required, of the smallest unital φ-invariant C^*-subalgebra of $\mathcal{M}(A)$ containing A. Beginning in Section 2, the C^*-algebra A in the cpc system (A, φ) is assumed unital (with unit element denoted p). Note in that section we introduce an "augmented" cpc system $(A_q, \widetilde{\varphi})$ associated with a cpc system (A, φ) where the C^*-algebra A_q is not unital, however there the cpc map $\widetilde{\varphi}$ is easily seen to be strict.

A simple illustration of Proposition 1.7 below occurs if A is unital (with unit p). Let $(T, \pi) : \mathcal{E}_\varphi \to C$ be a representation of the correspondence $\mathcal{E}_\varphi$ associated with the cpc system (A, φ) in a C^*-algebra C. Setting $T(p \otimes_\varphi p) = \mathbf{T} \in C$ it follows, using $T^*(x)T(y) = \pi(\langle x, y\rangle_A)$ for $x, y \in \mathcal{E}$, that $\mathbf{T}^*\mathbf{T} = \pi(\langle p \otimes_\varphi p, p \otimes_\varphi p\rangle_A) = \pi(\varphi(p))$ and

$$\mathbf{T}^*\pi(a)\mathbf{T} = \pi(\langle p \otimes_\varphi p, a \otimes_\varphi p\rangle_A) = \pi(\varphi(a)).$$

Since $\|\varphi\| \leq 1$, $\mathbf{T}$ is a contraction in C, and $\mathbf{T}$ implements the map φ in the image of A in C. Note also that since $\pi(p)\mathbf{T} = \mathbf{T}\pi(p) = \mathbf{T}$ we have that $\pi(p)$ is the unit for the C^*-algebra $C^*(T, \pi)$. Since T is an A bimodule map it is clear that the C^*-algebra $C^*(T, \pi)$ generated by $T(\mathcal{E}) \cup \pi(B)$ is the C^*-algebra $C^*(\mathbf{T}, \pi)$ generated by the contraction $\mathbf{T}$ and the C^*-algebra A.

In Proposition 1.7 we formalize this observation for general (A, φ) with φ strict. Recall that a $*$-homomorphism $\pi : A \to C$ of C^*-algebras is nondegenerate if $\pi : A \to \mathcal{L}(C)$ (so viewing C as a module over itself), is nondegenerate: so $\pi(A)C$ is dense in C. The argument follows in a similar fashion to Proposition 3.10 of [11].

Proposition 1.7. *Let $\mathcal{E}_\varphi$ be the C^*-correspondence over A associated with a cp system (A, φ) where φ is strict. There is a one-to-one correspondence between representations $(T, \pi) : \mathcal{E}_\varphi \to C$ (with π nondegenerate) and pairs $(\mathbf{T}, \pi)$ (which can viewed as representations of (A, φ)) where $\mathbf{T} \in \mathcal{M}(C)$,*

$$T(a \otimes_\varphi b) = \pi(a)\mathbf{T}\pi(b), \qquad a, b \in A$$

$$\mathbf{T} = s - \lim_{\lambda\in\Lambda}\lim_{\mu\in\Lambda} T(e_\lambda \otimes e_\mu)$$

where the left hand side limit is taken in the strict topology, and $\{e_\lambda \mid \lambda \in \Lambda\}$ of A is an approximate unit of A. We have

$$\mathbf{T}^*T(a \otimes_\varphi b) = \pi(\varphi(a)b) \text{ for } a, b \in A.$$

If $\mathbf{T} \in C$ then $C^*(T, \pi) = C^*(\mathbf{T}, \pi)$, the C^*-algebra generated by the contraction $\mathbf{T}$ and the C^*-algebra A.

2. The correspondence $\mathcal{E}_{\widetilde{\varphi}}$ of an augmentation $(A_q, \widetilde{\varphi})$

From now on we assume that the C^*-algebra A in our initial cpc system (A, φ) is unital, with unit p. With $\mathbb{C}$ the C^*-algebra of complex numbers consider the free product ([1]) C^*-algebra $\mathbb{C} * A$, denoted A_q, a non unital C^*-algebra. With q denoting the unit of $\mathbb{C}$ the span of finite words $\widehat{q}a_1qa_2q...qa_l\widehat{q}$, where the $a_k \in A$, (throughout the symbol $\widehat{\ }$ indicates that the designated element may or may not be present) forms a dense subalgebra of A_q. There are $*$-homomorphisms $\iota : A \to \mathbb{C} * A$ and $\epsilon : \mathbb{C} * A \to A$ with $\epsilon \circ \iota = Id_A$, where Id refers to the identity map on the designated space A, ι is the natural inclusion and ϵ is described by $\epsilon(\widehat{q}a_1qa_2q...qa_l\widehat{q}) = a_1a_2...a_l$ and $\epsilon(q) = p$. The map ι will not always be made explicit.

For A unital and $\varphi : A \to A$ a cpc map set $\widetilde{\varphi} : A_q \to A_q$ by $\widetilde{\varphi}(q) = p$, $\widetilde{\varphi}(\widehat{q}a_1qa_2q...qa_l\widehat{q}) = \varphi(a_1)\varphi(a_2)...\varphi(a_l)$ (where $a_k \in A$) and extend linearly. The map $\widetilde{\varphi}$ when restricted to the copy of A in A_q is equal to φ. It follows from known results that $\widetilde{\varphi}$ yields a cpc map of A_q (cf. [8]). Note that $\widetilde{\varphi}(qa) = \widetilde{\varphi}(aq) = \widetilde{\varphi}(a)$, $\widetilde{\varphi}(aqb) = \widetilde{\varphi}(a)\widetilde{\varphi}(b)$ for $a, b \in A_q$, and that the image of $\widetilde{\varphi}$ is a subspace of A, so $p\widetilde{\varphi}(a) = \widetilde{\varphi}(a)p = \widetilde{\varphi}(a)$ for $a \in A_q$. The resulting cpc system $(A_q, \widetilde{\varphi})$ may be viewed as an augmentation of (A, φ).

Consider the C^*-correspondence $\mathcal{E}_{\widetilde{\varphi}} = A_q \otimes_{\widetilde{\varphi}} A_q$ over A_q associated with the cpc system $(A_q, \widetilde{\varphi})$.

Notation 2.1. *If (A, φ) is a cpc system, with augmented cpc system $(A_q, \widetilde{\varphi})$, the $*$-homomorphism $\phi_{\widetilde{\varphi}}$ describing the left action of A_q on the Hilbert module $\mathcal{E}_{\widetilde{\varphi}}$ is denoted $\widetilde{\phi} : A_q \to \mathcal{L}(\mathcal{E}_{\widetilde{\varphi}})$.*

Note that although we assumed that A is unital the augmented C^*-algebra A_q is not unital. Although not required for the existence of the correspondence based on the system $(A_q, \widetilde{\varphi})$ it follows that the cpc map $\widetilde{\varphi} : A_q \to \mathcal{L}(A_q)$ is in fact strict. For example, if $\{e_\lambda \mid \lambda \in \Lambda\}$ is an approximate unit for A_q, qe_λ converges (in norm) to q. Then, since $\widetilde{\varphi}$ is (norm) continuous, and $\widetilde{\varphi}(e_\lambda) = \widetilde{\varphi}(qe_\lambda)$, $\widetilde{\varphi}(e_\lambda)$ must converge in A_q to $\widetilde{\varphi}(q) = p$, so $\widetilde{\varphi}(e_\lambda)$ converges strictly.

First note some relations for simple tensors in $\mathcal{E}_{\widetilde{\varphi}}$.

Proposition 2.2. *Let (A, φ) be a cpc system and $(A_q, \widetilde{\varphi})$ its augmented cpc system. Then, for $k, m, n \in A_q$,*
a. $m \otimes_{\widetilde{\varphi}} n = mq \otimes_{\widetilde{\varphi}} pn = mq \otimes_{\widetilde{\varphi}} n = m \otimes_{\widetilde{\varphi}} pn$
b. $kqm \otimes_{\widetilde{\varphi}} n = k \otimes_{\widetilde{\varphi}} \widetilde{\varphi}(m)n$.

Proof. Let $a \otimes_{\widetilde{\varphi}} b$ be a simple tensor in $\mathcal{E}_{\widetilde{\varphi}}$. We have

$$\begin{aligned} \langle mq \otimes_{\widetilde{\varphi}} pn, a \otimes_{\widetilde{\varphi}} b \rangle &= \langle pn, \widetilde{\varphi}(qm^*a)b \rangle_{A_q} = n^* p \widetilde{\varphi}(m^*a)b \\ &= n^* \widetilde{\varphi}(m^*a)b = \langle n, \widetilde{\varphi}(m^*a)b \rangle_{A_q} \end{aligned}$$

which equals $\langle m \otimes_{\widetilde{\varphi}} n, a \otimes_{\widetilde{\varphi}} b \rangle$. Similarly $\langle mq \otimes_{\widetilde{\varphi}} n, a \otimes_{\widetilde{\varphi}} b \rangle$ and $\langle m \otimes_{\widetilde{\varphi}} pn, a \otimes_{\widetilde{\varphi}} b \rangle$ also equal $\langle m \otimes_{\widetilde{\varphi}} n, a \otimes_{\widetilde{\varphi}} b \rangle$.

For part b

$$\begin{aligned} \langle kqm \otimes_{\widetilde{\varphi}} n, a \otimes_{\widetilde{\varphi}} b \rangle &= \langle n, \widetilde{\varphi}((kqm)^*a)b \rangle_{A_q} = \langle n, \widetilde{\varphi}(m^*)\widetilde{\varphi}(k^*a)b \rangle_{A_q} \\ &= n^* \widetilde{\varphi}(m)^* \widetilde{\varphi}(k^*a)b = \langle \widetilde{\varphi}(m)n, \widetilde{\varphi}(k^*a)b \rangle_{A_q} \\ &= \langle k \otimes_{\widetilde{\varphi}} \widetilde{\varphi}(m)n, a \otimes_{\widetilde{\varphi}} b \rangle . \end{aligned}$$

Both parts follow after noting that the span of the elements $a \otimes_{\widetilde{\varphi}} b$ is dense in $\mathcal{E}_{\widetilde{\varphi}}$. ☐

Given a representation $(\widetilde{T}, \widetilde{\pi}) : \mathcal{E}_{\widetilde{\varphi}} \to C$ of the correspondence $\mathcal{E}_{\widetilde{\varphi}}$ in a C^*-algebra C there is a partial isometry $\widetilde{\mathbf{T}}$ in C implementing the augmented cp system.

Proposition 2.3. *Let $(\widetilde{T}, \widetilde{\pi}) : \mathcal{E}_{\widetilde{\varphi}} \to C$ be a representation of the correspondence $\mathcal{E}_{\widetilde{\varphi}}$. Then there is a partial isometry $\widetilde{\mathbf{T}}$ in C with initial projection $\widetilde{\pi}(p)$, final projection a subprojection of $\widetilde{\pi}(q)$ and*

$$\widetilde{\mathbf{T}}^* \widetilde{\pi}(a) \widetilde{\mathbf{T}} = \widetilde{\pi}(\widetilde{\varphi}(a)) \text{ for } a \in A_q.$$

Proof. Setting $\widetilde{\mathbf{T}} = \widetilde{T}(q \otimes_{\widetilde{\varphi}} p)$ the covariance conditions yield

$$\widetilde{\mathbf{T}}^* \widetilde{\mathbf{T}} = \widetilde{\pi}(\langle q \otimes_{\widetilde{\varphi}} p, q \otimes_{\widetilde{\varphi}} p \rangle_{A_q}) = \widetilde{\pi}(\widetilde{\varphi}(q)) = \widetilde{\pi}(p)$$

and (using Proposition 2.2 a) $\widetilde{\mathbf{T}}^*\widetilde{\pi}(a)\widetilde{\mathbf{T}} = \widetilde{\pi}(\langle q \otimes_{\widetilde{\varphi}} p, aq \otimes_{\widetilde{\varphi}} p\rangle_{A_q}) = \widetilde{\pi}(\widetilde{\varphi}(a))$. Since $\widetilde{\pi}(p)$ is a projection $\widetilde{\mathbf{T}}$ is a partial isometry in C with initial projection $\widetilde{\pi}(p)$ implementing the map $\widetilde{\varphi}$ on the image of A_q in C. Also $\widetilde{\pi}(q)\widetilde{\mathbf{T}}(q \otimes_{\widetilde{\varphi}} p) = \widetilde{\mathbf{T}}$, implying $\widetilde{\mathbf{T}}$ has final projection $\widetilde{\mathbf{T}}\widetilde{\mathbf{T}}^* \leq \widetilde{\pi}(q)$. □

Remark 2.4. The element $\widetilde{\mathbf{T}}$ is the same element identified in Proposition 1.7 for a given representation $(\widetilde{T}, \widetilde{\pi}) : \mathcal{E}_{\widetilde{\varphi}} \to C$, when $\widetilde{\pi}$ is nondegenerate. Namely given $\{e_\lambda \mid \lambda \in \Lambda\}$ an approximate unit for A_q we have $\langle e_\lambda \otimes_{\widetilde{\varphi}} e_\mu, a \otimes_{\widetilde{\varphi}} b\rangle = \langle e_\mu, \widetilde{\varphi}(e_\lambda^* a)b\rangle_{A_q}$ for $a \otimes_{\widetilde{\varphi}} b$ a simple tensor in $\mathcal{E}_{\widetilde{\varphi}}$. Since the A_q-valued inner product and $\widetilde{\varphi}$ are norm continuous, we see the latter has limit $\widetilde{\varphi}(a)b$. However, $\langle q \otimes_{\widetilde{\varphi}} p, a \otimes_{\widetilde{\varphi}} b\rangle_{A_q} = \langle p, \widetilde{\varphi}(qa)b\rangle_{A_q}$ is also equal to $\widetilde{\varphi}(a)b$, and it follows that $e_\lambda \otimes_{\widetilde{\varphi}} e_\mu$ converges in norm to $q \otimes_{\widetilde{\varphi}} p$ in $\mathcal{E}_{\widetilde{\varphi}}$. Norm continuity of the linear map $\widetilde{T} : \mathcal{E}_{\widetilde{\varphi}} \to C$ implies the element $\widetilde{\mathbf{T}}$ identified in Proposition 1.7 is actually in C and $\widetilde{\mathbf{T}} = \widetilde{T}(q \otimes_{\widetilde{\varphi}} p)$.

