Adv. Oper. Theory 2 (2017), no. 4, 547–583
http://doi.org/10.22034/aot.1709-1231
ISSN: 2538-225X (electronic)
http://aot-math.org

L^p FOURIER TRANSFORMATION ON NON-UNIMODULAR LOCALLY COMPACT GROUPS

MARIANNE TERP

Communicated by M. S. Moslehian

ABSTRACT. Let G be a locally compact group with modular function Δ and left regular representation λ. We define the L^p Fourier transform of a function $f \in L^p(G)$, $1 \leq p \leq 2$, to be essentially the operator $\lambda(f)\Delta^{\frac{1}{q}}$ on $L^2(G)$ (where $\frac{1}{p}+\frac{1}{q}=1$) and show that a generalized Hausdorff–Young theorem holds. To do this, we first treat in detail the spatial L^p spaces $L^p(\psi_0)$, $1 \leq p \leq \infty$, associated with the von Neumann algebra $M = \lambda(G)''$ on $L^2(G)$ and the canonical weight ψ_0 on its commutant. In particular, we discuss isometric isomorphisms of $L^2(\psi_0)$ onto $L^2(G)$ and of $L^1(\psi_0)$ onto the Fourier algebra $A(G)$. Also, we give a characterization of positive definite functions belonging to $A(G)$ among all continuous positive definite functions.

INTRODUCTION

Suppose that G is an abelian locally compact group with dual group $\hat{G}$. Then the Hausdorff–Young theorem states that if $f \in L^p(G)$, where $1 \leq p \leq 2$, then its Fourier transform $\mathcal{F}(f)$ belongs to $L^q(\hat{G})$, where $\frac{1}{p}+\frac{1}{q}=1$ (cf. [23, p. 117]). In the case of Fourier series, i.e. when G is the circle group and $\hat{G}$ the integers, this is a classical result due to F. Hausdorff and W. H. Young. [24, p. 101]. An extension of this theorem to all unimodular locally compact groups was given by R. A. Kunze [14]. In this paper we shall treat the case of general, i.e. not necessarily unimodular, locally compact group.

Date: Received: Aug. 19, 2017; Accepted: Sep. 19, 2017.
2010 *Mathematics Subject Classification.* Primary 39B82; Secondary 44B20, 46C05.
Key words and phrases. L^p Fourier transformation, locally compact group, Fourier algebra, positive definite function.

In order to describe our results, we first briefly recall those of [14]. Suppose that f is an integrable function on a unimodular group G. Then we consider the Fourier transform $\mathcal{F}(f)$ to be the operator $\lambda(f)$ of left convolution by f on $L^2(G)$. (As pointed out by Kunze [14], this point of view is justified by the fact that in the abelian case $\lambda(f)$ is unitarily equivalent to the operator on $L^2(\hat{G})$ of multiplication by the (ordinary) Fourier transform $\hat{f}$. The Fourier transformation maps $L^1(G)$ into the space $L^\infty(G')$, defined as the von Neumann algebra M generated by $\lambda(L^1(G))$. More generally, one can define $\lambda(f)$ as an (unbounded) operator on $L^2(G)$ even for functions f not in $L^1(G)$. It then turns out that λ maps each $L^p(G)$, $1 \leq p \leq 2$, norm-decreasingly into a certain space $L^q(G')$ of closed densely defined operators on $L^2(G)$ (where $\frac{1}{p} + \frac{1}{q} = 1$). This is the Hausdorff–Young theorem. Kunze introduced the spaces $L^q(G')$ as spaces of measurable operators (in the sense of [21]) with respect to the canonical gage on M [14, p. 533]. An equivalent but simpler way of introducing the $L^q(G')$ is to consider the trace φ_0 on M characterized by $\varphi_0(\lambda(h) * \lambda(h)) = \|h\|_2^2$ for certain functions h, and then take $L^q(G')$ to be $L^q(M, \varphi_0)$ as defined by E. Nelson [15], viewing it as a space of "φ_0-measurable" operators [15, Theorem 5]. (In either case, the L^q spaces obtained are isomorphic to the abstract L^q spaces of J. Dixmier [5] associated with a trace on a von Neumann algebra.)

In the general (non-unimodular) case, φ_0 is no longer a trace, and the lack of adequate spaces L^q into which the $L^p(G)$ were to be mapped for a long time prevented the formulation of a Hausdorff–Young theorem, except for some special cases ([7, §8], [20, Proposition 15]). In [10], however, U. Haagerup constructed abstract L^p spaces corresponding to an arbitrary von Neumann algebra, and combining methods from [10] with the recent theory of spatial derivatives by A. Connes [2], M. Hilsum has developed a spatial theory of L^p spaces [12]. If M is a von Neumann algebra acting on a Hilbert space H and ψ is a weight on its commutant M', then the elements of $L^p(M, H, \psi)$ are (in general unbounded) operators on H satisfying a certain homogeneity property with respect to ψ. We shall see that when using these spaces (in the particular case of $M = \lambda(G)''$, $H = L^2(G)$, and $\psi =$ the canonical weight on M') and when defining the L^p Fourier transform of an L^p function f to be the operator $\xi \to f^* \Delta^{\frac{1}{q}} \xi$ on $L^2(G)$ (where Δ is the modular function of the group), one gets a nice L^p Fourier transformation theory and in particular a Hausdorff–Young theorem.

The paper is organized as follows. In Section 1 we fix the notations and describe our set-up. In Section 2, we study the L^p spaces of [12] in our particular case; we give a reformulation of the α-homogeneity property appearing in [2] that does not involve modular automorphism groups and we characterize $L^p(\psi_0)$ operators among all $(-\frac{1}{p})$-homogeneous operators. In Section 3, we treat the case $p = 2$ and obtain explicit expressions for the L^2 Fourier transformation $\mathcal{F}_2 = \mathcal{P}$, called the Plancherel transformation, as well as for its inverse.

Next, in Section 4, we deal with the case of a general $p \in [1, 2]$; we define the L^p Fourier transformation $\mathcal{F}_p$, and using interpolation (specifically, the three lines theorem) we prove our version of the Hausdorff–Young theorem.

Finally, in Section 5, we define an L^p Fourier cotransformation $\overline{\mathcal{F}}_p$ taking $L^p(\psi_0)$, $1 \leq p \leq 2$, into $L^q(G)$ and we investigate the relations between co-transformation and Fourier inversion. A detailed study of the $p = 1$ case gives a new characterization of $A(G)_+$ functions among all continuous positive definite functions on G.

1. Preliminaries and notation

Let G be a locally compact group with left Haar measure dx. We denote by $\mathcal{K}$(G) the set of continuous functions on G with compact support and by $L^p(G)$, $1 \leq p \leq \infty$, the ordinary Lebesgue spaces with respect to dx. The modular function Δ on G is given by

$$\int f(xa^{-1})dx = \Delta(a) \int f(x)dx$$

for all $f \in \mathcal{K}(G)$ and $a \in G$. For functions f on G we put

$$\check{f}(x) = f(x^{-1}), \quad \tilde{f}(x) = \overline{f(x^{-1})}, \quad f^*(x) = \Delta^{-1}(x)\overline{f(x^{-1})}$$

and

$$(Jf)(x) = \Delta^{-\frac{1}{2}}(x)\overline{f(x^{-1})}$$

for all $x \in G$. More generally, for each $p \in [1, \infty]$, we define

$$(J_p f)(x) = \Delta^{-1/p}(x)\overline{f(x^{-1})}, \quad x \in G.$$

Then in particular $J_1 f = f^*$, $J_2 f = Jf$, $J_\infty f = \tilde{f}$. Note that for each $p \in [1, \infty]$, the operation J_p is a conjugate linear isometric involution of $L^p(G)$.

We shall often make use of the following non-unimodular version of Young's inequalities for convolution:

Lemma 1.1. *(Young's convolution inequalities.) Let $p_1, p_2, p \in [1, \infty]$ and $\frac{1}{p_1} + \frac{1}{q_1} = 1$. Assume that $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p} = 1$. Then for all $f_1 \in L^{p_1}(G)$ and $f_2 \in L^{p_2}(G)$ the convolution product $f_1 * \Delta^{\frac{1}{q_1}} f_2$ exists and belongs to $L^p(G)$, and*

$$\|f_1 * \Delta^{\frac{1}{q_1}} f_2\|_p \leq \|f_1\|_{p_1} \|f_2\|_{p_2}.$$

This theorem is well-known in the unimodular case as well as in the special cases $(p_1, p_2, p) = (p_1, q_1, \infty)$ (where it follows from Hölder's inequality), $(p_1, p_2, p) = (1, p, p)$ or $(p_1, p_2, p) = (p, 1, p)$ [11, (20.14)], The general case has also been noted [13, Remark 2.2]. It can be proved by modifying the proof of [11, (20.18)] or by interpolation from the special cases mentioned above.

For operators T on the Hilbert space $L^2(G)$ we use the notation $D(T)$ (domain of T), $R(T)$ (range of T), $N(T)$ (kernel of T). If T is preclosed, we denote by $[T]$ the closure of T. If T is a positive self-adjoint operator and P the projection onto $N(T)^\perp$, then by definition T^{it}, $t \in \mathbb{R}$, is the partial isometry coinciding with the unitary $(TP)^{it}$ on $N(T)^\perp$ and O on $N(T)$. By convention, when speaking of operators, "bounded" always means "bounded and everywhere defined".

We denote by λ and ρ the left and right regular representations of G on $L^2(G)$, i.e. the unitary representations given by

$$(\lambda(x)f)(y) = f(x^{-1}y),$$

$$(\rho(x)f)(y) = \Delta^{\frac{1}{2}}(x)f(yx),$$

for all $x, y \in G$ and $f \in L^2(G)$. The corresponding representations of the algebra $L^1(G)$ (as in [4, 13.3]) are given by

$$\lambda(h)f = h * f \quad \text{and} \quad \rho(h)f = f * \Delta^{-\frac{1}{2}}\check{h}$$

for all $h \in L^1(G)$ and $f \in L^2(G)$.