3. Restricting representations of $\mathcal{E}_{\widetilde{\varphi}}$

We introduce maps involving an intermediate correspondence $\mathcal{E}_\varphi \otimes_\iota A_q$ in order to investigate relationships between the correspondences $\mathcal{E}_\varphi$ over A, and $\mathcal{E}_{\widetilde{\varphi}}$ over A_q.

Proposition 3.1. *The map $j : \mathcal{E}_\varphi \to \mathcal{E}_\varphi \otimes_\iota A_q$ defined by $j(m) = m \otimes_\iota p$ ($m \in \mathcal{E}_\varphi$) is an inner product preserving map of C^*-correspondences.*

Proof. Viewing A_q as a correspondence over itself, so A_q is viewed via left multiplication as $\mathcal{K}(A_q)$, the map $\iota : A \to A_q$ may be interpreted as a $*$-homomorphism $\iota : A \to \mathcal{L}(A_q)$. The map ι is clearly injective.

Consider the (inner) tensor product $\mathcal{E}_\varphi \otimes_\iota A_q$, an A-A_q correspondence with an A_q-valued inner product (and left action again given by ϕ) determined by

$$\begin{aligned}\langle (r \otimes_\varphi u) \otimes_\iota a, (s \otimes_\varphi v) \otimes_\iota b\rangle &= \left\langle a, \iota(\langle r \otimes_\varphi u, s \otimes_\varphi v\rangle_A)b\right\rangle_{A_q}\\ &= a^*\iota(u^*\varphi(r^*s)v)b\end{aligned}$$

for $r, s, u, v \in A$ and $a, b \in A_q$. With a, b the unit p we obtain the inner product in $\mathcal{E}_\varphi$. □

Additionally, introduce a map of correspondences $V : \mathcal{E}_\varphi \otimes_\iota A_q \to \mathcal{E}_{\widetilde{\varphi}}$ by setting $V((r \otimes_\varphi u) \otimes_\iota a) = r \otimes_{\widetilde{\varphi}} ua$ (strictly speaking this is $\iota(r) \otimes_{\widetilde{\varphi}} \iota(u)a$) for $r, u \in A$ and $a \in A_q$, and extending linearly. For $r, s, u, v \in A$ and $a, b \in A_q$,

$$\begin{aligned}\langle V(r \otimes_\varphi u) \otimes_\iota a), V((s \otimes_\varphi v) \otimes_\iota b)\rangle &= \langle r \otimes_{\widetilde{\varphi}} ua, s \otimes_{\widetilde{\varphi}} vb\rangle\\ &= \langle ua, \widetilde{\varphi}(r^*s)vb\rangle_{A_q}\\ &= a^*u^*\varphi(r^*s)vb\end{aligned}$$

which is the above described A_q-valued inner product in $\mathcal{E}_\varphi \otimes_\iota A_q$. Therefore V extends to an inner product preserving map, also denoted V, of the two A_q-Hilbert modules $\mathcal{E}_\varphi \otimes_\iota A_q$ and $\mathcal{E}_{\widetilde{\varphi}}$. The range of V must therefore be a closed

Hilbert submodule of $\mathcal{E}_{\widetilde{\varphi}}$. The map V also intertwines the two left actions of A; namely

$$V \circ \phi(a) = \widetilde{\phi}(\iota(a)) \circ V \text{ for } a \in A.$$

Noting that $\widetilde{\phi}(p) \in \mathcal{L}(\mathcal{E}_{\widetilde{\varphi}})$ is an adjointable idempotent (here in fact a projection) it follows that its range is closed and an orthogonally complemented Hilbert submodule of $\mathcal{E}_{\widetilde{\varphi}}$ (Corollary 3.3 [13]). We show that this submodule is the range of V.

Proposition 3.2. *The map V is adjointable, so $V \in \mathcal{L}(\mathcal{E}_{\varphi} \otimes_{\iota} A_q, \mathcal{E}_{\widetilde{\varphi}})$, and is an isometry with complemented range; $VV^* = \widetilde{\phi}(p)$.*

Proof. For $r, u \in A$ and $a \in A_q$ notice $\widetilde{\phi}(p)V((r \otimes_{\varphi} u) \otimes_{\iota} a) = pr \otimes_{\widetilde{\varphi}} ua = V((r \otimes_{\varphi} u) \otimes_{\iota} a)$, so $\operatorname{ran}(V)$ is contained in $\operatorname{ran}(\widetilde{\phi}(p))$. To show that $\operatorname{ran}(V)$ contains $\operatorname{ran}(\widetilde{\phi}(p))$ consider an element $\widetilde{\phi}(p)(m \otimes_{\widetilde{\varphi}} n) = pm \otimes_{\widetilde{\varphi}} n$ where $m, n \in A_q$ and $m = \widehat{q}m_1qa_2q...qm_l\widehat{q}$ where the $m_k \in A$. If $m = qm_1qa_2q...qm_l\widehat{q}$ then Proposition 2.2 implies

$$\begin{aligned} pm \otimes_{\widetilde{\varphi}} n &= pqm_1qa_2q...qm_l\widehat{q} \otimes_{\widetilde{\varphi}} n \\ &= p \otimes_{\widetilde{\varphi}} \widetilde{\varphi}(m_1)\widetilde{\varphi}(m_2)...\widetilde{\varphi}(m_l)n \\ &= V((p \otimes_{\varphi} (\widetilde{\varphi}(m_1)\widetilde{\varphi}(m_2)...\widetilde{\varphi}(m_l))) \otimes_{\iota} n) \end{aligned}$$

while if $m = m_1qa_2q...qm_l\widehat{q}$ then

$$\begin{aligned} pm \otimes_{\widetilde{\varphi}} n &= pm_1qm_2q...qm_l\widehat{q} \otimes_{\widetilde{\varphi}} n \\ &= m_1 \otimes_{\widetilde{\varphi}} \widetilde{\varphi}(m_2)...\widetilde{\varphi}(m_l)n \\ &= V((m_1 \otimes_{\varphi} \widetilde{\varphi}(m_2)...\widetilde{\varphi}(m_l)) \otimes_{\iota} n). \end{aligned}$$

Continuity of $\widetilde{\phi}(p)$ implies the span of elements $pm \otimes_{\widetilde{\varphi}} n$ is dense in $\operatorname{ran}(\widetilde{\phi}(p))$, and since $\operatorname{ran}(V)$ is closed, the desired containment holds, and $\operatorname{ran}(V) = \operatorname{ran}(\widetilde{\phi}(p))$.

Since $\operatorname{ran}(\widetilde{\phi}(p))$ is an orthogonally complemented Hilbert submodule of $\mathcal{E}_{\widetilde{\varphi}}$, $\operatorname{ran}(V)$ must be complemented. It follows (Proposition. 3.6, [13]) that $V \in \mathcal{L}(\mathcal{E}_{\varphi} \otimes_{\iota} A_q, \mathcal{E}_{\widetilde{\varphi}})$ and $V^*V = Id_{\mathcal{E}_{\varphi} \otimes_{\iota} A_q}$. □

The map $V \circ j : \mathcal{E}_{\varphi} \to \mathcal{E}_{\widetilde{\varphi}}$ is the natural inner product preserving map with

$$V \circ j(r \otimes_{\varphi} u) = r \otimes_{\widetilde{\varphi}} u \text{ for } r, u \in A.$$

Definition 3.3. For $(\widetilde{T}, \widetilde{\pi}) : \mathcal{E}_{\widetilde{\varphi}} \to C$ a representation of $\mathcal{E}_{\widetilde{\varphi}}$ in a C^*-algebra C define a restricted pair of maps on the correspondence $\mathcal{E}_{\varphi}$

$$(\widetilde{T}_r, \widetilde{\pi}_r) : \mathcal{E}_{\varphi} \to C$$

by $\widetilde{\pi}_r = \widetilde{\pi} \circ \iota$ and $\widetilde{T}_r = \widetilde{T} \circ V \circ j$; so

$$\widetilde{\pi}_r(a) = \widetilde{\pi}(\iota(a)) \text{ and } \widetilde{T}_r(r \otimes_{\varphi} u) = \widetilde{T}(r \otimes_{\widetilde{\varphi}} u)$$

for $a \in A$ and $r \otimes_{\varphi} u \in \mathcal{E}_{\varphi}$.

Remark 3.4. For an element $pm \otimes_{\widetilde{\varphi}} n \in \widetilde{\phi}(p)(\mathcal{E}_{\widetilde{\varphi}})$, where $m, n \in A_q$, the computations in the proof of Proposition 3.2 show that $(pm \otimes_{\widetilde{\varphi}} n) = V(x \otimes_\iota n) = V(j(x))n$ for some $x \in \mathcal{E}_\varphi$. Therefore

$$\widetilde{\pi}(p)\widetilde{T}(m \otimes_{\widetilde{\varphi}} n) = \widetilde{T}(pm \otimes_{\widetilde{\varphi}} n) = \widetilde{T}(V(j(x))n) = \widetilde{T}(V(j(x)))\widetilde{\pi}(n) = \widetilde{T}_r(x)\widetilde{\pi}(n).$$

It follows that $\widetilde{\pi}(p)\widetilde{T}(\mathcal{E}_\varphi) \subseteq \widetilde{T}_r(\mathcal{E}_\varphi)\widetilde{\pi}(A_q)$.

The following result shows that the covariance conditions for $(\widetilde{T}, \widetilde{\pi})$ yield covariance conditions for $(\widetilde{T}_r, \widetilde{\pi}_r)$, justifying use of the term restricted representation.

Proposition 3.5. *If $(\widetilde{T}, \widetilde{\pi}) : \mathcal{E}_{\widetilde{\varphi}} \to C$ is a representation of $\mathcal{E}_{\widetilde{\varphi}}$ in a C^*-algebra C the restricted pair $(\widetilde{T}_r, \widetilde{\pi}_r) : \mathcal{E}_\varphi \to C$ is a representation of $\mathcal{E}_\varphi$ with image in the corner C^*-algebra $\widetilde{\pi}(p)C\widetilde{\pi}(p)$.*

Proof. For $a, b \in A$ and $r \otimes_\varphi u$, $s \otimes_\varphi v \in \mathcal{E}_\varphi$ we have

$$\widetilde{\pi}_r(a)\widetilde{T}_r(r \otimes_\varphi u)\widetilde{\pi}_r(b) = \widetilde{\pi}(a)\widetilde{T}(r \otimes_{\widetilde{\varphi}} u)\widetilde{\pi}(b) = \widetilde{T}(ar \otimes_{\widetilde{\varphi}} ub) = \widetilde{T}_r(ar \otimes_\varphi ub).$$

Also

$$\begin{aligned}\widetilde{T}_r^*(r \otimes_\varphi u)\widetilde{T}_r(s \otimes_\varphi v) &= \widetilde{T}^*(r \otimes_{\widetilde{\varphi}} u)\widetilde{T}(s \otimes_{\widetilde{\varphi}} v) = \widetilde{\pi}(\langle r \otimes_{\widetilde{\varphi}} u, s \otimes_{\widetilde{\varphi}} v\rangle) \\ &= \widetilde{\pi}(\langle r \otimes_\varphi u, s \otimes_\varphi v\rangle) = \widetilde{\pi}_r(\langle r \otimes_\varphi u, s \otimes_\varphi v\rangle),\end{aligned}$$

where the third equality follows from $V \circ j$ preserving the inner product.