We denote by M the von Neumann algebra of operators on $L^2(G)$ generated by $\lambda(G)$ (or $\lambda(\mathcal{K}(G))$, or $\lambda(L^1(G))$). In other words, M is the left von Neumann algebra of $\mathcal{K}(G)$, where $\mathcal{K}(G)$ is considered as a left Hilbert algebra [3, Definition 2.1] with convolution, involution $*$, and the ordinary inner product in $L^2(G)$. The commutant M' of M is the von Neumann algebra generated by $\rho(G)$, and $M' = JMJ$.

A function $\xi \in L^2(G)$ is called left (resp. right) bounded if left (resp. right) convolution with ξ on $\mathcal{K}(G)$ extends to a bounded operator on $L^2(G)$, i.e. if there exists a bounded operator $\lambda(\xi)$ (resp. $\lambda'(\xi)$) such that $\forall k \in \mathcal{K}(G) : \lambda(\xi)k = \xi * k$ (resp. $\lambda'(\xi)k = k * \xi$). The set of left (resp. right) bounded $L^2(G)$-functions is denoted $\mathfrak{A}_l$ (resp. $\mathfrak{A}_r$). Obviously, $\mathcal{K}(G) \subseteq \mathfrak{A}_l, \mathcal{K}(G) \subseteq \mathfrak{A}_r$, and for $\xi \in \mathcal{K}(G)$ we have $\lambda'(\xi) = \rho(\Delta^{-\frac{1}{2}}\check{\xi})$. Note that $\xi \in L^2(G)$ is left bounded if and only if the operator $\eta \to \lambda'(\eta)\xi : \mathfrak{A}_r \to L^2(G)$ extends to a bounded operator on $L^2(G)$; if this is the case, we have $\lambda(\xi)\eta = \lambda'(\eta)\xi$ for all $\eta \in \mathfrak{A}_r$. (Our definition of left-boundedness therefore agrees with [1, Définition 2.1]). If $\xi \in \mathfrak{A}_l$ and $T \in M$, then $T\xi \in \mathfrak{A}_l$ and $\lambda(T\xi) = T\lambda(\xi)$.

We denote by φ_0 the canonical weight on M [1, Définition 2.12]. Then the weight ψ_0 on M' given by $\psi_0(y) = \varphi_0(JyJ)$ for all $y \in (M')_+$ is called the canonical weight on M'. The corresponding modular automorphism groups are given by

$$\sigma_t^{\varphi_0}(x) = \Delta^{it}x\Delta^{-it}, x \in M,$$

$$\sigma_t^{\psi_0}(y) = \Delta^{-it}y\Delta^{it}, y \in M',$$

for all $t \in \mathbb{R}$. Here, Δ denotes the multiplication operator on $L^2(G)$ by the function Δ (note that we shall not distinguish in our notation between the function Δ and the corresponding multiplication operator). With this definition, Δ is in fact the modular operator of $\mathcal{K}(G)$ (as defined in [3, Lemma 2.2]).

It follows from the defining property of φ_0 [1, Théorème 2.11] that for all $y \in M'$ we have

$$\psi_0(y * y) = \begin{cases} \|\eta\|_2^2 & \text{if } y = \lambda'(\eta) \text{ for some } \eta \in \mathfrak{A}_r, \\ \infty & \text{otherwise} \end{cases}$$

We identify the Hilbert space completion H_{ψ_0} of $n_{\psi_0} = \{y \in M' | \psi_0(y * y) < \infty\}$ with $L^2(G)$ via $\eta \to \lambda'(\eta)$.

Now recall that by definition [2, Definition 1], $D(L^2(G), \psi_0)$ is the set of $\xi \in L^2(G)$ such that $y \mapsto y\xi : n_{\psi_0} \to L^2(G)$ extends to a bounded operator $R^{\psi_0}(\xi) : H_{\psi_0} \to L^2(G)$, i.e., in view of the identification of H_{ψ_0} with $L^2(G)$, such that $\eta \mapsto \lambda'(\eta)\xi : \mathfrak{A}_r \to L^2(G)$ extends to a bounded operator on $L^2(G)$. Thus $D(L^2(G), \psi_0) = \mathfrak{A}_l$, and for all $\xi \in D(L^2(G), \psi_0)$ we have $R^{\psi_0}(\xi) = \lambda(\xi)$.

If φ is a normal semi-finite weight on M, then by definition [2], $\frac{d\varphi}{d\psi_0}$ is the unique positive self-adjoint operator T satisfying

$$\forall \xi \in \mathfrak{A}_l : \varphi(\lambda(\xi)\lambda(\xi)^*) = \begin{cases} \|T^{\frac{1}{2}}\xi\|^2 & \text{if } \xi \in D(T^{\frac{1}{2}}) \\ \infty & \text{otherwise} \end{cases}$$

and

$$T^{\frac{1}{2}} = [T^{\frac{1}{2}}|_{\mathfrak{A}_l \cap D(T^{\frac{1}{2}})}].$$

In particular, we have

$$\frac{d\varphi_0}{d\psi_0} = \Delta$$

(cf. [2, Lemma 10 (b)] together with the proof of [2, Lemma 10 (a)].

If φ is a functional, then by the definition of $\frac{d\varphi}{d\psi_0}$ we have $\mathfrak{A}_l \subseteq D\left((\frac{d\varphi}{d\psi_0})^{\frac{1}{2}}\right)$ and $(\frac{d\varphi}{d\psi_0})^{\frac{1}{2}} = \left[(\frac{d\varphi}{d\psi_0})^{\frac{1}{2}}|_{\mathfrak{A}_l}\right]$. Finally, we note that the predual space M_* of the von Neumann algebra M may be viewed as a space of functions on the group in the following manner: for each $\varphi \in M_*$, define $u : G \to \mathbb{C}$ by

$$u(x) = \varphi(\lambda(x)), x \in G.$$

Then u is a continuous function on the group determining φ completely. The linear space of such functions, normed by $\|u\| = \|\varphi\|$, is exactly the Fourier algebra $A(G)$ of G introduced by P. Eymard [6] (this follows from [6, Théorème (3.10)]).

The identification of $A(G)$ with M_* is such that

$$\langle \varphi, \lambda(f) \rangle = \int \varphi(x) f(x) dx$$

for all $\varphi \in M_* \simeq A(G)$ and all $f \in L^1(G)$.

Recall that by [4, 13.4.4] a continuous function φ on G is positive definite if and only if

$$\forall \xi \in \mathcal{K}(G) : \int \varphi(x)(\xi * \xi^*)(x) dx \geq 0,$$

i.e. if and only if

$$\forall \xi \in \mathcal{K}(G) : \int \int \varphi(yx^{-1})\xi(y)\overline{\xi(x)} dy dx \geq 0.$$

If $\varphi \in A(G)$, then φ is positive definite if and only if the corresponding functional $\varphi \in M_*$ is positive. We denote by $A(G)_+$ the set of positive definite $\varphi \in A(G)$.

2. Homogeneous operators on $L^2(G)$ and the spaces $L^p(\psi_0)$

Definition 2.1. Let $\alpha \in \mathbb{R}$. An operator T on $L^2(G)$ is called α-homogeneous if

$$\forall x \in G : \rho(x)T \subseteq \Delta^{-\alpha}(x) T \rho(x).$$

Remark 2.2. (1) The O-homogeneous operators are precisely the operators affiliated with M.

(2) If T is α-homogeneous, then actually $\rho(x)T = \Delta^{-\alpha}(x)T\rho(x)$ for all $x \in G$ (to see this, replace x by x^{-1} in the definition).

(3) If T and S are both α-homogeneous, then $T + S$ is α-homogeneous. If T is α-homogeneous and S is β-homogeneous, then TS is $(\alpha+\beta)$-homogeneous. If T is densely defined and α-homogeneous, then T^* is also α-homogeneous. If T is positive self-adjoint and α-homogeneous and $\beta \in \mathbb{R}_+$, then T^β is $(\alpha\beta)$-homogeneous (use $\rho(x)T^\beta\rho(x^{-1}) = (\rho(x)T\rho(x^{-1}))^\beta$).

(4) If T is α-homogeneous for some $\alpha \in \mathbb{R}$, then the projection onto $N(T)^\perp$ belongs to M (since $N(T)$ is invariant under all $\rho(x), x \in G$).

(5) If a preclosed operator T is α-homogeneous, then its closure $[T]$ is also α-homogeneous.

(6) For each $\alpha \in \mathbb{R}$, $\Delta^{-\alpha}$ is α-homogeneous.

Lemma 2.3. *Let T be a closed densely defined operator on $L^2(G)$ with polar decomposition $T = U|T|$. Let $\alpha \in \mathbb{R}$. Then T is α-homogeneous if and only if $U \in M$ and $|T|$ is α-homogeneous.*

Proof. If T is α-homogeneous, then, by Remark 2.2(3), $|T| = (T^*T)^{\frac{1}{2}}$ is also α-homogeneous. Then for all $x \in G$ and $\xi \in D(|T|)$ we have $\rho(x)U|T|\xi = \rho(x)T\xi = \Delta^{-\alpha}(x)T\rho(x)\xi = \Delta^{-\alpha}(x)U|T|\rho(x)\xi = U\rho(x)|T|\xi$, i.e. $\rho(x)U \subseteq U\rho(x)$ on $R(|T|)$. Since the projection onto $R(|T|) = N(|T|)^\perp$ belongs to M, we conclude that U commutes with all $\rho(x)$; thus $U \in M$.