Since $\widetilde{T}_r(r \otimes_\varphi u) = \widetilde{T}_r(pr \otimes_\varphi up) = \widetilde{\pi}(p)\widetilde{T}(r \otimes_{\widetilde{\varphi}} u)\widetilde{\pi}(p)$ (and similarly for $\widetilde{\pi}_r$) the last assertion follows. □

Investigating the ideal of compact operators in the C^*-algebra of adjointable operators of a correspondence is crucial for understanding the structure of the Cuntz–Pimsner algebras associated with a correspondence. We next describe relationships between the ideals $\mathcal{K}(\mathcal{E}_\varphi)$ and $\mathcal{K}(\mathcal{E}_{\widetilde{\varphi}})$ and the $*$-homomorphisms $\Psi_{\widetilde{T}_r}$ and $\Psi_{\widetilde{T}}$ arising from representations $(\widetilde{T}, \widetilde{\pi})$ of the correspondence $\mathcal{E}_{\widetilde{\varphi}}$ over A_q.

Consider the induced map $\iota_* : \mathcal{L}(\mathcal{E}_\varphi) \to \mathcal{L}(\mathcal{E}_\varphi \otimes_\iota A_q)$ (described on simple tensors of $\mathcal{E}_\varphi$ by $\iota_*(t)(m \otimes_\iota a) = tm \otimes_\iota a$ for $t \in \mathcal{L}(\mathcal{E}_\varphi)$), which is injective since $\iota : A \to \mathcal{L}(A_q)$ is injective. It is known by general properties of the (inner) tensor product that $\iota_*(\mathcal{K}(\mathcal{E}_\varphi)) \subseteq \mathcal{K}(\mathcal{E}_\varphi \otimes_\iota A_q)$ (Proposition 4.7, [13]), although here, using the assumption that A is unital, it is straightforward to check this; $\iota_*(\theta_{m,n}) = \theta_{j(m),j(n)} = \theta_{m\otimes p, n\otimes p}$ for $m, n \in \mathcal{E}_\varphi$.

The isometry V defines the canonical $*$-homomorphism $\Phi : \mathcal{L}(\mathcal{E}_\varphi \otimes_\iota A_q) \to \mathcal{L}(\mathcal{E}_{\widetilde{\varphi}})$ mapping $t \to VtV^*$, so necessarily the ideal of compacts $\mathcal{K}(\mathcal{E}_\varphi \otimes_\iota A_q)$ must be mapped to the compacts $\mathcal{K}(\mathcal{E}_{\widetilde{\varphi}})$ of $\mathcal{L}(\mathcal{E}_{\widetilde{\varphi}})$;

$$\Phi(\mathcal{K}(\mathcal{E}_\varphi \otimes_\iota A_q)) \subseteq \mathcal{K}(\mathcal{E}_{\widetilde{\varphi}}).$$

In fact $\Phi(\theta_{m,n}) = \theta_{Vm,Vn}$. Since $\iota_* : \mathcal{L}(\mathcal{E}_\varphi) \to \mathcal{L}(\mathcal{E}_\varphi \otimes_\iota A_q)$ also maps the ideal $\mathcal{K}(\mathcal{E}_\varphi)$ of $\mathcal{L}(\mathcal{E}_\varphi)$ to $\mathcal{K}(\mathcal{E}_\varphi \otimes_\iota A_q)$, the $*$-homomorphism $\Phi \circ \iota_* : \mathcal{L}(\mathcal{E}_\varphi) \to \mathcal{L}(\mathcal{E}_{\widetilde{\varphi}})$ maps $\mathcal{K}(\mathcal{E}_\varphi)$ to $\mathcal{K}(\mathcal{E}_{\widetilde{\varphi}})$, and the composition $\Psi_{\widetilde{T}} \circ \Phi \circ \iota_*$ is defined. It follows from $V^*V = Id_{\mathcal{E}_\varphi \otimes_\iota A_q}$ that Φ is injective, and so the $*$-homomorphism $\Phi \circ \iota_*$, a composition of injections, must be injective.

Recall that the $*$-homomorphism $\Psi_T : \mathcal{K}(\mathcal{E}) \to C$ arising from any representation (T,π) of a correspondence $\mathcal{E}$ in a C^*-algebra C is determined by mapping $\theta_{x,y} \to T(x)T^*(y)$.

Proposition 3.6. *If $(\widetilde{T},\widetilde{\pi}) : \mathcal{E}_{\widetilde{\varphi}} \to C$ is a representation in a C^*-algebra C then $\Phi\circ\iota_*(\mathcal{K}(\mathcal{E}_\varphi)) \subseteq \mathcal{K}(\mathcal{E}_{\widetilde{\varphi}})$, and the $*$-homomorphisms $\Psi_{\widetilde{T}}\circ\Phi\circ\iota_*$ and $\Psi_{\widetilde{T}_r} : \mathcal{K}(\mathcal{E}_\varphi) \to C$ are equal.*

Proof. The preceding paragraphs show that $\Phi \circ \iota_*(\theta_{m,n}) = \theta_{Vj(m),Vj(n)}$ for $m, n \in \mathcal{E}_\varphi$, so

$$\Psi_{\widetilde{T}} \circ \Phi \circ \iota_*((\theta_{m,n})) = \widetilde{T}(Vj(m))\widetilde{T}^*(Vj(n)) = \widetilde{T}_r(m)\widetilde{T}_r^*(n) = \Psi_{\widetilde{T}_r}(\theta_{m,n}).$$

□

Proposition 3.7. *Let (A,φ) be a cpc system and $(A_q,\widetilde{\varphi})$ the associated augmented cpc system. The $*$-homomorphisms $\Phi \circ \iota_* \circ \phi$ and $\widetilde{\phi} \circ \iota$ which map A to $\mathcal{L}(\mathcal{E}_{\widetilde{\varphi}})$ are equal, and $\ker\phi = \ker\widetilde{\phi} \cap A$.*

Proof. For $a \in A$ we show that $\Phi \circ \iota_*(\phi(a)) = \widetilde{\phi}(a)$. Since $V : \mathcal{E}_\varphi \otimes_\iota A_q \to \mathcal{E}_{\widetilde{\varphi}}$ satisfies $VV^* = \widetilde{\phi}(p)$ (Proposition 3.2) and $\widetilde{\phi}(a)\widetilde{\phi}(p) = \widetilde{\phi}(a)$ it is enough to show that $V\iota_*(\phi(a)) = \widetilde{\phi}(a)V$. However, for a simple tensor $(r \otimes_\varphi u) \otimes_\iota n \in \mathcal{E}_\varphi \otimes_\iota A_q$,

$$\begin{aligned} V(\iota_*(\phi(a)))((r \otimes_\varphi u) \otimes_\iota n) &= V((ar \otimes_\varphi u) \otimes_\iota n) = ar \otimes_{\widetilde{\varphi}} un \\ &= \widetilde{\phi}(a)(r \otimes_{\widetilde{\varphi}} un) = \widetilde{\phi}(a)V((r \otimes_\varphi u) \otimes_\iota n). \end{aligned}$$

The last statement follows from the injectivity of $\Phi \circ \iota_*$. □

Corollary 3.8. *Let (A,φ) be a cpc system and $(A_q,\widetilde{\varphi})$ the associated augmented cpc system. The ideal $\phi^{-1}(\mathcal{K}(\mathcal{E}_\varphi)) = J(\mathcal{E}_\varphi)$ of A is contained in $\widetilde{\phi}^{-1}(\mathcal{K}(\mathcal{E}_{\widetilde{\varphi}})) \cap A$, so $\iota(J(\mathcal{E}_\varphi)) \subseteq J(\mathcal{E}_{\widetilde{\varphi}})$.*

Proof. If $a \in J(\mathcal{E}_\varphi)$ then $\phi(a) \in \mathcal{K}(\mathcal{E}_\varphi)$ is mapped by $\Phi\circ\iota_*$ to $\mathcal{K}(\mathcal{E}_{\widetilde{\varphi}})$ (Proposition 3.6). Since $J(\mathcal{E}_{\widetilde{\varphi}}) = \widetilde{\phi}^{-1}(\mathcal{K}(\mathcal{E}_{\widetilde{\varphi}}))$ Proposition 3.7 implies $\iota(a) \in J(\mathcal{E}_{\widetilde{\varphi}})$. □

Let I be an ideal contained in the ideal $J(\mathcal{E}_{\widetilde{\varphi}}) = \widetilde{\phi}^{-1}(\mathcal{K}(\mathcal{E}_{\widetilde{\varphi}}))$ of A_q and $(\widetilde{T},\widetilde{\pi}) : \mathcal{E}_{\widetilde{\varphi}} \to C^*(\widetilde{T},\widetilde{\pi})$ a representation of $\mathcal{E}_{\widetilde{\varphi}}$ coisometric on I. The above results imply coisometric conditions for its associated restricted representation $(\widetilde{T}_r,\widetilde{\pi}_r) : \mathcal{E}_\varphi \to \widetilde{\pi}(p)C^*(\widetilde{T},\widetilde{\pi})\widetilde{\pi}(p)$.

Corollary 3.9. *Let (A,φ) be a cpc system and $(A_q,\widetilde{\varphi})$ the associated augmented cpc system. Let I be an ideal in $J(\mathcal{E}_{\widetilde{\varphi}}) = \widetilde{\phi}^{-1}(\mathcal{K}(\mathcal{E}_{\widetilde{\varphi}}))$ and $(\widetilde{T},\widetilde{\pi}) : \mathcal{E}_{\widetilde{\varphi}} \to C^*(\widetilde{T},\widetilde{\pi})$ a representation of $\mathcal{E}_{\widetilde{\varphi}}$ coisometric on I. The restricted representation*

$$(\widetilde{T}_r,\widetilde{\pi}_r) : \mathcal{E}_\varphi \to \widetilde{\pi}(p)C^*(\widetilde{T},\widetilde{\pi})\widetilde{\pi}(p)$$

is a representation of $\mathcal{E}_\varphi$ which is coisometric on $\iota^{-1}(I) \cap J(\mathcal{E}_\varphi)$.

Proof. For $a \in \iota^{-1}(I)\cap J(\mathcal{E}_\varphi)$, Proposition 3.6 implies $\Psi_{\widetilde{T}_r}(\phi(a)) = \Psi_{\widetilde{T}}\circ\Phi\circ\iota_*(\phi(a))$ while Proposition 3.7 shows that this $= \Psi_{\widetilde{T}}(\widetilde{\phi}(\iota(a)))$. Since $\iota(a) \in I$, this in turn is equal to $\widetilde{\pi}(\iota(a)) = \widetilde{\pi}_r(a)$. □

Lemma 3.10. *If $a, b \in A_q$ then $\widetilde{\phi}(aqb^*) = \theta_{a\otimes_{\widetilde{\varphi}}p, b\otimes_{\widetilde{\varphi}}p} \in \mathcal{K}(\mathcal{E}_{\widetilde{\varphi}})$. In particular $\widetilde{\phi}(q) = \theta_{q\otimes_{\widetilde{\varphi}}p, q\otimes_{\widetilde{\varphi}}p}$, and $q \in J(\mathcal{E}_{\widetilde{\varphi}})$.*

Proof. For $m \otimes_{\widetilde{\varphi}} n \in \mathcal{E}_{\widetilde{\varphi}}$,

$$\begin{aligned}\theta_{a\otimes_{\widetilde{\varphi}}p, b\otimes_{\widetilde{\varphi}}p}(m \otimes_{\widetilde{\varphi}} n) &= a \otimes_{\widetilde{\varphi}} p \langle b \otimes_{\widetilde{\varphi}} p, m \otimes_{\widetilde{\varphi}} n\rangle = a \otimes_{\widetilde{\varphi}} p \langle p, \widetilde{\varphi}(b^*m)n\rangle_{A_q} \\ &= a \otimes_{\widetilde{\varphi}} \widetilde{\varphi}(b^*m)n = aqb^*m \otimes_{\widetilde{\varphi}} n \\ &= \widetilde{\phi}(aqb^*)(m \otimes_{\widetilde{\varphi}} n)\end{aligned}$$

where the last equality follows by Proposition 2.2. □

Proposition 3.11. *Let I be an ideal in $J(\mathcal{E}_{\widetilde{\varphi}})$ containing the idempotent q and $(\widetilde{T}, \widetilde{\pi}) : \mathcal{E}_{\widetilde{\varphi}} \to C$ a representation in a C^*-algebra C which is coisometric on I. Then the partial isometry $\widetilde{\mathbf{T}} = \widetilde{T}(q \otimes_{\widetilde{\varphi}} p)$ with initial projection $\widetilde{\pi}(p)$ has final projection $\widetilde{\pi}(q)$, and both these projections are full in the C^*-subalgebra $C^*(\widetilde{T}, \widetilde{\pi})$ of C.*