The "if"-part follows directly from Remarks 2.2(3) and 2.2(1). □

Lemma 2.4. *Let T be a closed densely defined operator on $L^2(G)$ and $\alpha \in \mathbb{C}$. If*

$$\forall x \in G : \rho(x)T \subseteq \Delta^{-\alpha}(x)T\rho(x),$$

then

$$\forall f \in \mathcal{K}(G) : \lambda'(f)T \subseteq T\lambda'(\Delta^\alpha f).$$

Proof. Let $f \in \mathcal{K}(G)$ and $\xi \in D(T)$. Then for all $\eta \in D(T^*)$ we have

$$\begin{aligned}(\rho(f)T\xi|\eta) &= \int f(x)(\rho(x)T\xi|\eta)dx \\ &= \int f(x)\Delta^{-\alpha}(x)(T\rho(x)\xi|\eta)dx \\ &= \int \Delta^{-\alpha}(x)f(x)(\rho(x)\xi|T*\eta)dx \\ &= (\rho(\Delta^{-\alpha}f)\xi|T^*\eta).\end{aligned}$$

This shows that $\rho(\Delta^{-\alpha}f)\xi \in D(T^{**}) = D(T)$, and $T\rho(\Delta^{-\alpha}f)\xi = \rho(f)T\xi$ for all $\xi \in D(T)$, i.e.

$$\rho(f)T \subseteq T\rho(\Delta^{-\alpha}f).$$

Hence for all $f \in \mathcal{K}(G)$ we have

$$\lambda'(f)T = \rho(\Delta^{-\frac{1}{2}}\check{f}) \subseteq T\rho(\Delta^{-\alpha}\Delta^{-\frac{1}{2}}\check{f}) = T\lambda'(\Delta^\alpha f).$$

□

Lemma 2.5. *Let T be a closed densely defined operator on $L^2(G)$, α-homogeneous for some $\alpha \in \mathbb{R}$. Let $\xi \in \mathfrak{A}_l$. Then for all $t \in \mathbb{R}$ we have $|T|^{it}\xi \in \mathfrak{A}_l$ and*

$$\|\lambda(|T|^{it}\xi)\| \leq \|\lambda(\xi)\|.$$

Proof. By Lemma 2.3, we have $\rho(x)|T|\rho(x^{-1}) = \Delta^{-\alpha}(x)|T|$ for all $x \in G$, whence $\rho(x)|T|^{it}\rho(x^{-1}) = \Delta^{-i\alpha t}(x)|T|^{it}$ for all $x \in G$ and all $t \in \mathbb{R}$. Then, applying the preceding lemma to $|T|^{it}$, we obtain for all $\eta \in \mathcal{K}(G)$ that

$$|T|^{it}\xi * \eta = \lambda'(\eta)|T|^{it}\xi = |T|^{it}\lambda'(\Delta^{i\alpha t}\eta)\xi = |T|^{it}\lambda(\xi)\Delta^{i\alpha t}\eta$$

and thus

$$\| \, |T|^{it}\xi * \eta\|_2 \leq \| \, |T|^{it}\| \, \|\lambda(\xi)\| \, \|\Delta^{i\alpha t}\eta\|_2 \leq \|\lambda(\xi)\| \, \|\eta\|_2.$$

We conclude that $|T|^{it}\xi$ is left bounded and that

$$\|\lambda(|T|^{it}\xi)\| \leq \|\lambda(\xi)\|.$$

□

Remark 2.6. In particular, $\Delta^{it}\xi \in \mathfrak{A}_l$ with $\|\lambda(\Delta^{it}\xi)\| \leq \|\lambda(\xi)\|$ for all $\xi \in \mathfrak{A}_l$ and $t \in \mathbb{R}$

Our next lemma shows that α-homogeneity as defined here is equivalent to homogeneity of degree α with respect to ψ_0 as defined in [2, Definition 17].

Lemma 2.7. *Let $\alpha \in \mathbb{R}$, and let T be a closed densely defined operator on $L^2(G)$ with polar decomposition $T = U|T|$. Then the following conditions are equivalent:*

(i) *T is α-homogeneous,*
(ii) *$U \in M$ and $\forall y \in M' \quad \forall t \in \mathbb{R} : \sigma^{\psi_0}_{\alpha t}(y)|T|^{it} = |T|^{it}y$.*

Proof. By Lemma 2.3, we may assume that T is positive self-adjoint.

Denote by P the projection onto $N(T)^{\perp}$. If either (i) or (ii) holds, then P is in M, and thus the subspace $PL^2(G)$ is invariant under all operators considered. Therefore, we may suppose that $P \in M$, and the lemma is proved when we have shown the equivalence of

$$\forall x \in G : \rho(x)T\rho(x^{-1})P = \Delta^{-\alpha}(x)TP \tag{2.1}$$

and

$$\forall t \in \mathbb{R} \quad \forall y \in M' : \sigma^{\psi_0}_{\alpha t}(y)P = T^{it}yT^{-it}P. \tag{2.2}$$

Now for all $x \in G$ we have

$$\sigma^{\psi_0}_{\alpha t}(\rho(x)) = \Delta^{-i\alpha t}\rho(x)\Delta^{i\alpha t} = \Delta^{i\alpha t}(x)\rho(x)$$

since

$$(\Delta^{-i\alpha t}\rho(x)\Delta^{i\alpha t}f)(z) = \Delta^{-it}(z)\Delta^{\frac{1}{2}}(x)\Delta^{it}(zx)f(zx) = \Delta^{-it}(x)(\rho(x)f)(z)$$

for all $f \in L^2(G)$ and all $x, z \in G$. Then, since M' is generated by the $\rho(x)$, the condition (2.2) is equivalent to

$$\forall x \in G \quad \forall t \in \mathbb{R} : \Delta^{i\alpha t}(x)\rho(x)P = T^{it}\rho(x)T^{-it}P$$

or (changing t into $-t$)

$$\forall x \in G \quad \forall t \in \mathbb{R} : \rho(x)T^{it}\rho(x)P = \Delta^{-i\alpha t}(x)T^{it}P,$$

which in turn is equivalent to (2.1). □

Now, by [2, Theorem 13] a positive self-adjoint operator on $L^2(G)$ is (-1)-homogeneous if and only if it has the form $\frac{d\varphi}{d\psi_0}$ for a (necessarily unique) normal semi-finite weight φ on M.

We define the "integral with respect to ψ_0" of a positive self-adjoint (-1)-homogeneous operator T as

$$\int T d\psi_0 = \varphi(1) \in [0, \infty],$$

where $T = \frac{d\varphi}{d\psi_0}$. If $\int T d\psi_0 < \infty$, i.e. if φ is a functional, we shall say that T is integrable. (These definitions agree with those given in [2, remarks following Corollary 18].)

For each $p \in [1, \infty)$, we denote by $L^p(\psi_0)$ the set of closed densely defined $(-\frac{1}{p})$-homogeneous operators T on $L^2(G)$ satisfying

$$\int |T|^p d\psi_0 < \infty.$$

(Note that $|T|^p$ is (-1)-homogeneous, so that $\int |T|^p d\psi_0$ is defined.) We put $L^\infty(\psi_0) = M$.

The spaces $L^p(\psi_0)$ introduced here are special cases of the spatial L^p-spaces of M. Hilsum [12]. We recall their main properties (note, however, that our notation differs from that of [12] in that we maintain throughout the distinction between operators and their closures):

If $T, S \in L^p(\psi_0)$, then $T + S$ is densely defined and preclosed, and the closure $[T + S]$ belongs to $L^p(\psi_0)$. With the obvious scalar multiplication and the sum $(T, S) \mapsto [T+S]$, $L^p(\psi_0)$ is a linear space, and even a Banach space with the norm $\|.\|_p$ defined by $\|T\|_p = (\int |T|^p d\psi_0)^{1/p}$ if $p \in [1, \infty)$ and $\|T\|_p = \|T\|$ (operator norm) if $p = \infty$. The operation $T \mapsto T^*$ is an isometry of $L^p(\psi_0)$ onto $L^p(\psi_0)$. We denote $L^p(\psi_0)_+$ the set of positive self-adjoint operators belonging to $L^p(\psi_0)$

By linearity, $T \mapsto \int T d\psi_0$ defined on $L^1(\psi_0)_+$ extends to a linear form on the whole of $L^1(\psi_0)$ satisfying $\int T^* d\psi_0 = \overline{\int T d\psi_0}$ and $|\int T d\psi_0| \leq \|T\|_1$ for all $T \in L^1(\psi_0)$.

Let $p_1, p_2, p \in [1, \infty]$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$. If $T \in L^{p_1}(\psi_0)$ and $S \in L^{p_2}(\psi_0)$, then the operator TS is densely defined and preclosed, its closure $[TS]$ belongs to $L^p(\psi_0)$, and

$$\|[TS]\|_p \leq \|T\|_{p_1} \|S\|_{p_2}.$$

In particular, if $T \in L^p(\psi_0)$ and $S \in L^q(\psi_0)$, where $\frac{1}{p} + \frac{1}{q} = 1$, then $[TS] \in L^1(\psi_0)$ and $\|[TS]\|_1 \leq \|T\|_p \|S\|_q$ (Hölder's inequality); furthermore, $\int [TS] d\psi_0 = \int [ST] d\psi_0$.

If $p \in [1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$, then we identify $L^q(\psi_0)$ with the dual space of $L^p(\psi_0)$ by means of the form $(T, S) \mapsto \int [TS] d\psi_0$, $T \in L^p(\psi_0)$. $S \in L^q(\psi_0)$. In particular, $L^1(\psi_0)$ is the predual of $M = L^\infty(\psi_0)$. The space $L^2(\psi_0)$ is a Hilbert space with the inner product $(T|S)_{L^2(\psi_0)} = \int [S * T] d\psi_0$.