Proof. By definition ([1]) we need to show that the ideals

$$J_p = C^*(\widetilde{T}, \widetilde{\pi})\widetilde{\pi}(p)C^*(\widetilde{T}, \widetilde{\pi}) \text{ and } J_q = C^*(\widetilde{T}, \widetilde{\pi})\widetilde{\pi}(q)C^*(\widetilde{T}, \widetilde{\pi})$$

generated by the projections $\widetilde{\pi}(p)$ and $\widetilde{\pi}(q)$ are all of $C^*(\widetilde{T}, \widetilde{\pi})$. It is enough to show that the generators $\widetilde{T}(\mathcal{E}_{\widetilde{\varphi}}) \cup \widetilde{\pi}(A_q)$ of $C^*(\widetilde{T}, \widetilde{\pi})$ are in these ideals. Proposition 2.3 showed that $\widetilde{\mathbf{T}}$ has initial projection $\widetilde{\pi}(p)$, so $\widetilde{\mathbf{T}}\widetilde{\pi}(p) = \widetilde{\mathbf{T}}$, while the final projection $\widetilde{\mathbf{T}}\widetilde{\mathbf{T}}^* \leq \widetilde{\pi}(q)$ and so $\widetilde{\pi}(q)\widetilde{\mathbf{T}} = \widetilde{\mathbf{T}}$. Therefore $\widetilde{\mathbf{T}}$ is contained in both ideals, and therefore its initial projection $\widetilde{\pi}(p)$, and its final projection lie in both ideals. Since $\widetilde{\pi}(p)$ is the unit of $\widetilde{\pi}(A)$, $\widetilde{\pi}(A)$ must be contained in both ideals.

Since $\widetilde{\mathbf{T}}\widetilde{\pi}(p) = \widetilde{\mathbf{T}}$ and $\widetilde{T}(m \otimes_{\widetilde{\varphi}} n) = \widetilde{\pi}(m)\widetilde{T}(q \otimes_{\widetilde{\varphi}} p)\widetilde{\pi}(n)$ it follows that the image $\widetilde{T}(\mathcal{E}_{\widetilde{\varphi}})$ is contained in both ideals. To show that $\widetilde{\pi}(A_q)$ is in these ideals it remains to show that $\widetilde{\pi}(q)$ is the final projection of $\widetilde{\mathbf{T}}$. However, the hypothesis and coisometric condition imply $\widetilde{\pi}(q) = \psi_{\widetilde{T}}(\widetilde{\phi}(q))$, while the latter is equal to $\psi_{\widetilde{T}}(\theta_{q\otimes_{\widetilde{\varphi}}p, q\otimes_{\widetilde{\varphi}}p}) = \widetilde{T}(q \otimes_{\widetilde{\varphi}} p)\widetilde{T}^*(q \otimes_{\widetilde{\varphi}} p) = \widetilde{\mathbf{T}}\widetilde{\mathbf{T}}^*$ by Lemma 3.10. □

Given a representation $(\widetilde{T}, \widetilde{\pi}) : \mathcal{E}_{\widetilde{\varphi}} \to C^*(\widetilde{T}, \widetilde{\pi})$ of $\mathcal{E}_{\widetilde{\varphi}}$ coisometric on I, apply Corollary 3.9 to its restricted representation $(\widetilde{T}_r, \widetilde{\pi}_r) : \mathcal{E}_\varphi \to \widetilde{\pi}(p)C^*(\widetilde{T}, \widetilde{\pi})\widetilde{\pi}(p)$; it is a representation of the correspondence $\mathcal{E}_\varphi$ coisometric on $\iota^{-1}(I) \cap J(\mathcal{E}_\varphi)$. Let (T_φ, π_φ) denote the universal representation of $\mathcal{E}_\varphi$ coisometric on $\iota^{-1}(I) \cap J(\mathcal{E}_\varphi)$ in the relative Cuntz–Pimsner C^*-algebra $\mathcal{O}(\iota^{-1}(I) \cap J(\mathcal{E}_\varphi), \mathcal{E}_\varphi)$. The universal property (for representations of $\mathcal{E}_\varphi$ coisometric on $\iota^{-1}(I) \cap J(\mathcal{E}_\varphi)$) yields a $*$-homomorphism γ (depending on the chosen initial representation $(\widetilde{T}, \widetilde{\pi})$ of $\mathcal{E}_{\widetilde{\varphi}}$)

$$\gamma : \mathcal{O}(\iota^{-1}(I) \cap J(\mathcal{E}_\varphi), \mathcal{E}_\varphi) \to \widetilde{\pi}(p)C^*(\widetilde{T}, \widetilde{\pi})\widetilde{\pi}(p)$$

to a corner of the C^*-algebra $C^*(\widetilde{T}, \widetilde{\pi})$ with

$$(\widetilde{T}_r, \widetilde{\pi}_r) = \gamma \circ (T_\varphi, \pi_\varphi).$$

Proposition 3.12. *Let I be an ideal in $J(\mathcal{E}_{\widetilde{\varphi}})$ containing q and $(\widetilde{T}, \widetilde{\pi}) : \mathcal{E}_{\widetilde{\varphi}} \to C^*(\widetilde{T}, \widetilde{\pi})$ a representation of $\mathcal{E}_{\widetilde{\varphi}}$ coisometric on I. The $*$-homomorphism*

$$\gamma : \mathcal{O}(\iota^{-1}(I) \cap J(\mathcal{E}_\varphi), \mathcal{E}_\varphi) \to \widetilde{\pi}(p)C^*(\widetilde{T}, \widetilde{\pi})\widetilde{\pi}(p)$$

defined by the universal property is surjective.

Proof. By definition the C^*-algebra $C^*(\widetilde{T}, \widetilde{\pi})$ is generated by $\widetilde{T}(\mathcal{E}_{\widetilde{\varphi}}) \cup \widetilde{\pi}(A_q)$. Since $(\widetilde{T}_r, \widetilde{\pi}_r) = \gamma \circ (T_\varphi, \pi_\varphi)$ it is enough to show that the algebra generated by $\widetilde{T}_r(\mathcal{E}_\varphi) \cup \widetilde{\pi}_r(A)$ contains $\widetilde{\pi}(p)\widetilde{T}(\mathcal{E}_{\widetilde{\varphi}})\widetilde{\pi}(p) \cup \widetilde{\pi}(p)\widetilde{\pi}(A_q)\widetilde{\pi}(p)$.

For the C^*-subalgebra A of A_q, $\widetilde{\pi}_r(A) = \widetilde{\pi}(p)\widetilde{\pi}(A)\widetilde{\pi}(p)$, so to show that the algebra generated by $\widetilde{T}_r(\mathcal{E}_\varphi) \cup \widetilde{\pi}_r(A)$ contains $\widetilde{\pi}(p)\widetilde{\pi}(A_q)\widetilde{\pi}(p)$ it is enough to show that the element $\widetilde{\pi}(p)\widetilde{\pi}(q)\widetilde{\pi}(p)$ is contained in this algebra. First note that (Proposition 2.2)

$$\widetilde{T}_r(p \otimes_\varphi p) = \widetilde{T}(p \otimes_{\widetilde{\varphi}} p) = \widetilde{T}(pq \otimes_{\widetilde{\varphi}} p) = \widetilde{\pi}(p)\widetilde{T}(q \otimes_{\widetilde{\varphi}} p).$$

Setting $\widetilde{\mathbf{T}} = \widetilde{T}(q \otimes_{\widetilde{\varphi}} p)$, and $\widetilde{\mathbf{T}}_r = \widetilde{T}_r(p \otimes_\varphi p)$, we have

$$\widetilde{\mathbf{T}}_r \widetilde{\mathbf{T}}_r^* = \widetilde{\pi}(p)\widetilde{\mathbf{T}}\widetilde{\mathbf{T}}^*\widetilde{\pi}(p) = \widetilde{\pi}(p)\widetilde{\pi}(q)\widetilde{\pi}(p)$$

by Proposition 3.11.

By Remark 3.4, $\widetilde{\pi}(p)\widetilde{T}(\mathcal{E}_{\widetilde{\varphi}}) \subseteq \widetilde{T}_r(\mathcal{E}_\varphi)\widetilde{\pi}(A_q)$. However, the latter is equal to $(\widetilde{T}_r(\mathcal{E}_\varphi)\widetilde{\pi}(p))\widetilde{\pi}(A_q)$, and therefore

$$\widetilde{\pi}(p)\widetilde{T}(\mathcal{E}_{\widetilde{\varphi}})\widetilde{\pi}(p) \subseteq \widetilde{T}_r(\mathcal{E}_\varphi)\widetilde{\pi}(p)\widetilde{\pi}(A_q)\widetilde{\pi}(p)$$

which is contained in the algebra generated by $\widetilde{T}_r(\mathcal{E}_\varphi) \cup \widetilde{\pi}_r(A)$. □

4. An augmented representation

Given a representation $(T, \pi) : \mathcal{E}_\varphi \to C$ of the A-A correspondence $\mathcal{E}_\varphi$ in a C^*-algebra C, with $\pi(p) = Id_C$, there is an augmented, or induced, representation

$$(T_q, \pi_q) : \mathcal{E}_{\widetilde{\varphi}} \to \mathrm{M}_2(C)$$

of the A_q-A_q correspondence $\mathcal{E}_{\widetilde{\varphi}}$.

To define this representation first consider the element $\mathbf{T}_q \in \mathrm{M}_2(C)$ formed from the contraction $\mathbf{T} = T(p \otimes_\varphi p) \in C$ (as in [4]):

$$\mathbf{T}_q = \begin{bmatrix} \mathbf{T} & 0 \\ \sqrt{\pi(p) - \mathbf{T}^*\mathbf{T}} & 0 \end{bmatrix}.$$

Since $T_q^* T_q$ is a projection in $\mathrm{M}_2(C)$, T_q is a partial isometry. Next define a $*$-representation $\pi_q : A_q \to \mathrm{M}_2(C)$ by setting

$$\pi_q(a) = \begin{bmatrix} \pi(a) & 0 \\ 0 & 0 \end{bmatrix} \text{ for } a \in A, \text{ and}$$

$$\pi_q(\widehat{q}a_1qa_2q...qa_l\widehat{q}) = \widehat{\mathbf{T}_q\mathbf{T}_q^*}\pi_q(a_1)\mathbf{T}_q\mathbf{T}_q^*\pi_q(a_2)\mathbf{T}_q\mathbf{T}_q^*...\mathbf{T}_q\mathbf{T}_q^*\pi_q(a_l)\widehat{\mathbf{T}_q\mathbf{T}_q^*}$$

on words in A_q, so $\pi_q(q) = \mathbf{T}_q\mathbf{T}_q^*$, and $\mathbf{T}_q$ is a partial isometry with initial projection $\pi_q(p)$ and final projection $\pi_q(q)$. Extend π_q linearly to a dense subalgebra of A_q. The norm on A_q ensures this is a representation bounded by 1, and π_q extends to a representation, also denoted π_q, of A_q in $\mathrm{M}_2(C)$.

Note that

$$\pi_q(q) = \begin{bmatrix} \mathbf{T}\mathbf{T}^* & \mathbf{T}\sqrt{\pi(p) - \mathbf{T}^*\mathbf{T}} \\ (\sqrt{\pi(p) - \mathbf{T}^*\mathbf{T}})\mathbf{T}^* & \pi(p) - \mathbf{T}^*\mathbf{T} \end{bmatrix}.$$

The following shows that the partial isometry $\mathbf{T}_q$ implements the pair $(A_q, \widetilde{\varphi})$ under the representation π_q.

Lemma 4.1. *With $\mathbf{T}_q \in \mathrm{M}_2(C)$ and $\pi_q : A_q \to \mathrm{M}_2(C)$ defined as above,*

$$\mathbf{T}_q^* \pi_q(m) \mathbf{T}_q = \pi_q(\widetilde{\varphi}(m))$$

for $m \in A_q$.