Remark 2.8. Suppose that G is unimodular. Then the $\alpha-$*homogeneous* operators for any α are simply the operators affiliated with M and the canonical weight

φ_0 on M is a trace. We claim that $\int T d\psi_0 = \varphi_0(T)$ for all positive self-adjoint operators T affiliated with M, where we have written $\varphi_0(T)$ for the value of $\varphi = \varphi_0(T.)$ at 1 (with $\varphi_0(T.)$ defined as in [17, §4]). To see this, recall that $\frac{d\varphi_0}{d\psi_0} = \Delta = 1$, so that using [2, Theorem 9, (2)], we have

$$T^{it} = (D\varphi : D\varphi_0)_t = \left(\frac{d\varphi}{d\psi_0}\right)^{it}\left(\frac{d\varphi_0}{d\psi_0}\right)^{-it} = \left(\frac{d\varphi}{d\psi_0}\right)^{it}$$

for all $t \in \mathbb{R}$. Thus $T = \frac{d\varphi}{d\psi_0}$, and $\int T d\psi_0 = \varphi(1) = \varphi_0(T)$. (When proving $T = \frac{d\varphi}{d\psi_0}$, we implicitly assumed that T is injective so that $\varphi = \varphi_0(T.)$ is faithful. In the general case, denote by $Q \in M$ the projection onto $N(T)$, note that $T+Q$ is positive self-adjoint, affiliated with M, and injective, and verify that

$$T + Q = \frac{d\varphi_0((T+Q).)}{d\psi_0} = \frac{d\varphi_0(T.)}{d\psi_0} + \frac{d\varphi_0(Q.)}{d\psi_0}.$$

Since the supports of $\frac{d\varphi_0(T.)}{d\psi_0}$ and $\frac{d\varphi_0(Q.)}{d\psi_0}$ are $1-Q$ and Q, respectively, we conclude that $T = \frac{d\varphi_0(T.)}{d\psi_0}$ as desired.) It follows that in this case the spaces $L^p(\psi_0)$ reduce the ordinary $L^p(M, \varphi_0)$ (discussed in the introduction).

Returning to the general case, we now proceed to a more detailed study of the spaces $L^p(\psi_0)$. For this, we shall need the following slightly generalized version of [12, II, Proposition 2].

Lemma 2.9. *Suppose that T is a positive self-adjoint operator on $L^2(G)$ and $\alpha-$homogeneous for some $\alpha \in \mathbb{R}$. Let $\xi \in \mathfrak{A}_l$. Then for each $n \in \mathbb{N}$ there exists $\xi_n \in \mathfrak{A}_l \cap \left(\cap_{\beta\in\mathbb{R}_+} D(T^\beta)\right)$ such that*
(i) $\forall n \in \mathbb{N} : \|\lambda(\xi_n)\| \le \|\lambda(\xi)\|$,
(ii) $\xi_n \to \xi$ as $n \to \infty$,
(iii) $T^\beta \xi_n \to T^\beta \xi$ as $n \to \infty$ whenever ξ and $\beta \in \mathbb{R}_+$ satisfy $\xi \in D(T^\beta)$.

Proof. For each $n \in N$, define $f_n : [0,\infty) \to \mathbb{C}$ by

$$f_n(x) = \begin{cases} \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2} x^{\frac{it}{\sqrt{n}}} dt & \text{if } x > 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Since for all $x \in [0,\infty)$ we have $|f_n(x)| \le \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2} dt = 1$, the operators $f_n(T)$ are bounded. For each $\eta \in \mathbb{N}$, put $\xi_n = f_n(T)\xi$.

To prove that the ξ_n belong to $\mathfrak{A}_l$ and satisfy (i), denote by P the projection onto $N(T)^\perp$ and observe that for all $\eta \in \mathcal{K}(G)$ we have

$$\begin{aligned} f_n(T)P\xi * \eta &= \lambda'(\eta) f_n(T) P\xi \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2}\lambda'(\eta) T^{\frac{it}{\sqrt{n}}}\xi dt \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2} T^{\frac{it}{\sqrt{n}}}\lambda'(\Delta^{\frac{i\alpha t}{\sqrt{n}}}\eta)\xi dt \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2} T^{\frac{it}{\sqrt{n}}}(\xi * \Delta^{\frac{i\alpha t}{\sqrt{n}}}\eta) dt, \end{aligned}$$

where we have used Lemma 2.4. It follows that

$$\|f_n(T)P\xi * \eta\|_2 \leq \frac{1}{\sqrt{\pi}} \int e^{-t^2} \|\lambda(\xi)\| \, \|\Delta^{\frac{i\alpha t}{\sqrt{n}}}\eta\|_2 dt \leq \|\lambda(\xi)\| \, \|\eta\|_2.$$

On the other hand,

$$\|(1-P)\xi * \eta\|_2 \leq \|\lambda((1-P)\xi)\| \, \|\eta\|_2 \leq \|\lambda(\xi)\| \, \|\eta\|_2,$$

since $P \in M$.

In all, $f_n(T)\xi = f_n(T)P\xi + (1-P)\xi$ belongs to $\mathfrak{A}_l$ and $\|\lambda(f_n(T)\xi)\| \leq \|\lambda(\xi)\|$.

Now, to see that $\xi_n \in D(T^\beta)$ for all $\beta \in \mathbb{R}_+$, note that

$$\begin{aligned} f_n(x) &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} e^{\frac{it}{\sqrt{n}} \log x} \, dt \\ &= e^{-\frac{1}{4n}(\log x)^2} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(t - \frac{i}{2\sqrt{n}} \log x)^2} dt \\ &= e^{-\frac{1}{4n}(\log x)^2} \end{aligned}$$

for all $x > 0$. Then $x \mapsto x^\beta f_n(x) = e^{(\beta \log x - \frac{1}{4n}(\log x)^2)}$ is bounded, so that $T^\beta f_n(T)$ is a bounded operator, and thus $f_n(T)\xi \in D(T^\beta)$.

Since f_n is bounded and $f_n(x) \to 1$ as $n \to \infty$ for all $x \in [0, \infty)$, we have

$$f_n(T)\zeta \to \zeta \quad \text{as} \quad n \to \infty$$

for all ζ. From this, we immediately get (ii) and (iii). Indeed, $\xi_n = f_n(T)\xi \to \xi$, and if $\xi \in D(T^\beta)$, then

$$T^\beta \xi_n = T^\beta f_n(T)\xi = f_n(T)T^\beta \xi \to T^\beta \xi.$$

□

Proposition 2.10. *Let T be a closed densely defined $(-1)-$homogeneous operator on $L^2(G)$. Then the following conditions are equivalent:*
(i) $T \in L^1(\psi_0)$,
(ii) there exists a constant $C \geq 0$ such that

$$\forall \xi \in \mathfrak{A}_l \cap D(T) \quad \forall \eta \in \mathfrak{A}_l : |(T\xi|\eta)| \leq C\|\lambda(\xi)\| \, \|\lambda(\eta)\|,$$

(iii) there exists a constant $C \geq 0$ such that

$$\forall \xi \in \mathfrak{A}_l \cap D(|T|^{\frac{1}{2}}) : \| \, |T|^{\frac{1}{2}}\xi\|^2 \leq C\|\lambda(\xi)\|^2,$$

(iv) there exists an approximate identity $(\xi_i)_{i \in I}$ in $K(G)_+$ such that all $\xi_i \in D(|T|^{\frac{1}{2}})$ and

$$\liminf_{i \in I} \| \, |T|^{\frac{1}{2}}\xi_i\| < \infty.$$

If $T \in L^1(\psi_0)$, then $\mathfrak{A}_l \subseteq D(|T|^{\frac{1}{2}})$, and for any approximate identity $(\xi_i)_{i \in I}$ in $\mathcal{K}(G)_+$ we have

$$\|T\|_1 = \lim_{i \in I} \||T|^{\frac{1}{2}}\xi_i\|^2.$$

Furthermore, $\|T\|_1$ is the smallest C satisfying (ii) and the smallest C satisfying (iii).

Proof. Let $T = U|T|$ be the polar decomposition of T.

First, suppose that $T \in L^1(\psi_0)$. Then $|T| \in L^1(\psi_0)_+$, and therefore $|T| = \frac{d\varphi}{d\psi_0}$ for some positive functional φ on M. Recall that $\mathfrak{A}_l \subseteq D(|T|^{\frac{1}{2}})$. Thus for all $\xi \in \mathfrak{A}_l \cap D(T)$ and $\eta \in \mathfrak{A}_l$ we have

$$\begin{aligned} |(T\xi|\eta)| &= \left|(|T|^{\frac{1}{2}}\xi \mid |T|^{\frac{1}{2}}U^*\eta)\right| \\ &= |\varphi(\lambda(\xi)\lambda(U^*\eta))| \\ &\leq \|\varphi\| \, \|\lambda(\xi)\| \, \|\lambda(U^*\eta)\| \\ &\leq \|T\|_1 \|\lambda(\xi)\| \, \|\lambda(\eta)\|, \end{aligned}$$

i.e. (ii) holds.

Next, suppose that T satisfies (ii). Then for all $\xi \in \mathfrak{A}_l \cap D(|T|)$ we have

$$\begin{aligned} \| \, |T|^{\frac{1}{2}}\xi\|^2 &= |(T\xi|U\xi)| \\ &\leq C\|\lambda(\xi)\| \, \|\lambda(U\xi)\| \\ &\leq C\|\lambda(\xi)\|^2. \end{aligned}$$

Now if $\xi \in \mathfrak{A}_l \cap D(|T|^{\frac{1}{2}})$, there exist (by Lemma 2.9) $\xi_n \in \mathfrak{A}_l \cap D(|T|)$ such that $|T|^{\frac{1}{2}}\xi_n \to |T|^{\frac{1}{2}}\xi$ and $\|\lambda(\xi_n)\| \leq \|\lambda(\xi)\|$. Since

$$\| \, |T|^{\frac{1}{2}}\xi_n\|^2 \leq C\|\lambda(\xi_n)\|^2 \leq C\|\lambda(\xi)\|^2,$$

we conclude that $\| \, |T|^{\frac{1}{2}}\xi\|^2 \leq C\|\lambda(\xi)\|^2$. Thus (iii) is proved.