Proof. A computation shows that

$$\mathbf{T}_q^* \pi_q(a) \mathbf{T}_q = \begin{bmatrix} \mathbf{T}^*\pi(a)\mathbf{T} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \pi(\varphi(a)) & 0 \\ 0 & 0 \end{bmatrix} = \pi_q(\varphi(a)) \text{ for } a \in A.$$

Given a word $m = \widehat{q} a_1 q a_2 q ... q a_l \widehat{q}$ in A_q we have

$$\begin{aligned} \mathbf{T}_q^* \pi_q(m) \mathbf{T}_q &= \mathbf{T}_q^* \widehat{\mathbf{T}_q \mathbf{T}_q^*} \pi_q(a_1) \mathbf{T}_q \mathbf{T}_q^* \pi_q(a_2) \mathbf{T}_q \mathbf{T}_q^* ... \mathbf{T}_q \mathbf{T}_q^* \pi_q(a_l) \widehat{\mathbf{T}_q \mathbf{T}_q^*} \\ &= (\mathbf{T}_q^* \pi_q(a_1) \mathbf{T}_q)(\mathbf{T}_q^* \pi_q(a_2) \mathbf{T}_q) ... (\mathbf{T}_q^* \pi_q(a_l) \mathbf{T}_q) \\ &= \pi_q(\varphi(a_1)) ... \pi_q(\varphi(a_l)) = \pi_q(\widetilde{\varphi}(m)). \end{aligned}$$

Density of the linear span of words in A_q and continuity finish the claim. □

Proposition 4.2. *If $(T, \pi) : \mathcal{E}_\varphi \to C$ is a representation in a C^*-algebra C there is a representation $(T_q, \pi_q) : \mathcal{E}_{\widetilde{\varphi}} \to \mathrm{M}_2(C)$ of the augmented correspondence $\mathcal{E}_{\widetilde{\varphi}}$ with $T_q(q \otimes_{\widetilde{\varphi}} p)$ the partial isometry $\mathbf{T}_q \in \mathrm{M}_2(C)$.*

Proof. The previous paragraphs describe a $*$-representation $\pi_q : A_q \to \mathrm{M}_2(C)$ and an element $\mathbf{T}_q \in \mathrm{M}_2(C)$. Define a linear map $S : A_q \otimes_{\mathrm{alg}} A_q \to \mathrm{M}_2(C)$ by mapping $a \otimes b$ to $\pi_q(a) \mathbf{T}_q \pi_q(b)$. Note $S(q \otimes_{\widetilde{\varphi}} p) = \pi_q(q) \mathbf{T}_q \pi_q(p) = \mathbf{T}_q$. The previous Lemma implies

$$\begin{aligned} \langle S(m \otimes n), S(a \otimes b) \rangle_{\mathrm{M}_2(C)} &= \pi_q(n^*) \mathbf{T}_q^* \pi_q(m^*) \pi_q(a) \mathbf{T}_q \pi_q(b) \\ &= \pi_q(n^*) \pi_q(\widetilde{\varphi}(m^* a)) \pi_q(b) \\ &= \pi_q(\langle m \otimes_{\widetilde{\varphi}} n, a \otimes_{\widetilde{\varphi}} b \rangle). \end{aligned}$$

Therefore S determines a linear map (bounded), denoted by $T_q : \mathcal{E}_{\widetilde{\varphi}} \to \mathrm{M}_2(C)$, and (T_q, π_q) is clearly a covariant representation of $\mathcal{E}_{\widetilde{\varphi}}$ with $T_q(q \otimes_{\widetilde{\varphi}} p)$ the partial isometry $\mathbf{T}_q$. □

Definition 4.3. For K an ideal of A set K_q to be the ideal of A_q generated by $\iota(K) \cup \{q\}$.

It follows from Lemma 3.10 and Corollary 3.8 that if $K \subseteq J(\mathcal{E}_\varphi)$ then $K_q \subseteq J(\mathcal{E}_{\widetilde{\varphi}})$. If $K = 0$ then K_q is the singly generated ideal of A_q generated by q.

Proposition 4.4. *Let $(T,\pi):\mathcal{E}_\varphi \to C$ be a representation in a C^*-algebra C and $(T_q,\pi_q):\mathcal{E}_{\widetilde{\varphi}} \to \mathrm{M}_2(C)$ its associated augmented representation of $\mathcal{E}_{\widetilde{\varphi}}$. If (T,π) is coisometric on an ideal $K \subseteq J(\mathcal{E}_\varphi)$ then (T_q,π_q) is coisometric on the ideal $K_q \subseteq J(\mathcal{E}_{\widetilde{\varphi}})$.*

Proof. The comment after Definition 4.3 shows that $K_q \subseteq J(\mathcal{E}_{\widetilde{\varphi}})$.

Lemma 3.10 yields

$$\begin{aligned}\psi_{T_q}(\widetilde{\phi}(q)) &= \psi_{T_q}(\theta_{q\otimes_{\widetilde{\varphi}}p,q\otimes_{\widetilde{\varphi}}p}) = T_q(q\otimes_{\widetilde{\varphi}} p)T_q^*(q\otimes_{\widetilde{\varphi}} p)\\ &= \mathbf{T}_q\mathbf{T}_q^* = \pi_q(q),\end{aligned}$$

so (T_q,π_q) is coisometric on the ideal of A_q generated by q.

It remains to show that (T_q,π_q) is coisometric on $\iota(K)$, i.e., that $\psi_{T_q}(\widetilde{\phi}(\iota(a)) = \pi_q(\iota(a))$ for $a \in K$. First consider the restricted linear map $(T_q)_r : \mathcal{E}_\varphi \to \mathrm{M}_2(C)$. For $s, v \in A$ compute that

$$\begin{aligned}(T_q)_r(s\otimes_\varphi v) &= T_q(s\otimes_{\widetilde{\varphi}} v) = \pi_q(s)\mathbf{T}_q\pi_q(v)\\ &= \begin{bmatrix}\pi(s) & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{T} & 0\\ \sqrt{\pi(p)-\mathbf{T}^*\mathbf{T}} & 0\end{bmatrix}\begin{bmatrix}\pi(v) & 0\\ 0 & 0\end{bmatrix}\\ &= \begin{bmatrix}\pi(s)T(p\otimes p)\pi(v) & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}T(s\otimes_\varphi v) & 0\\ 0 & 0\end{bmatrix}.\end{aligned}$$

It follows that

$$\begin{aligned}\psi_{(T_q)_r}(\theta_{r\otimes_\varphi u, s\otimes_\varphi v}) &= (T_q)_r(r\otimes_\varphi u)(T_q)_r^*(s\otimes_\varphi v)\\ &= \begin{bmatrix}T(r\otimes_\varphi u) & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}T^*(s\otimes_\varphi v) & 0\\ 0 & 0\end{bmatrix}\\ &= \begin{bmatrix}\psi_T(\theta_{r\otimes_\varphi u, s\otimes_\varphi v}) & 0\\ 0 & 0\end{bmatrix}.\end{aligned}$$

Therefore $\psi_{(T_q)_r} = \pi_q \circ \psi_T$ on $\mathcal{K}(\mathcal{E}_\varphi)$, so $\psi_{(T_q)_r}(\phi(a)) = \pi_q(\psi_T(\phi(a)))$ for $a \in K$.

By assumption (T,π) is coisometric on K, so $\psi_T(\phi(a)) = \pi(a)$ for $a \in K$. Hence, for $a \in K$,

$$\psi_{T_q}(\widetilde{\phi}(\iota(a)) = \psi_{T_q}(\Phi\circ\iota_*\circ\phi(a)) = \psi_{(T_q)_r}(\phi(a)) = \pi_q(\iota(a)),$$

where Propositions 3.7 and 3.6 are used for the first two equalities. ☐

Establishing the isomorphism statement of next theorem involves the following constructed $*$-homomorphism δ. Starting with the universal representation

$$(T_\varphi,\pi_\varphi):\mathcal{E}_\varphi \to C^*(T_\mathcal{E},\pi_\mathcal{E}) = \mathcal{O}(K,\mathcal{E}_\varphi)$$

of $\mathcal{E}_\varphi$ coisometric on $K \subseteq J(\mathcal{E}_\varphi)$, form its associated augmented representation $(T_q,\pi_q):\mathcal{E}_{\widetilde{\varphi}} \to \mathrm{M}_2(\mathcal{O}(K,\mathcal{E}_\varphi))$ of $\mathcal{E}_{\widetilde{\varphi}}$ coisometric on K_q (Proposition 4.4). Let

$$(T_{\widetilde{\varphi}},\pi_{\widetilde{\varphi}}):\mathcal{E}_{\widetilde{\varphi}} \to C^*(T_{\widetilde{\varphi}},\pi_{\widetilde{\varphi}}) = \mathcal{O}(K_q,\mathcal{E}_{\widetilde{\varphi}})$$

denote the universal representation of $\mathcal{E}_{\widetilde{\varphi}}$ coisometric on K_q. The universal property for representations of $\mathcal{E}_{\widetilde{\varphi}}$ coisometric on K_q yields a $*$-homomorphism

$$\delta' : \mathcal{O}(K_q,\mathcal{E}_{\widetilde{\varphi}}) \to \mathrm{M}_2(\mathcal{O}(K,\mathcal{E}_\varphi))$$

with

$$(T_q, \pi_q) = \delta' \circ (T_{\widetilde{\varphi}}, \pi_{\widetilde{\varphi}}).$$

Consider the cut down by the projection $\pi_{\widetilde{\varphi}}(p)$ of δ' on its domain $\mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})$ and by $\pi_q(p)$ on its codomain $\mathrm{M}_2(\mathcal{O}(K, \mathcal{E}_\varphi))$ to obtain a unital $*$-homomorphism

$$\delta : \pi_{\widetilde{\varphi}}(p)\mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})\pi_{\widetilde{\varphi}}(p) \to \mathcal{O}(K, \mathcal{E}_\varphi).$$

We will see that δ is a $*$-isomorphism, which leads to the following Theorem.

Theorem 4.5. *Let $\varphi : A \to A$ be a completely positive contractive map of a unital C^*-algebra A. For K an ideal in $J(\mathcal{E}_\varphi)$ the Cuntz–Pimsner C^*-algebra $\mathcal{O}(K, \mathcal{E}_\varphi)$ is isomorphic to a full corner of $\mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})$ and these relative Cuntz–Pimsner C^*-algebras are Morita equivalent.*

Proof. Recall K_q is the ideal in $J(\mathcal{E}_{\widetilde{\varphi}})$ generated by $\iota(K) \cup q$ (Definition 4.3). Consider the universal representation $(T_{\widetilde{\varphi}}, \pi_{\widetilde{\varphi}}) : \mathcal{E}_{\widetilde{\varphi}} \to \mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})$ of $\mathcal{E}_{\widetilde{\varphi}}$ coisometric on K_q. Proposition 3.11 states $\pi_{\widetilde{\varphi}}(p)\mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})\pi_{\widetilde{\varphi}}(p)$ is a full corner of $\mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})$, so the Morita equivalence follows once an isomorphism $\mathcal{O}(K, \mathcal{E}_\varphi) \cong \pi_{\widetilde{\varphi}}(p)\mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})\pi_{\widetilde{\varphi}}(p)$ is established.

Since there are representations of A_q where the C^*-subalgebra generated by words involving q has zero intersection with $\iota(A)$, and since ι is an injective homomorphism, the ideal $\iota^{-1}(K_q) \cap J(\mathcal{E}_\varphi)$ of A is $\iota^{-1}(\iota(K)) \cap J(\mathcal{E}_\varphi) = K \cap J(\mathcal{E}_\varphi) = K$. The remarks preceding Proposition 3.12 (setting the ideal I to be K_q) yield, for the universal representation $(T_{\widetilde{\varphi}}, \pi_{\widetilde{\varphi}})$ of $\mathcal{E}_{\widetilde{\varphi}}$ coisometric on K_q, a $*$-homomorphism

$$\gamma : \mathcal{O}(K, \mathcal{E}_\varphi) \to \pi_{\widetilde{\varphi}}(p)\mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})\pi_{\widetilde{\varphi}}(p)$$

satisfying $(T_{\widetilde{\varphi}r}, \pi_{\widetilde{\varphi}r}) = \gamma \circ (T_\varphi, \pi_\varphi)$, where (T_φ, π_φ) is the universal representation of $\mathcal{E}_\varphi$ coisometric on K.

Consider the composition $\delta \circ \gamma : \mathcal{O}(K, \mathcal{E}_\varphi) \to \mathcal{O}(K, \mathcal{E}_\varphi)$. For $a \in A$ compute that $\delta \circ \gamma(\pi_\varphi(a)) = \delta(\pi_{\widetilde{\varphi}r}(a)) = \delta(\pi_{\widetilde{\varphi}}(a))$ which is equal to the cut down of $\pi_q(a) = (\delta' \circ \pi_{\widetilde{\varphi}})(a)$ by $\pi_q(p)$, namely $\pi_\varphi(a)$. Also $\delta \circ \gamma \circ T_\varphi = \delta \circ T_{\widetilde{\varphi}r}$ which is the cut down of $T_q = \delta' \circ T_{\widetilde{\varphi}}$ by $\pi_q(p)$, namely T_φ. Thus $\delta \circ \gamma = Id_{\mathcal{O}(K, \mathcal{E}_\varphi)}$ and therefore γ is injective. Since γ is surjective (Proposition 3.12) it is an isomorphism. □

Remark 4.6. If the C^*-algebra A is separable then $\mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})$ is separable, and by Brown's theorem ([5]) the full corner C^*-subalgebra $\pi_{\widetilde{\varphi}}(p)\mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})\pi_{\widetilde{\varphi}}(p)$ and $\mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})$ are stably isomorphic. Therefore, in this situation, $\mathcal{O}(K, \mathcal{E}_\varphi)$ and $\mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})$ are stably isomorphic.