Now suppose that T satisfies (iii). First we show that this implies $\mathfrak{A}_l \subseteq D(|T|^{\frac{1}{2}})$. Let $\xi \in \mathfrak{A}_l$. Then by Lemma 2.9 there exist $\xi_n \in \mathfrak{A}_l \cap D(|T|^{\frac{1}{2}})$ such that $\xi_n \to \xi$ and $\|\lambda(\xi_n)\| \leq \|\lambda(\xi)\|$. Then for all $\eta \in D(|T|^{\frac{1}{2}})$ we have

$$\begin{aligned} \left|(|T|^{\frac{1}{2}}\xi_n|\eta)\right| &\leq \| \, |T|^{\frac{1}{2}}\xi_n\| \, \|\eta\| \\ &\leq C^{1/2}\|\lambda(\xi_n)\| \, \|\eta\| \\ &\leq C^{1/2}\|\lambda(\xi)\| \, \|\eta\| \end{aligned}$$

and

$$(|T|^{\frac{1}{2}}\xi_n|\eta) = (\xi_n| \, |T|^{\frac{1}{2}}\eta) \to (\xi \mid |T|^{\frac{1}{2}}\eta).$$

We conclude that

$$\forall \eta \in D(|T|^{\frac{1}{2}}) : \left|(\xi \mid |T|^{\frac{1}{2}}\eta)\right| \leq C^{1/2}\|\lambda(\xi)\| \, \|\eta\|.$$

Thus $\xi \in D(|T|^{\frac{1}{2}})$ as wanted.

Now, still assuming (iii), let us prove (iv). Let $(\xi_i)_{i\in I}$ be any approximate identity in $\mathcal{K}(G)_+$. Then automatically all $\xi_i \in \mathcal{K}(G) \subseteq \mathfrak{A}_l \subseteq D(|T|^{\frac{1}{2}})$, and $\|\lambda(\xi_i)\| \leq \|\xi_i\|_1 = 1$ so that

$$\| \, |T|^{\frac{1}{2}}\xi_i\|^2 \leq C\|\lambda(\xi_i)\|^2 \leq C,$$

whence $\liminf_{i\in I} \| \, |T|^{\frac{1}{2}}\xi_i\| \leq C^{\frac{1}{2}} < \infty$.

Finally, suppose that T satisfies (iv) for some $(\xi_i)_{i\in I}$. Note that since $\int(\xi_i * \xi_i^*)(x)dx = 1, (\xi_i * \xi_i^*)_{i\in I}$ is again an approximate identity in $\mathcal{K}(G)_+$. Therefore,

$\lambda(\xi_i)\lambda(\xi_i)^* = \lambda(\xi_i * \xi_i^*)$ convergence strongly, and hence weakly, to 1 in M. Since all $\|\lambda(\xi_i)\lambda(\xi_i)^*\| \leq 1$, this convergence is also σ-weak, and by the σ-weak lower semicontinuity of φ, this implies

$$\begin{aligned}\varphi(1) &\leq \liminf_{i\in I} \varphi(\lambda(\xi_i)\lambda(\xi_i)^*) \\ &= \liminf_{i\in I} \|\,|T|^{\frac{1}{2}}\xi_i\|^2 \\ &\leq C \liminf_{i\in I} \|\lambda(\xi_i)\|^2 \\ &\leq C < \infty\,.\end{aligned}$$

Since $\varphi(1) = \int |T| d\psi_0 < \infty$, we have $T \in L^1(\psi_0)$, i.e. (i) holds.

Note that once $\varphi(1) < \infty$ is established, φ is known to be σ-weakly lower continuous and thus

$$\varphi(1) = \lim_{i\in I} \varphi(\lambda(\xi_i)\lambda(\xi_i)^*) = \lim_{i\in I} \|\,|T|^{\frac{1}{2}}\xi_i\|^2$$

for any approximate identity $(\xi_i)_{i\in I}$, i.e.

$$\|T\|_1 = \lim_{i\in I} \|\,|T|^{\frac{1}{2}}\xi_i\|^2.$$

In the course of the proof we observed that $\|T\|_1$ may be used as the constant C in (ii), that every constant C satisfying (ii) also satisfies (iii), and that any C satisfying (iii) is bigger than $\lim_{i\in I} \|\,|T|^{\frac{1}{2}}\xi_i\|^2$, i.e. bigger than $\|T\|_1$. This proves the remarks that end Proposition 2.10 □

As an immediate corollary, we have:

Proposition 2.11. *Let T be a closed densely defined $(-\frac{1}{2})$-homogeneous operator on $L^2(G)$. Then the following conditions are equivalent:*

(i) $T \in L^2(\psi_0)$,

(ii) *there exists a constant $C \geq 0$ such that*

$$\forall \xi \in \mathfrak{A}_l \cap D(T) : \|T\xi\| \leq C\|\lambda(\xi)\|,$$

(iii) *there exists an approximate identity $(\xi_i)_{i\in I}$ in $\mathcal{K}(G)_+$ such that all $\xi_i \in D(T)$ and*

$$\liminf_{i\in I} \|T\xi_i\| < \infty.$$

If $T \in L^2(\psi_0)$, then $\mathfrak{A}_l \subseteq D(T)$, and for any approximate identity $(\xi_i)_{i\in I}$ in $\mathcal{K}(G)_+$ we have

$$\|T\|_2 = \lim_{i\in I} \|T\xi_i\|;$$

furthermore, $\|T\|_2$ is the smallest constant C satisfying (ii).

We now come to the case of a general $p \in [1, \infty)$. Suppose that $T \in L^P(\psi_0)$ and $S \in L^q(\psi_0)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then by [12, II, Proposition 5, 1], we have

$$(T\xi|S\eta) = \langle [S * T], \lambda(\xi)\lambda(\eta)^* \rangle$$

for all $\xi \in \mathfrak{A}_l \cap D(T)$ and $\eta \in \mathfrak{A}_l \cap D(S)$. (Here, $\langle .,. \rangle$ denotes the form giving the duality of $L^1(\psi_0)$ and M.) Using Hölder's inequality, we get

$$|(T\xi|S\eta)| \leq \|[S^*T]\|_1 \|\lambda(\xi)\lambda(\eta)^*\| \leq \|T\|_p \|S\|_q \|\lambda(\xi)\| \, \|\lambda(\eta)\|$$

for all such ξ and η. This kind of inequality in fact characterizes $L^p(\psi_0)$-operators among all $(-\frac{1}{p})-$homogeneous operators:

Proposition 2.12. *Let $p \in [1, \infty]$ and define q by $\frac{1}{p} + \frac{1}{q} = 1$. Let T be a closed densely defined $(-\frac{1}{p})-$homogeneous operator on $L^2(G)$. Then the following conditions are equivalent:*

(i) $T \in L^p(\psi_0)$,
(ii) *there exists a constant $C \geq 0$ such that*

$$\forall S \in L^q(\psi_0)\ \forall \xi \in \mathfrak{A}_l \cap D(T)\ \forall \eta \in \mathfrak{A}_l \cap D(S) : |(T\xi|S\eta)| \leq C\|S\|_q \|\lambda(\xi)\|\,\|\lambda(\eta)\|.$$

If $T \in L^p(\psi_0)$, then $\|T\|_p$ is the smallest C satisfying (ii).

Proof. In view of the remarks preceding this proposition, we just have to show that if T satisfies (ii) for some constant C, then $T \in L^p(\psi_0)$, and $\|T\|_p \leq C$.

Therefore suppose that T with polar decomposition $T = U|T|$ satisfies (ii). Then also

$$|(|T|\xi|S\eta)| = |(T\xi|U^*S\eta)| \leq C\|[U^*S]\|_q \|\lambda(\xi)\|\,\|\lambda(\eta)\| \leq C\|S\|_q \|\lambda(\xi)\|\,\|\lambda(\eta)\|$$

for all S, ξ, and η chosen as in (ii). Then we may assume that T is positive self-adjoint.

Let $S \in L^q(\psi_0)$ and $\eta \in \mathfrak{A}_l \cap D(T^{\frac{1}{2}}S)$. We claim that for all $\xi \in \mathfrak{A}_l \cap D(T^{\frac{1}{2}})$ we have

$$|(T^{\frac{1}{2}}\xi|(T^{\frac{1}{2}}S\eta)| \leq C\|S\|_q \|\lambda(\xi)\|\,\|\lambda(\eta)\|. \tag{2.3}$$

If $\xi \in \mathfrak{A}_l \cap D(T)$, this follows directly from the hypothesis. In case of a general $\xi \in \mathfrak{A}_l \cap D(T^{\frac{1}{2}})$, choose (by Lemma 2.9) $\xi_n \in \mathfrak{A}_l \cap D(T)$ such that $T^{\frac{1}{2}}\xi_n \to T^{\frac{1}{2}}\xi$ and $\|\lambda(\xi_n)\| \leq \|\lambda(\xi)\|$. Then (2.3) follows by passing to the limit.

Now since T is $-\frac{1}{p}$-homogeneous, there exist $T_i \in L^p(\psi_0)_+$ satisfying $T_i^p \leq T^p$ and $\int T^p d\psi_0 = \sup T_i^P d\psi_0$. (To see this, recall that $T^p = \frac{d\varphi}{d\psi_0}$ for some normal semi-finite weight φ on M; put $T_i = (\frac{d\varphi_i}{d\psi_0})^{1/p}$ where φ_i are positive normal functionals such that $\varphi_i \nearrow \varphi$; then $\frac{d\varphi_i}{d\psi_0} \leq \frac{d\varphi}{d\psi_0}$ by [2, Proposition 8], and $\int T^p d\psi_0 = \varphi(1) = \sup \varphi_i(1) = \sup \int T_i^p d\psi_0$.)

Since the function $t \to t^{1/p}$ is operator monotone on $[0, \infty)$ (by [16, Proposition 1.3.8]), we have $T_i \leq T$, i.e. $D(T_i^{\frac{1}{2}}) \supseteq D(T^{\frac{1}{2}})$ and

$$\forall \xi \in D(T^{\frac{1}{2}}) : \|T_i^{\frac{1}{2}}\xi\| \leq \|T^{\frac{1}{2}}\xi\|,$$

for each $i \in I$ (cf. also the remark following this proof).

For each i, let B_i be the bounded operator characterized by $B_i T^{\frac{1}{2}}\xi = T_i^{\frac{1}{2}}\xi$ for all $\xi \in D(T^{\frac{1}{2}})$ and $B_i\xi = 0$ for all $\xi \in R(T^{\frac{1}{2}})^{\perp}$. Then $\|B_i\| \leq 1$. Since $B_i T^{\frac{1}{2}} \subseteq T_i^{\frac{1}{2}}$, and since $T^{\frac{1}{2}}$ and $T_i^{\frac{1}{2}}$ are $(-\frac{1}{p})$-homogeneous, B_i is 0-homogeneous, i.e. $B_i \in M$. Put $A_i = B_i^*$. Then $A_i \in M$, $\|A_i\| \leq 1$, and

$$T_i^{\frac{1}{2}} \subseteq T^{\frac{1}{2}} A_i.$$

Using this, the fact that

$$T_i^{p-1} = T_i^{\frac{p}{q}} \in L^q(\psi_0) \quad \text{with} \quad \|T_i^{p-1}\|_q = \|T_i\|_p^{p-1},$$

and (2.3), we find that for all $\xi \in \mathfrak{A}_l \cap \left(\cap_{\beta\in\mathbb{R}_+} D(T_i^\beta)\right)$, we have

$$\begin{aligned}\|T_i^{\frac{p}{2}}\xi\|^2 &= (T_i^{\frac{1}{2}}\xi|T_i^{\frac{1}{2}}T_i^{p-1}\xi)\\ &= (T^{\frac{1}{2}}A_i\xi|T^{\frac{1}{2}}A_iT_i^{p-1}\xi)\\ &\le C\|[A_iT_i^{p-1}]\|_q\|\lambda(A_i\xi)\|\,\|\lambda(\xi)\|\\ &\le C\|A_i\|\,\|T_i^{p-1}\|_q\|A_i\|\,\|\lambda(\xi)\|^2\\ &= C\|T_i\|_p^{p-1}\|\lambda(\xi)\|^2.\end{aligned}$$

By means of Lemma 2.9, we conclude that the estimate

$$\|T_i^{\frac{p}{2}}\|^2 \le C\|T_i\|_p^{p-1}\|\lambda(\xi)\|^2$$

holds for all $\xi \in \mathfrak{A}_l \cap D(T_i^{p/2})$. Thus by Proposition 2.10,

$$\|T_i\|_p^p = \|T_i^p\|_1 \le C\|T_i\|_p^{p-1},$$

i.e.

$$\|T_i\|_p \le C.$$

Since this holds for all i, we have

$$\int T^p d\psi_0 = \sup \int T_i^p d\psi_0 \le C^p < \infty;$$

thus $T \in L^p(\psi_0)$ and $\|T_i\|_p \le C$. □

Remark 2.13. we have used the fact that if a continuous function f on $[0,\infty)$ is operator monotone in the sense that $R \le S$ implies $f(R) \le f(S)$ for all positive bounded operators R and S, then the same is true for all - possibly unbounded - positive self-adjoint R and S. To see this, suppose that $R \le S$. Then for all $\varepsilon \in \mathbb{R}_+$, we have $R(1+\varepsilon R)^{-1} \le S(1+\varepsilon S)^{-1}$ by [17, §4], and hence $f(R(1+\varepsilon R)^{-1}) \le f(S(1+\varepsilon S)^{-1})$. Now if $\xi \in D(f(S)^{\frac{1}{2}})$, we have by the spectral theory

$$\begin{aligned}(f(R(1+\varepsilon R)^{-1})\xi|\xi) &\le (f(S+(1+\varepsilon S)^{-1})\xi|\xi)\\ &\to \|f(S)^{\frac{1}{2}}\xi\|^2 \quad \text{as} \quad \varepsilon \to 0.\end{aligned}$$

Again by the spectral theory, we conclude that $\xi \in D(f(R)^{\frac{1}{2}})$ and that

$$\|f(R)^{\frac{1}{2}}\xi\|^2 = \lim_{\varepsilon\to 0}(f(R(1+\varepsilon R)^{-1})\xi|\xi) \le \|f(S)^{\frac{1}{2}}\xi\|^2.$$

In all, we have proved that $f(R) \le f(S)$.

Recall from [12, §1, Théorème 4, 1)], that if T_1 and T_2 belong to some $L^p(\psi_0)$, $1 \le p < \infty$, and if $T_2 \subseteq T_1$, then $T_1 = T_2$. Actually, a stronger result holds:

Lemma 2.14. *Let $p \in [1,\infty]$. Let $T_1 \in L^p(\psi_0)$ and let T_2 be a closed densely defined $(-\frac{1}{p})$-homogeneous operator on $L^2(G)$. If $T_2 \subseteq T_1$ or $T_1 \subseteq T_2$, then $T_1 = T_2$.*

Proof. First suppose that $T_2 \subseteq T_1$. If $p = \infty$, the result is well-known (a closed densely defined operator having a bounded and everywhere defined extension is equal to that extension). If $p \in [1,\infty)$, we conclude by Proposition 2.12 that also $T_2 \in L^p(\psi_0)$, and thus by [12, §1, Théorème 4, 1)], $T_1 = T_2$. (Alternatively, this can be proved directly, i.e. without using Proposition 2.12, by the methods of the proof of [12, §1, Théorème 4, 1)].)

If $T_1 \subseteq T_2$, apply the first part of the proof to $T_2^* \subseteq T_1^*$. □

A specific form of this lemma will be crucial to much of the following:

Proposition 2.15. *Let $p \in [1,\infty]$.*

1) *Let T and S be closed densely defined $(-\frac{1}{p})$-homogeneous operators on $L^2(G)$ with $\mathcal{K}(G) \subseteq D(T)$ and $\mathcal{K}(G) \subseteq D(S)$. Suppose that $T\xi = S\xi$ for all $\xi \in \mathcal{K}(G)$. If one of the operators, say T, belongs to $L^p(\psi_0)$, we may conclude that $T = S$.*
2) *If $T \in L^p(\psi_0)$ and $\mathcal{K}(G) \subseteq D(T)$, then $T = [T|_{\mathcal{K}(G)}]$.*

Proof. (of both parts). Suppose that $T \in L^p(\psi_0)$. Then $T|_{\mathcal{K}(G)}$, being a restriction of a $(-\frac{1}{p})$-homogeneous operator to a right invariant subspace, is itself $(-\frac{1}{p})$-homogeneous. Therefore also $[T|_{\mathcal{K}(G)}]$ is $(-\frac{1}{p})$-homogeneous. Since $[T|_{\mathcal{K}(G)}] \subseteq T$, we conclude by the above lemma that $T = [T|_{\mathcal{K}(G)}]$. This proves 2). As for 1), note that $S \supseteq S|_{\mathcal{K}(G)} = T|_{\mathcal{K}(G)}$, and thus $S \supseteq [T|_{\mathcal{K}(G)}] = T$. Again we conclude $S = T$. □

Finally, for later reference, we summarize in a lemma some remarks of Hilsum [12]:

Lemma 2.16. *Let $q \in [2,\infty)$. Let $T \in L^q(\psi_0)$. Then $\mathfrak{A}_l \subseteq D(T)$, and for all $\xi \in \mathfrak{A}_l$ we have*

$$\|T\xi\| \leq \|T\|_q \|\lambda(\xi)\|^{2/q} \|\xi\|^{1-2/q}.$$

Proof. Since $|T|^{\frac{q}{2}} \in L^2(\psi_0)$, we have $\mathfrak{A}_l \subseteq D(|T|^{\frac{q}{2}})$. Now let $\xi \in \mathfrak{A}_l$. Then by the spectral theory $\xi \in D(|T|)$ and

$$\begin{aligned} \|\,|T|\xi\|^2 &\leq (\|\,|T|^{\frac{q}{2}}\xi\|^2)^{2/q}.(\|\xi\|^2)^{1-2/q} \\ &\leq (\|\,|T|^q\|_1 \|\lambda(\xi)\|^2)^{2/q}.\|\xi\|^{2(1-2/q)} \\ &= (\|T\|_q \|\lambda(\xi)\|^{2/q} \|\xi\|^{1-2/q})^2. \end{aligned}$$

□

3. The Plancherel transformation

Given any functions $f \in L^2(G)$ and $\xi \in L^2(G)$, the convolution product $f * \Delta^{\frac{1}{2}}\xi$ exists and belongs to $L^\infty(G)$. Thus the following definition makes sense:

Definition 3.1. Let $f \in L^2(G)$. The Plancherel transform $\mathcal{P}(f)$ of f is the operator on $L^2(G)$ given by

$$\mathcal{P}(f)\xi = f * \Delta^{\frac{1}{2}}\xi, \quad \xi \in D(\mathcal{P}(f)),$$

where

$$D(\mathcal{P}(f)) = \{\xi \in L^2(G) | f * \Delta^{\frac{1}{2}}\xi \in L^2(G)\}.$$

Theorem 3.2. *(Plancherel).*

(1) *Let $f \in L^2(G)$. Then $\mathcal{P}(f)$ belongs to $L^2(\psi_0)$, and*

$$\|\mathcal{P}(f)\|_2 = \|f\|_2.$$

(2) *The Plancherel transformation $\mathcal{P} : L^2(G) \to L^2(\psi_0)$ is a unitary transformation of $L^2(G)$ onto $L^2(\psi_0)$.*

Proof. (1) First note that $\mathcal{P}(f)$ is $(-\frac{1}{2})$-homogeneous: for all $x, y \in G$ and $\xi \in D(\mathcal{P}(f))$, we have

$$\begin{aligned}\rho(x)(\mathcal{P}(f)\xi)(y) &= \Delta^{\frac{1}{2}}(x)(f * \Delta^{\frac{1}{2}}\xi)(yx) \\ &= \Delta^{\frac{1}{2}}(x) \int f(z)\Delta^{\frac{1}{2}}(z^{-1}yx)\xi(z^{-1}yx)dz \\ &= \Delta^{\frac{1}{2}}(x) \int f(z)\Delta^{\frac{1}{2}}(z^{-1}y)(\rho(x)\xi)(z^{-1}y)dz \\ &= \Delta^{\frac{1}{2}}(x)(f * \Delta^{\frac{1}{2}}\rho(x)\xi)(y),\end{aligned}$$

i.e. $\rho(x)\mathcal{P}(f) \subseteq \Delta^{\frac{1}{2}}\mathcal{P}(f)\rho(x)$.