The proof of Theorem 4.5 implies that the $*$-homomorphism

$$\delta : \pi_{\widetilde{\varphi}}(p)\mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})\pi_{\widetilde{\varphi}}(p) \to \mathcal{O}(K, \mathcal{E}_\varphi)$$

is an isomorphism, so the cut down of the universal representation of $\mathcal{E}_{\widetilde{\varphi}}$ coisometric on K_q is, under the map δ, the universal representation (T_φ, π_φ) of $\mathcal{E}_\varphi$ coisometric on K.

Corollary 4.7. *Assume $K \subseteq J_{\mathcal{E}_\varphi}$ and let $(T_{\widetilde{\varphi}}, \pi_{\widetilde{\varphi}})$ be the universal representation of $\mathcal{E}_{\widetilde{\varphi}}$. Then* $\ker(\pi_{\widetilde{\varphi}}) \subseteq \{a \in A_q \mid \widetilde{\varphi}(a^*a) = 0\}$.

Proof. Assume $a \in \ker(\pi_{\widetilde{\varphi}})$, equivalently $a^*a \in \ker(\pi_{\widetilde{\varphi}})$. Then $0 = \mathbf{T}^*_{\widetilde{\varphi}}\pi_{\widetilde{\varphi}}(a^*a)\mathbf{T}_{\widetilde{\varphi}} = \pi_{\widetilde{\varphi}}(p)\pi_{\widetilde{\varphi}}(\widetilde{\varphi}(a^*a))\pi_{\widetilde{\varphi}}(p)$ where $\mathbf{T}_{\widetilde{\varphi}}$ denotes, as usual, the partial isometry $T_{\widetilde{\varphi}}(q\otimes_{\widetilde{\varphi}}p)$ with initial projection $\pi_{\widetilde{\varphi}}(p)$. Using the notation in the above theorem, $0 = (\delta\circ\pi_{\widetilde{\varphi}})(\widetilde{\varphi}(a^*a))$, which is the cut down of $\pi_q(\widetilde{\varphi}(a^*a))$ by $\pi_q(p)$, namely $\pi_\varphi(\widetilde{\varphi}(a^*a))$. Since $K \subseteq J_{\mathcal{E}_\varphi}$, π_φ is injective on A and the statement follows. □

5. The partial isometry case

The following briefly considers the case where the given cpc system (A, φ) maps the unit p of A to a projection $\varphi(p)$ of A; this is the case for example if φ is a $*$-endomorphism of A, or if φ is a retraction from A to a C^*-subalgebra of A ([13] p. 55). If $\varphi(p)$ is a projection the implementing contraction $\mathbf{T} = T(p\otimes_\varphi p)$ for any representation (T, π) of the correspondence $\mathcal{E}_\varphi$ is necessarily a partial isometry with initial projection $\mathbf{T}^*\mathbf{T} = \pi(\varphi(p))$. It is shown that the augmented system $(A_q, \widetilde{\varphi})$ reflects some structure of the original system. Such systems (A, φ) also provide basic examples where representations of the augmented correspondence $\mathcal{E}_{\widetilde{\varphi}}$ over A_q which are coisometric on K_q cannot be injective.

Lemma 5.1. *Consider (A, φ) where $\varphi(p) = e$ is a projection of A. Then*

$$\varphi(r) = e\varphi(r)e \text{ for all } r \in A.$$

Proof. (cf. [13], Proposition 5.10) If $0 \le r$ with $\|r\| \le 1$ then $0 \le \varphi(r) \le \varphi(p) = e$, so $0 \le (p-e)\varphi(r)(p-e) \le (p-e)e(p-e) = 0$. Decomposing $\varphi(r)$ with respect to e and viewing it as the square of an element in A it follows that the equality follows for $r \ge 0$. By linearity the result holds for all $r \in A$. □

Lemma 5.2. *Consider (A, φ) where $\varphi(p) = e$ is a projection of A. If $(A_q, \widetilde{\varphi})$ is the augmented cpc system then $\widetilde{\varphi}(pa) = \varphi(p)\widetilde{\varphi}(a) = \widetilde{\varphi}(a)\varphi(p) = \widetilde{\varphi}(ap)$ for all $a \in A_q$ (so p is in the multiplicative domain of $\widetilde{\varphi}$).*

Proof. It is enough to check this when $a = q$ and when $a = \widehat{q}a_1qa_2q...qa_l\widehat{q} \in A_q$ where the $a_i \in A$. Since $\widetilde{\varphi}(q) = p$ the unit of A, the first case when $a = q$ follows from the definition of $\widetilde{\varphi}$. The previous lemma implies $\varphi(r) = e\varphi(r) = \varphi(r)e$ for all $r \in A$, therefore in the second case,

$$\widetilde{\varphi}(a) = \varphi(a_1)\varphi(a_2)...\varphi(a_l) = \varphi(p)\widetilde{\varphi}(a) = \widetilde{\varphi}(a)\varphi(p),$$

showing that the possible values for $\widetilde{\varphi}(pa)$ and $\widetilde{\varphi}(ap)$ are all equal. □

Lemma 5.3. *Let $(A_q, \widetilde{\varphi})$ be the augmented cpc system associated with (A, φ). If $a, b \in A_q$ with $qa - bq \in \ker(\widetilde{\phi})$ then*

$$\widetilde{\varphi}(a^*)\widetilde{\varphi}(a) - \widetilde{\varphi}(a^*)\widetilde{\varphi}(b) - \widetilde{\varphi}(b^*)\widetilde{\varphi}(a) + \widetilde{\varphi}(b^*b) = 0.$$

If $qa - aq \in \ker(\widetilde{\phi})$ then $\widetilde{\varphi}(a^*)\widetilde{\varphi}(a) = \widetilde{\varphi}(a^*a)$.

Proof. The proof of Lemma 1.5 shows that $\ker(\widetilde{\phi})$ is contained in the left ideal $\{c \in A_q \mid \widetilde{\varphi}(c^*c) = 0\}$. With $c = qa - bq$ the first identity is $\widetilde{\varphi}(c^*c) = 0$. The second one follows by setting $a = b$. □

Proposition 5.4. *For (A,φ) a cpc system, let $\widetilde{\phi} : A_q \to \mathcal{L}(\mathcal{E}_{\widetilde{\varphi}})$ be the $*$-homomorphism defining the left action of A_q on the correspondence $\mathcal{E}_{\widetilde{\varphi}}$. Then*
a. $\varphi(p) = e$ is a projection of A if and only if $\widetilde{\phi}(pq) = \widetilde{\phi}(qp)$.
b. $\varphi(p) = p$ if and only if $\widetilde{\phi}(q) = \widetilde{\phi}(pq)$.

Proof. The identity $\widetilde{\phi}(pq) = \widetilde{\phi}(qp)$ holds if and only if $\langle pqr \otimes_{\widetilde{\varphi}} u, s \otimes_{\widetilde{\varphi}} v\rangle = \langle qpr \otimes_{\widetilde{\varphi}} u, s \otimes_{\widetilde{\varphi}} v\rangle$ for all simple tensors $r \otimes_{\widetilde{\varphi}} u$ and $s \otimes_{\widetilde{\varphi}} v$ in $\mathcal{E}_{\widetilde{\varphi}}$. If $\varphi(p) = e$ is a projection of A then by Lemma 5.2 the left hand side

$$\begin{aligned}\langle pqr \otimes_{\widetilde{\varphi}} u, s \otimes_{\widetilde{\varphi}} v\rangle &= \langle u, \widetilde{\varphi}(r^*qps)v\rangle_{A_q} = \langle u, \widetilde{\varphi}(r)^*\widetilde{\varphi}(ps)v\rangle \\ &= \langle u, \widetilde{\varphi}(r)^*\varphi(p)\widetilde{\varphi}(s)v\rangle \,.\end{aligned}$$

This, however, is also equal to $\langle qpr \otimes_{\widetilde{\varphi}} u, s \otimes_{\widetilde{\varphi}} v\rangle$. If $\varphi(p) = p$ the above right hand side further simplifies to $\langle u, \widetilde{\varphi}(r)^*\widetilde{\varphi}(s)v\rangle$ which is equal to $\langle qr \otimes_{\widetilde{\varphi}} u, s \otimes_{\widetilde{\varphi}} v\rangle$, so $\widetilde{\phi}(q) = \widetilde{\phi}(pq)$.

Conversely, if $\widetilde{\phi}(qp - pq) = 0$ then the second statement of Lemma 5.3 implies $\varphi(p)\varphi(p) = \varphi(p)$, and $\varphi(p)$ is a projection. This Lemma also implies (after setting $a = q$ and $b = p$) that if $\widetilde{\phi}(q - pq) = 0$ then $\varphi(p) = p$. □

Theorem 5.5. *Let (A,φ) be a cpc system and K an ideal of A contained in $J_{\mathcal{E}_\varphi} = \phi^{-1}(\mathcal{K}(\mathcal{E})) \cap (\ker\phi)^\perp$. Let $(T_{\widetilde{\varphi}}, \pi_{\widetilde{\varphi}}) : \mathcal{E}_{\widetilde{\varphi}} \to \mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})$ be the universal representation of $\mathcal{E}_{\widetilde{\varphi}}$ coisometric on K_q.*

a. $\varphi(p) = e$ is a projection if and only if the projections $\pi_{\widetilde{\varphi}}(p) = \pi_{\widetilde{\varphi}}(q)$ commute.

b. φ is unital if and only if $\pi_{\widetilde{\varphi}}(A_q)$ is unital and $\pi_{\widetilde{\varphi}}(p)$ is the unit in $\pi_{\widetilde{\varphi}}(A_q)$.

Proof. Since the ideal K is also contained in $J(\mathcal{E}_\varphi)$ then, as noted after Definition 4.3, K_q is contained in $J(\mathcal{E}_{\widetilde{\varphi}})$. Also, the coisometric hypothesis on K_q implies the ideal $K_q \cap (\ker\widetilde{\phi}) \subseteq \ker\pi_{\widetilde{\varphi}}$. Since both $pq - qp \in K_q$ and $q - qp \in K_q$, Proposition 5.4 implies that the condition $\varphi(p) = e$ is a projection of A is equivalent to $pq - qp \in K_q \cap (\ker\widetilde{\phi})$, while the condition $\varphi(p) = p$ is equivalent to $q - qp \in K_q \cap (\ker\widetilde{\phi})$. Therefore the conditions imply these elements are in $\ker\pi_{\widetilde{\varphi}}$, and both forward implications follow.

For the converse implications note that the hypothesis on K implies, by Corollary 4.7, that $\ker\pi_{\widetilde{\varphi}}$ is contained in $\{a \in A_q \mid \widetilde{\varphi}(a^*a) = 0\}$. Calculating $\widetilde{\varphi}(a^*a) = 0$ for $a = qp - pq \in \ker\pi_{\widetilde{\varphi}}$ yields $\varphi(p) - \varphi(p)^2 = 0$. The hypothesis for part b implies the hypothesis of part a, so $\varphi(p)$ is a projection in A. Then calculating $\widetilde{\varphi}(a^*a) = 0$ for $a = q - pq \in \ker\pi_{\widetilde{\varphi}}$ yields $p - \varphi(p) = 0$. □

This illustrates that there are ready examples where a representation of the augmented correspondence $\mathcal{E}_{\widetilde{\varphi}}$ which is coisometric on the ideal K_q of A_q may not be injective, even though its restriction to $\mathcal{E}_\varphi$ coisometric on K may be injective. For example the universal coisometric representation of $\mathcal{E}_\varphi$ is injective when $K \subseteq J_{\mathcal{E}_\varphi}$. One may interpret $\ker\pi_{\widetilde{\varphi}}$ as reflecting a lack of 'freeness' in the original system (A,φ).

We remark that the proof of Theorem 5.5 shows that the two forward implications hold if the ideal $K \subseteq J(\mathcal{E}_\varphi)$.

Corollary 5.6. *Let (A, φ) be a cpc system and K an ideal of A contained in $J(\mathcal{E}_\varphi)$. If φ is unital then the Cuntz–Pimsner C^*-algebra $\mathcal{O}(K, \mathcal{E}_\varphi)$ is isomorphic to the Cuntz–Pimsner algebra $\mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})$ of the augmented correspondence $\mathcal{E}_{\widetilde{\varphi}}$.*

Proof. By Theorem 4.5 the relative Cuntz–Pimsner C^*-algebra $\mathcal{O}(K, \mathcal{E}_\varphi)$ is isomorphic to the corner $\pi_{\widetilde{\varphi}}(p)\mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})\pi_{\widetilde{\varphi}}(p)$. The remark following Theorem 5.5 shows $\pi_{\widetilde{\varphi}}(p)$ is the identity of $\pi_{\widetilde{\varphi}}(A_q)$. Since $\mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}}) = C^*(\mathbf{T}_{\widetilde{\varphi}}, \pi_{\widetilde{\varphi}})$ (where $\mathbf{T}_{\widetilde{\varphi}} = T_{\widetilde{\varphi}}(q \otimes_{\widetilde{\varphi}} p))$ and the final projection of $\mathbf{T}_{\widetilde{\varphi}} = \pi_{\widetilde{\varphi}}(q) \leq \pi_{\widetilde{\varphi}}(p)$, we have that ($\mathbf{T}_{\widetilde{\varphi}}$ is an isometry and) $\pi_{\widetilde{\varphi}}(p)$ is the unit of $\mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})$. □

Consider now the further special case of φ equal to a $*$-endomorphism β of A. The following observation appears standard (cf. [11] Lemma 3.25).

Lemma 5.7. *Let (A, φ) be the cpc system where φ is a $*$-endomorphism β of A. Then $p \otimes \beta(r)u = r \otimes u$ in the A-A correspondence $\mathcal{E}_\beta$. The ideal $J(\mathcal{E}_\beta) = A$.*

Proof. Let $r, u, s, v \in A$. The first equality follows by noting, since β is an endomorphism, that

$$\langle r \otimes_\beta u, s \otimes_\beta v\rangle = \langle u, \beta(r^*s)v\rangle_A = u^*\beta(r^*)\beta(s)v$$

while

$$\langle p \otimes_\beta \beta(r)u, s \otimes_\beta v\rangle = \langle \beta(r)u, \beta(ps)v\rangle_A = u^*\beta(r^*)\beta(s)v.$$

It follows that $\theta_{p\otimes_\beta p, p\otimes_\beta p}$ maps $(r \otimes_\beta u)$ to

$$p \otimes_\beta p \langle p \otimes_\beta p, r \otimes_\beta u\rangle = p \otimes \beta(r)u = r \otimes u.$$

and therefore $\theta_{p\otimes_\beta p, p\otimes_\beta p} = \phi(p)$, the identity map in $\mathcal{L}(\mathcal{E}_\beta)$. Thus $p \in J(\mathcal{E}_\beta)$ and $J(\mathcal{E}_\beta) = A$. □

For a representation $(T, \pi) : \mathcal{E}_\beta \to C^*(T, \pi)$ of $\mathcal{E}_\beta$ Lemma 5.7 (cf. [11] Proposition 3.26) implies that

$$\mathbf{T}\mathbf{T}^* = \psi_T(\theta_{p\otimes_\beta p, p\otimes_\beta p}) = \psi_T(\phi(p)),$$

for $\mathbf{T} = T(p \otimes_\beta p)$. If in addition the representation (T, π) is coisometric on $J(\mathcal{E}_\beta) = A$ then $\mathbf{T}\mathbf{T}^* = \pi(p)$.

If β is an injective $*$-endomorphism of A then the ideal $J_{\mathcal{E}_\beta} = J(\mathcal{E}_\beta) = A$, so if a representation (T, π) is coisometric on the ideal $J_{\mathcal{E}_\beta}$ then $\mathbf{T}\mathbf{T}^* = \pi(p)$ and the partial isometry $\mathbf{T}$ must be a coisometry in $C^*(T, \pi)$. It follows from Theorem 5.5 that if β is a unital injective $*$-endomorphism then the coisometry $\mathbf{T}$ implementing β is also an isometry, so is necessarily a unitary in $C^*(T, \pi)$.

Consider the augmented system $(A_q, \widetilde{\beta})$ if φ is a $*$-endomorphism β of A. First note that $\widetilde{\beta}$ is then also a $*$-endomorphism of A_q with $\widetilde{\beta}(q) = p$.

Lemma 5.8. *Let (A, φ) be a cpc system with φ a $*$-endomorphism β of A. Then $\widetilde{\phi}(q)$ is the identity of $\mathcal{L}(\mathcal{E}_{\widetilde{\beta}})$.*

Proof. It is sufficient to show that $m \otimes_{\widetilde{\beta}} n = qm \otimes_{\widetilde{\beta}} n$ in $\mathcal{E}_{\widetilde{\beta}}$ for $m, n \in A_q$. For $a \otimes_{\widetilde{\beta}} b$ a simple tensor in $\mathcal{E}_{\widetilde{\beta}}$ calculate $\left\langle m \otimes_{\widetilde{\beta}} n, a \otimes_{\widetilde{\beta}} b\right\rangle = \left\langle n, \widetilde{\beta}(m^*a)b\right\rangle_{A_q}$. Using that $\widetilde{\beta}$ is a $*$-endomorphism this is equal to

$$\left\langle n, \widetilde{\beta}(m)^*\widetilde{\beta}(a)b\right\rangle_{A_q}$$

which in turn is equal to

$$\left\langle \widetilde{\beta}(m)n, \widetilde{\beta}(a)b\right\rangle_{A_q} = \left\langle q \otimes_{\widetilde{\beta}} \widetilde{\beta}(m)n, a \otimes_{\widetilde{\beta}} b\right\rangle.$$

Now apply part b of Proposition 2.2 which shows that $q \otimes_{\widetilde{\beta}} \widetilde{\beta}(m)n = qm \otimes_{\widetilde{\beta}} n$ in $\mathcal{E}_{\widetilde{\beta}}$. □

Proposition 5.9. *Consider (A, φ) where φ is an injective $*$-endomorphism β of A. Let $K = J_{\mathcal{E}_\beta}$. Then $K_q = A_q$, and if $(\widetilde{T}, \widetilde{\pi}) : \mathcal{E}_{\widetilde{\beta}} \to C^*(\widetilde{T}, \widetilde{\pi})$ is a representation of $\mathcal{E}_{\widetilde{\beta}}$ coisometric on K_q, then $\widetilde{\mathbf{T}} = \widetilde{T}(q \otimes p)$ is a coisometry in $C^*(\widetilde{T}, \widetilde{\pi})$. Furthermore, if β is unital, then $\widetilde{\pi}(p) = \widetilde{\pi}(q)$ and $\widetilde{\mathbf{T}} = \widetilde{T}(q \otimes p)$ is a unitary in $C^*(\widetilde{T}, \widetilde{\pi})$.*

Proof. By Definition 4.3 and the remark following it we have $K_q \subseteq J(\mathcal{E}_{\widetilde{\beta}})$ for K any ideal of $J(\mathcal{E}_\beta)$. Since β is an injective $*$-endomorphism of A, $J_{\mathcal{E}_\beta} = J(\mathcal{E}_\beta) = A$ and K_q must be all of A_q, so in particular must contain p.

Since $\widetilde{\phi}(q) = Id_{\mathcal{L}(\mathcal{E}_{\widetilde{\beta}})}$ by Lemma 5.8, $\widetilde{\phi}(p) = \widetilde{\phi}(qp) = \widetilde{\phi}(pq)$. Therefore if $(\widetilde{T}, \widetilde{\pi})$ is coisometric on K_q, so $\psi_{\widetilde{T}} \circ \widetilde{\phi} = \widetilde{\pi}$ on A_q, then $\widetilde{\pi}(p) = \widetilde{\pi}(p)\widetilde{\pi}(q) = \widetilde{\pi}(q)\widetilde{\pi}(p)$. Thus $\widetilde{\pi}(p) \leq \widetilde{\pi}(q)$. To see that $\widetilde{\mathbf{T}}$ is a coisometry it is enough, using Proposition 3.11, to check that $\widetilde{\pi}(q)$ is the identity of $C^*(\widetilde{T}, \widetilde{\pi})$. However $\widetilde{\pi}(q)$ is the identity for $\widetilde{\pi}(A_q)$, also a left unit for $\widetilde{\mathbf{T}}$, and a right unit for $\widetilde{\mathbf{T}}$ since $\widetilde{\pi}(p)$ is.

If $\beta(p) = p$, then

$$\widetilde{\beta}(m^*pa) = \widetilde{\beta}(m)^*\widetilde{\beta}(p)\widetilde{\beta}(a) = \widetilde{\beta}(m)^*p\widetilde{\beta}(a) = \widetilde{\beta}(m)^*\widetilde{\beta}(a)$$

for $m, a \in A_q$. A computation similar to that in first part of Lemma 5.8 shows that $pm \otimes_{\widetilde{\beta}} n = m \otimes_{\widetilde{\beta}} n$, showing that $\widetilde{\phi}(p) = Id_{\mathcal{L}(\mathcal{E}_{\widetilde{\beta}})}$. Thus $\widetilde{\phi}(q) = \widetilde{\phi}(p)$, which implies $\widetilde{\pi}(p) = \widetilde{\pi}(q)$ by the coisometric condition, and $\widetilde{\mathbf{T}}$ is unitary. □

6. A quotient system (A_1, φ_1)

This section considers natural quotient systems (that depend on the coisometry ideal K of A) of the augmented cpc system $(A_q, \widetilde{\varphi})$. These systems modify the free aspects of the algebra A_q to reflect properties of the original system.

First recall that for ${}_B\mathcal{E}_B$ a C^*-correspondence over a C^*-algebra B an ideal I of B is said to be $\mathcal{E}$-invariant if $\phi(I)\mathcal{E} \subseteq \mathcal{E}I$, where $\mathcal{E}I = \{xb \mid x \in \mathcal{E}, b \in I\}$ is a correspondence over I. Note that $\mathcal{E}I = \{x \mid \langle x, y\rangle_B \in I \text{ for all } y \in \mathcal{E}\}$ ([7]). We include the following proof for completeness although it is generally known (cf. [10] Lemma 5.10(i)).

Lemma 6.1. *Let (T, π) be a representation of the C^*-correspondence $\mathcal{E}_\varphi$ over A associated with a cpc system (A, φ). The ideal $I = \ker \pi$ of A is invariant under φ, and is $\mathcal{E}_\varphi$-invariant.*

Proof. Let $a \in I$. Then $\pi(\varphi(a)) = \mathbf{T}^*\pi(a)\mathbf{T} = 0$, so $\varphi(I) \subseteq I$.

Let $r \otimes_\varphi u$ be a simple tensor in $\mathcal{E}_\varphi$. To show that I is $\mathcal{E}_\varphi$-invariant it is enough to show that $\langle \phi_\varphi(a)(r \otimes_\varphi u), s \otimes_\varphi v\rangle \in I$ for all $s \otimes_\varphi v$ simple tensors in $\mathcal{E}_\varphi$. However this inner product is $u^*\varphi(r^*a^*s)v$, which is contained in I since $\varphi(I) \subseteq I$. □

In general an $\mathcal{E}$-invariant ideal I of B defines a correspondence $\mathcal{E}/\mathcal{E}I$ over B/I where the right Hilbert module structure (cf. [7]) is given by (where [ ] denotes the appropriate quotient class)

$$[x][a] = [xa] \text{ for } a \in B, x \in \mathcal{E},$$

$$\langle [e], [f]\rangle_{B/I} = [\langle e, f\rangle_B] \text{ for } e, f \in \mathcal{E}$$

and where the left action $\phi_{B/I}$ on $\mathcal{E}/\mathcal{E}I$ is given by

$$\phi_{B/I}([b])([x]) = [\phi(b)x] \ \text{ for } b \in B, x \in \mathcal{E}.$$

Let $K \subseteq J(\mathcal{E}_\varphi)$ be an ideal and $(T_{\widetilde{\varphi}}, \pi_{\widetilde{\varphi}}) : \mathcal{E}_{\widetilde{\varphi}} \to C^*(T_{\widetilde{\varphi}}, \pi_{\widetilde{\varphi}}) = \mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})$ the universal representation of $\mathcal{E}_{\widetilde{\varphi}}$ coisometric on K_q. The previous section provides some basic examples where $\ker \pi_{\widetilde{\varphi}}$ is nonzero for various choices of K. Lemma 6.1 shows that the ideal $\ker \pi_{\widetilde{\varphi}}$ is invariant under $\widetilde{\varphi}$ and is $\mathcal{E}_{\widetilde{\varphi}}$-invariant. The following cpc system $(A_1, \varphi_1)_K$ (denoted (A_1, φ_1) if the ideal K is understood) is therefore well defined.