We next show that $\mathcal{P}(f)$ is closed. Suppose that $\xi_n \to \xi$ in $L^2(G)$ and $\mathcal{P}(f)\xi_n \to \eta$ in $L^2(G)$, where all the $\xi_n \in D(\mathcal{P}(f))$. Then $f * \Delta^{\frac{1}{2}}\xi_n \to f * \Delta^{\frac{1}{2}}\xi$ uniformly (by a simple case of Lemma 1.1). Since $f * \Delta^{\frac{1}{2}}\xi_n \to \eta$ in $L^2(G)$, we conclude that $\eta = f * \Delta^{\frac{1}{2}}\xi$. Thus $\xi \in D(\mathcal{P}(f))$ and $\mathcal{P}(f)\xi = \eta$, so that $\mathcal{P}(f)$ is closed. Obviously, $\mathcal{K}(G) \subseteq D(\mathcal{P}(f))$. In all, we have shown that $\mathcal{P}(f)$ is closed, densely defined, and $(-\frac{1}{2})$-homogeneous, so that we are now in a position to apply Proposition 2.11.

Let $(\xi_i)_{i \in I}$ be an approximate identity in $\mathcal{K}(G)_+$. Then

$$\mathcal{P}(f)\xi_i = f * \Delta^{\frac{1}{2}}\xi_i \to f \quad \text{in} \quad L^2(G).$$

Thus $\|\mathcal{P}(f)\xi_i\| \to \|f\|_2$. By Proposition 2.11 we conclude that $\mathcal{P}(f) \in L^2(\psi_0)$ and that

$$\|\mathcal{P}(f)\|_2 = \|f\|_2.$$

(2) the map $\mathcal{P}$ is linear: if $f_1, f_2 \in L^2(G)$, then $[\mathcal{P}(f_1) + \mathcal{P}(f_2)]$ and $\mathcal{P}(f_1 + f_2)$ obviously agree on $\mathcal{K}(G)$ and therefore by Proposition 2.15, we have

$$\mathcal{P}(f_1 + f_2) = [\mathcal{P}(f_1) + \mathcal{P}(f_2)].$$

Now, to prove that $\mathcal{P}$ is surjective, let $T \in L^2(\psi_0)$. We shall show that there exists a function $f \in L^2(G)$ such that $T = \mathcal{P}(f)$. Let $(\varepsilon_i)_{i \in T}$ be an approximation identity in $\mathcal{K}(G)_+$. Then for all $\eta, \zeta \in \mathcal{K}(G)$ we have

$$\begin{aligned}(\eta * \Delta^{-\frac{1}{2}}\tilde{\zeta}|T\xi_i) &= (\eta|(T\xi_i) * \Delta^{\frac{1}{2}}\zeta) \\ &= (\eta|T(\xi_i * \zeta)) \\ &= (T * \eta|\xi_i * \zeta) \\ &\quad \to (T^*\eta|\zeta) = (\eta|T\zeta)\end{aligned}$$

where we have used the $(-\dfrac{1}{2})$-homogeneity of T and the fact that $\mathcal{K}(G) \subseteq D(T^*)$ since $T^* \in L^2(\psi_0)$. Thus we can define a linear functional F on the dense subspace $\mathcal{K}(G) * \mathcal{K}(G)$ of $L^2(G)$ by

$$F(\xi) = \lim_i (\xi|T\xi_i).$$

Since

$$|(\xi|T\xi_i)| \leq \|\xi\|_2 \|T\xi_i\|_2 \leq \|\xi\|_2 \|T\|_2 \|\lambda(\xi_i)\| \leq \|T\|_2 \|\xi\|_2,$$

this functional is bounded and therefore is given by some $f \in L^2(G)$:

$$\forall\ \xi \in \mathcal{K}(G) * \mathcal{K}(G) : F(\xi) = (\xi|f).$$

In particular, we have

$$(\eta|T\zeta) = F(\eta * \Delta^{-\frac{1}{2}}\tilde{\zeta}) = (\eta * \Delta^{-\frac{1}{2}}\tilde{\zeta}|f)$$

for all $\eta, \zeta \in \mathcal{K}(G)$. Since

$$(\eta * \Delta^{-\frac{1}{2}}\tilde{\zeta}|f) = (\eta|f * \Delta^{\frac{1}{2}}\zeta) = (\eta|\mathcal{P}(f)\zeta),$$

this implies

$$\forall\ \zeta \in \mathcal{K}(G) : T\zeta = \mathcal{P}(f)\zeta,$$

and we conclude, by Proposition 2.15, that $T = \mathcal{P}(f)$. ☐

Proposition 3.3. 1) *For all $T \in M$ and all $f \in L^2(G)$, we have*

$$\mathcal{P}(Tf) = [T\mathcal{P}(f)].$$

2) *For all $f \in L^2(G)$, we have*

$$\mathcal{P}(Jf) = \mathcal{P}(f)^*.$$

Proof. 1) Let $f \in L^2(G)$ and $T \in M$. Then $[T\mathcal{P}(f)]$ and $\mathcal{P}(Tf)$ both belong to $L^2(\psi_0)$, and for all $\xi \in \mathcal{K}(G)$ we have

$$\mathcal{P}(Tf)\xi = (Tf) * \Delta^{\frac{1}{2}}\xi = T(f * \Delta^{\frac{1}{2}}\xi) = [T\mathcal{P}(f)]\xi,$$

since T commutes with right convolution. By Proposition 2.15 we conclude that $\mathcal{P}(Tf) = [T\mathcal{P}(f)]$.

2) Let $f \in L^2(G)$. Then for all $\xi, \eta \in \mathcal{K}(G)$ we have

$$\begin{aligned}(\mathcal{P}(Jf)\xi|\eta) &= (Jf * \Delta^{\frac{1}{2}}\xi|\eta) \\ &= (Jf|\eta * \Delta^{-\frac{1}{2}}\tilde{\xi}) \\ &= (J(\eta * \Delta^{-\frac{1}{2}}\tilde{\zeta})|f) \\ &= (\xi * \Delta^{-\frac{1}{2}}\tilde{\eta}|f) \\ &= (\xi|f * \Delta^{\frac{1}{2}}\eta) \\ &= (\xi|\mathcal{P}(f)\eta),\end{aligned}$$

so that $\mathcal{P}(Jf)|_{\mathcal{K}(G)} \subseteq (\mathcal{P}(f)|_{\mathcal{K}(G)})^* = [\mathcal{P}(f)|_{\mathcal{K}(G)}]^* = \mathcal{P}(f)^*$ (since $\mathcal{P}(f) = [\mathcal{P}(f)|_{\mathcal{K}(G)}]$). We conclude by Proposition 2.15 that $\mathcal{P}(Jf) = \mathcal{P}(f)^*$. □

Proposition 3.4. *Let $f \in L^2(G)$. Then $\mathcal{P}(f) \geq 0$ if and only if*

$$\int f(x)(\xi * J\xi)(x)dx \geq 0$$

for all $\xi \in \mathcal{K}(G)$.

Proof. . For all $\xi \in \mathcal{K}(G)$ we have

$$\int f(x)(\xi * J\xi)(x)dx = (f|\bar{\xi} * \Delta^{-\frac{1}{2}}\check{\xi}) = (f * \bar{\xi}|\bar{\xi}) = (\mathcal{P}(f)\bar{\xi}|\bar{\xi}).$$

Since $\mathcal{P}(f) = [\mathcal{P}(f)|_{\mathcal{K}(G)}]$, we have $\mathcal{P}(f) \geq 0$ if and only if $(\mathcal{P}(f)\eta|\eta) \geq 0$ for all $\eta \in \mathcal{K}(G)$, and the result follows. □

By [10, Theorem 1.21 (3)] (or, to be precise, its spatial analogue obtained by the methods of [12, §1] connecting abstract [10] and spatial [12] L^p spaces), $L^2(\psi_0)_+$ is a selfdual cone in $L^2(\psi_0)$. By Proposition 3.4 and the unitarity of $\mathcal{P}$ we conclude that

$$P_0 = \{f \in L^2(G) \mid \forall \xi \in \mathcal{K}(G) : \int f(x)(\xi * J\xi)(x) \geq 0\}$$

is a selfdual cone in $L^2(G)$. Denote by P the ordinary selfdual cone in $L^2(G)$ associated with the achieved left Hilbert algebra $\mathfrak{A}_l \cap \mathfrak{A}_l^*$, i.e. let P be the closure in $L^2(G)$ of the set $\{\lambda(\xi)(J\xi)|\xi \in \mathfrak{A}_l \cap \mathfrak{A}_l^*\}$ (see [8, §1]). Since P is selfdual, we have

$$P = \{f \in L^2(G) \mid \forall \xi \in \mathfrak{A}_l \cap \mathfrak{A}_l^* : (f|\lambda(\xi)(J\xi)) \geq 0\}.$$

Thus $P \subseteq P_0$. Since P and P_0 are both selfdual, this implies that $P = P_0$. We have proved

Corollary 3.5. *A function $f \in L^2(G)$ belongs to the positive selfdual cone of $L^2(G)$ if and only if*

$$\forall \xi \in \mathcal{K}(G) : \int f(x)(\xi * J\xi)(x)dx \geq 0.$$

Remark 3.6. This result is similar to the characterization of the cone $P^\flat$ given in [18, p. 392] and proved in general in [9, Corollary 8]. The methods of [9] would also apply for our result. Our proof is based on the fact that $\mathcal{P}(f) = [\mathcal{P}(f)|_{\mathcal{K}(G)}]$.