Definition 6.2. Denote the ideal $\ker \pi_{\widetilde{\varphi}}$ of A_q by I_q. Define a quotient cpc system $(A_1, \varphi_1)_K$ as follows: A_1 is the quotient C^*-algebra A_q/I_q with $\chi : A_q \to A_1$ the natural quotient map, and φ_1 is given by $\varphi_1(\chi(m)) = \chi(\widetilde{\varphi}(m))$ for $m \in A_q$. Set $\mathcal{E}_1$ to be the correspondence $\mathcal{E}_{\varphi_1} = A_1 \otimes_{\varphi_1} A_1$ associated with the the cpc system (A_1, φ_1) with the left action denoted by ϕ_1.

It is clear that φ_1 is a cpc map on A_1. The cpc system (A_1, φ_1) depends on the ideal K of A initially specified for the coisometric relation. By construction the representation $\pi_{\widetilde{\varphi}}$ drops to an injective representation $\pi_1 : A_1 \to \mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})$.

Note that in certain natural cases A is injectively included in A_1 via the natural $*$-homomorphism. For example, if $K \subseteq J_{\mathcal{E}_\varphi}$then the universal representation $(T_\varphi, \pi_\varphi) : \mathcal{E}_\varphi \to \mathcal{O}(K, \mathcal{E}_\varphi)$ is injective. Now recall from the proof of Theorem 4.5 that there is an isomorphism $\gamma : \mathcal{O}(K, \mathcal{E}_\varphi) \to \pi_{\widetilde{\varphi}}(p)\mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})\pi_{\widetilde{\varphi}}(p)$ so that the restriction of the universal representation $(T_{\widetilde{\varphi}r}, \pi_{\widetilde{\varphi}r}) = \gamma \circ (T_\varphi, \pi_\varphi)$. It follows that the restricted representation $\pi_{\widetilde{\varphi}r}$ of A is injective, and therefore $A \cap \ker \pi_{\widetilde{\varphi}} = 0$.

Notation 6.3. *There is a well defined contractive linear map* $L : \mathcal{E}_{\widetilde{\varphi}} \to \mathcal{E}_1$, *described on simple tensors by* $m \otimes_{\widetilde{\varphi}} n \to \chi(m) \otimes_{\varphi_1} \chi(n)$, *which satisfies*

$$\langle L(m \otimes_{\widetilde{\varphi}} n), L(a \otimes_{\widetilde{\varphi}} b)\rangle_{A_1} = \chi(\langle m \otimes_{\widetilde{\varphi}} n, a \otimes_{\widetilde{\varphi}} b\rangle_{A_q}).$$

Proposition 6.4. *The correspondence* $\mathcal{E}_{\widetilde{\varphi}}/\mathcal{E}_{\widetilde{\varphi}}I_q$ *over* A_1 *formed via the* $\mathcal{E}_{\widetilde{\varphi}}$*-invariant ideal* I_q *is isomorphic to the correspondence* $\mathcal{E}_1$ *over* A_1 *associated with the cpc system* (A_1, φ_1).

Proof. The pair $(L, \chi) : \mathcal{E}_{\widetilde{\varphi}} \to \mathcal{E}_1$ is a morphism of correspondences (cf. [10] Definition 2.3), and therefore the relations

$$L(\widetilde{\phi}(a)(m \otimes_{\widetilde{\varphi}} n)) = \phi_1(\chi(a))L(m \otimes_{\widetilde{\varphi}} n)$$
$$L(m \otimes_{\widetilde{\varphi}} n)\chi(a) = L(m \otimes_{\widetilde{\varphi}} na).$$

hold for $a, b, m, n \in A_q$.

Let $x \in \mathcal{E}_{\widetilde{\varphi}}$. Since the image of L contains the dense subspace $A_1 \otimes_{\varphi_1} A_1$ of $\mathcal{E}_1$, it follows that the vector $L(x) = 0$ if and only if $\langle L(x), L(y)\rangle_{A_1} = 0$ for all $y \in \mathcal{E}_{\widetilde{\varphi}}$. Using the first relation above this is equivalent to $\chi(\langle x, y\rangle_{A_q}) = 0$, or since π_1 is injective, $\pi_{\widetilde{\varphi}}(\langle x, y\rangle_{A_q}) = \pi_1(\chi \langle x, y\rangle_{A_q}) = 0$ for all $y \in \mathcal{E}_{\widetilde{\varphi}}$. This is equivalent to $\langle x, y\rangle \in \ker \pi_{\widetilde{\varphi}} = I_q$ for all $y \in \mathcal{E}_{\widetilde{\varphi}}$, or equivalently, $x \in \mathcal{E}_{\widetilde{\varphi}} I_q$. Thus the kernel of L is $\mathcal{E}_{\widetilde{\varphi}} I_q$.

The first equality above implies that the linear map $U : \mathcal{E}_{\widetilde{\varphi}}/\mathcal{E}_{\widetilde{\varphi}} I_q \to \mathcal{E}_1$ defined on the quotient $\mathcal{E}_{\widetilde{\varphi}}/\mathcal{E}_{\widetilde{\varphi}} I_q$ via L is an isometry of Hilbert modules. Since L and therefore U has dense range, it (and therefore also L) is surjective. Therefore U is a unitary of correspondences. □

Remark 6.5. We thank the referee for pointing out that this amounts to an example of a general process which applies to a correspondence $\mathcal{E}$ and a given ideal $K \subseteq J(\mathcal{E}_\varphi)$ ([12] Section 5.1); here this process can be applied to the correspondence $\mathcal{E}_{\widetilde{\varphi}}$ over A_q. Proposition 6.4 above along with Theorem 5.4 of [12] imply that the ideal I_q is the 'reduction ideal' $(K_q)_\infty$ of [12], a recursively defined ideal equalling the smallest $\mathcal{E}_{\widetilde{\varphi}}$-invariant ideal in A_q satisfying an additional condition. We point out that this additional condition (for a correspondence $\mathcal{E}$ and the ideal $J_{\mathcal{E}}$) appears in [14] as the definition of a $\mathcal{E}$-saturated ideal.

Define $\Psi_L : \mathcal{K}(\mathcal{E}_{\widetilde{\varphi}}) \to \mathcal{K}(\mathcal{E}_1)$ by $\Psi_L(\theta_{x,y}) = \theta_{L(x),L(y)}$ for $x, y \in \mathcal{E}_{\widetilde{\varphi}}$. The surjectivity of L implies Ψ_L is surjective. It follows from the morphism properties of L (Proposition 6.4), and by verifying on the simple tensors in $\mathcal{E}_{\widetilde{\varphi}}$, that

$$\Psi_L(k) \circ L = L \circ k \text{ for } k \in \mathcal{K}(\mathcal{E}_{\widetilde{\varphi}}).$$

This identity, along with $L \circ \widetilde{\phi}(a) = \phi_1(\chi(a)) \circ L$ for $a \in A_q$ (from Proposition 6.4) yields

$$\Psi_L \circ \widetilde{\phi} = \phi_1 \circ \chi \text{ on } K_q,$$

and therefore $\chi(K_q) \subseteq J(\mathcal{E}_1)$.

Definition 6.6. For $K \subseteq J(\mathcal{E}_\varphi)$ an ideal and $(T_{\widetilde{\varphi}}, \pi_{\widetilde{\varphi}}) : \mathcal{E}_{\widetilde{\varphi}} \to \mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})$ the universal representation of $\mathcal{E}_{\widetilde{\varphi}}$ coisometric on K_q set $K_1 = \chi(K_q)$, an ideal of A_1 contained in $J(\mathcal{E}_1)$.

The universal representation $(T_{\widetilde{\varphi}}, \pi_{\widetilde{\varphi}})$ of $\mathcal{E}_{\widetilde{\varphi}}$ coisometric on K_q determines a representation (T_1, π_1) of $\mathcal{E}_1$ with image in $C^*(T_{\widetilde{\varphi}}, \pi_{\widetilde{\varphi}})$. First set $\pi_1 : A_1 \to \mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})$ as above, so $\pi_{\widetilde{\varphi}} = \pi_1 \circ \chi$, and define $T_1 : \mathcal{E}_1 \to \mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})$ by

$$T_1 \circ L = T_{\widetilde{\varphi}}.$$

This is a well defined linear map on $\mathcal{E}_{\varphi_1}$ since

$$T_{\widetilde{\varphi}}(m \otimes_{\widetilde{\varphi}} n) = \pi_{\widetilde{\varphi}}(m)T_{\widetilde{\varphi}}(q \otimes_{\widetilde{\varphi}} p)\pi_{\widetilde{\varphi}}(n) = \pi_1(\chi(m))T_{\widetilde{\varphi}}(q \otimes_{\widetilde{\varphi}} p)\pi_1(\chi(n)).$$

It is straightforward to check that $(T_1, \pi_1) : \mathcal{E}_{\varphi_1} \to C^*(T_{\widetilde{\varphi}}, \pi_{\widetilde{\varphi}}) = \mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})$ is a representation of $\mathcal{E}_1$ coisometric on the ideal K_1.

Proposition 6.7. *Let (A, φ) be a cpc system and (A_1, φ_1) its associated cpc system. For an ideal $K \subseteq J(\mathcal{E}_\varphi)$ then $K_1 \subseteq J_{\mathcal{E}_1}$.*

Proof. Let $(T_{\widetilde{\varphi}}, \pi_{\widetilde{\varphi}}) : \mathcal{E}_{\widetilde{\varphi}} \to \mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})$ denote the universal representation of $\mathcal{E}_{\widetilde{\varphi}}$ coisometric on K_q. It is enough to show that $K_1 \subseteq \ker(\phi_1)^\perp$ since, as already noted, $\chi(K_q) \subseteq J(\mathcal{E}_1)$. For $\chi(a) \in K_1$ where $a \in K_q$, and $\chi(b) \in \ker \phi_1$ where $b \in A_q$ it suffices to show that $\chi(a)\chi(b) = 0$, i.e., that $ab \in \ker \pi_{\widetilde{\varphi}}$. However $\chi(b)$ is in the ideal $\ker \phi_1$, so $\chi(ab) \in \ker \phi_1$ and $0 = \Psi_{T_1}(\phi_1(\chi(ab))$. The above identities show that this is equal to

$$\Psi_{T_1}(\Psi_L(\widetilde{\phi}(ab))) = \Psi_{T_1 \circ L}((\widetilde{\phi}(ab)) = \Psi_{T_{\widetilde{\varphi}}}((\widetilde{\phi}(ab)) = \pi_{\widetilde{\varphi}}(ab)$$

the latter equality following from $ab \in K_q$ the ideal of coisometry for $(T_{\widetilde{\varphi}}, \pi_{\widetilde{\varphi}})$. □

Theorem 6.8. *Let (A, φ) be a cpc system, $(A_q, \widetilde{\varphi})$ its augmented cpc system, $K \subseteq J(\mathcal{E}_\varphi)$ an ideal, and (A_1, φ_1) the associated quotient cpc system. The universal C^*-algebra $\mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})$ for representations of $\mathcal{E}_{\widetilde{\varphi}}$ coisometric on K_q is isomorphic to the universal C^*-algebra $\mathcal{O}(K_1, \mathcal{E}_1)$ for representations of the correspondence $\mathcal{E}_1$ coisometric on K_1, and $\mathcal{O}(K, \mathcal{E}_\varphi)$ and $\mathcal{O}(K_1, \mathcal{E}_1)$ are Morita equivalent C^*-algebras.*

Proof. By Theorem 4.5 Morita equivalence follows once $\mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})$ is shown to be isomorphic to $\mathcal{O}(K_1, \mathcal{E}_1)$.

Let $(T_{\widetilde{\varphi}}, \pi_{\widetilde{\varphi}})$ denote the universal representation $(T_{\widetilde{\varphi}}, \pi_{\widetilde{\varphi}}) : \mathcal{E}_{\widetilde{\varphi}} \to \mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})$ of $\mathcal{E}_{\widetilde{\varphi}}$ coisometric on K_q. Since I_q is the kernel of $\pi_{\widetilde{\varphi}}$, the ideal in $\mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})$ generated by $\pi_{\widetilde{\varphi}}(I_q)$ is zero, and therefore the quotient of $\mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})$ by this 0 ideal must be $\mathcal{O}(K_q, \mathcal{E}_{\widetilde{\varphi}})$. Applying the isomorphism part of Theorem 3.1 of [7], this quotient algebra is isomorphic to $\mathcal{O}(K_1, \mathcal{E}_1)$. □

The cpc system (A_1, φ_1) may be viewed as a natural extension of the initial cpc system (A, φ) which minimizes the extraneous free aspects of cpc system $(A_q, \widetilde{\varphi})$. There are many natural questions concerning the relationships of the cpc system (A_1, φ_1) to the cpc system (A, φ), including, for example, conditions determining it uniquely up to equivalence, that will be explored elsewhere.

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