Note 3.7. We have proved that $\mathcal{P} : L^2(G) \to L^2(\psi_0)$ carries the left regular representation on $L^2(G)$ into left multiplication on $L^2(\psi_0)$, takes J into $*$, and maps the positive selfdual cone of $L^2(G)$ onto $L^2(\psi_0)_+$. That a unitary transformation $L^2(G) \to L^2(\psi_0)$ having these properties exists (and is unique) also follows from [8, Theorem 2.3], since both representations of M are standard (by the spatial analogue of [10, Theorem 1.21, (3)]). In our approach, we have given a simple and direct definition of $\mathcal{P}$.

We can give an explicit description of the inverse of $\mathcal{P}$:

Proposition 3.8. *Let $T \in L^2(\psi_0)$, and let $(\xi_i)_{i\in I}$ be an approximate identity in $\mathcal{K}(G)_+$. Then*

$$\mathcal{P}^{-1}(T) = \lim_{i\in I} T\xi_i.$$

Proof. Let $f \in \mathcal{P}^{-1}(T)$. Then

$$T\xi_i = \mathcal{P}(f)\xi_i = f * \Delta^{\frac{1}{2}}\xi_i \to f$$

in $L^2(G)$. □

Remark 3.9. From Proposition 2.11 we already knew that for any approximate identity $(\xi_i)_{i\in I}$, the $\|T\xi_i\|$ tend to a limit and that this limit is independent of the choice of $(\xi_i)_{i\in I}$. Now, using that $L^2(\psi_0) = \mathcal{P}(L^2(G))$, we have proved that the same holds for the $T\xi_i$ themselves.

As a corollary, we have the following characterization of the inner product in $L^2(\psi_0)$, generalizing the formula for $\|T\|_2$ given in Proposition 2.11:

Corollary 3.10. *Let $T, S \in L^2(\psi_0)$. Then*

$$(T|S)_{L^2(\psi_0)} = \lim_{i\in I}(T\xi_i|S\xi_i)$$

for any approximate identity $(\xi_i)_{i\in I}$ in $\mathcal{K}(G)_+$.

Proof. Since $\mathcal{P}$ is unitary, we have

$$(T|S)_{L^2(\psi_0)} = (\mathcal{P}^{-1}(T)|\mathcal{P}^{-1}(S))_{L^2(G)} = \lim_{i\in I}(T\xi_i|S\xi_i)_{L^2(G)}.$$

□

4. The L^p Fourier transformations

Let $p \in [1, 2]$ and define $q \in [2, \infty]$ by $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Definition 4.1. Let $f \in L^p(G)$. The L^p Fourier transform of f is the operator $\mathcal{F}_p(f)$ on $L^2(G)$ given by

$$\mathcal{F}_p(f)\xi = f * \Delta^{\frac{1}{q}}\xi, \ \xi \in D(\mathcal{F}_p(f)),$$

where $D(\mathcal{F}_p(f)) = \{\xi \in L^2(G) \mid f * \Delta^{\frac{1}{q}}\xi \in L^2(G)\}$.

Note that by Lemma 1.1 the convolution product $f * \Delta^{\frac{1}{q}}\xi$ exists and belongs to $L^r(G)$, where $r \in [2, \infty]$ is given by $\dfrac{1}{p} + \dfrac{1}{2} - \dfrac{1}{r} = 1$, whenever $f \in L^p(G)$ and $\xi \in L^2(G)$, so that the definition of $D(\mathcal{F}_p(f))$ makes sense.

Remark 4.2. For $p=1$, we write $\mathcal{F}_1 = \mathcal{F}$; we have $\mathcal{F}(f)\xi = f*\xi$ and $D(\mathcal{F}(f)) = L^2(G)$, so that $\mathcal{F}(f)$ is simply $\lambda(f)$. For $p=2$, we have $\mathcal{F}_2(f) = \mathcal{P}(f)$.

Now again let $p \in [1,2]$. Let $f \in L^p(G)$. Then the operator $\mathcal{F}_p(f)$ is closed. To see this, suppose that $\xi_i \in D(\mathcal{F}_p(f))$ converges in $L^2(G)$ to some $\xi \in L^2(G)$ and $\mathcal{F}_p(f)\xi_i$ converges in $L^2(G)$ to some $\eta \in L^2(G)$. Now by Lemma 1.1 we have $\mathcal{F}_p(f)\xi_i = f * \Delta^{\frac{1}{q}}\xi_i \to f * \Delta^{\frac{1}{q}}\xi$ in $L^r(G)$ (where $\frac{1}{p}+\frac{1}{2}-\frac{1}{r}=1$). Therefore $f*\Delta^{\frac{1}{q}}\xi = \eta$, so that $f*\Delta^{\frac{1}{q}}\xi \in L^2(G)$, i.e. $\xi \in D(\mathcal{F}_p(f))$ and $\mathcal{F}_p(f)\xi = \eta$ as wanted.

Next we show that $\mathcal{F}_p(f)$ is $(-\frac{1}{q})$-homogeneous. For all $\xi \in D(\mathcal{F}_p(f))$ and all $x, y \in G$ we have

$$\begin{aligned}
\rho(x)(\mathcal{F}_p(f)\xi)(y) &= \Delta^{\frac{1}{2}}(x)(f*\Delta^{\frac{1}{q}}\xi)(yx)\\
&= \Delta^{\frac{1}{2}}(x)\int f(z)\Delta^{\frac{1}{q}}(z^{-1}yx)\xi(z^{-1}yx)dz\\
&= \Delta^{\frac{1}{q}}(x)\int f(z)\Delta^{\frac{1}{q}}(z^{-1}y)\Delta^{\frac{1}{2}}(x)\xi(z^{-1}yx)dz\\
&= \Delta^{\frac{1}{q}}(x)\int f(z)\Delta^{\frac{1}{q}}(z^{-1}y)(\rho(x)\xi)(z^{-1}y)dz\\
&= \Delta^{\frac{1}{q}}(x)(f*\Delta^{\frac{1}{q}}\rho(x)\xi)(y)\\
&= \Delta^{\frac{1}{q}}(x)(\mathcal{F}_p(f)\rho(x)\xi)(y),
\end{aligned}$$

i.e.

$$\rho(x)\mathcal{F}_p(f) \subseteq \Delta^{\frac{1}{q}}(x)\mathcal{F}_p(f)\rho(x)$$

for all $x \in G$ as wanted.

Finally, note that if $\xi \in L^2(G)\cap L^s(G)$ where $s \in [1,2]$ is given by $\frac{1}{p}+\frac{1}{s}-\frac{1}{2}=1$, then $\xi \in D(\mathcal{F}_p(f))$ by Lemma 1.1. In particular, $\mathcal{K}(G) \subseteq D(\mathcal{F}_p(f))$. In all, we have proved that for all $f \in L^p(G)$, $\mathcal{F}_p(f)$ is closed, densely defined, and $(-\frac{1}{q})$-homogeneous. We shall see, using the criterion from Proposition 2.12, that actually $\mathcal{F}_p(f) \in L^q(\psi_0)$. The proof is based on interpolation from the special cases

$$\mathcal{F} : L^1(G) \to L^\infty(\psi_0)$$

and

$$\mathcal{P} : L^2(G) \to L^2(\psi_0).$$

First we restrict our attention to $f \in \mathcal{K}(G)$

Lemma 4.3. *Let $p \in [1,2]$. Denote by A the closed strip $\{\alpha \in \mathbb{C} \mid \frac{1}{2} \le \mathrm{Re}(\alpha) \le 1\}$. Let $f \in \mathcal{K}(G)$ and $\xi \in \mathfrak{A}_l$. Then:*

(i) for each $\alpha \in A$, the convolution product

$$\xi_\alpha = \mathrm{sg}(f)|f|^{p\alpha} * \Delta^{1-\alpha}\xi$$

exists, and $\xi_\alpha \in L^2(G)$;

(*ii*) *the function*

$$\alpha \mapsto \xi_\alpha,\ \ \alpha \in A,$$

with values in $L^2(G)$ *is bounded;*

(*iii*) *for each* $\eta \in L^2(G)$*, the scalar function*

$$\alpha \mapsto (\xi_\alpha|\eta),\ \ \alpha \in A,$$

is continuous on A *and analytic in the interior of* A*.*

Proof. Write $g = \Delta^{-1/p}\check{f}$. Then

$$\forall \alpha \in A : \mathrm{sg}(f)|f|^{p\alpha} = \Delta^{-\alpha}(\mathrm{sg}(g)|g|^{p\alpha})^\vee.$$

Note that g as well as all $\mathrm{sg}(g)|g|^{p\alpha}, \alpha \in A$, belong to $\mathcal{K}(G)$.

For each $\eta \in \mathcal{K}(G)$, we define

$$H_\eta(\alpha) = \int \xi(x)(\mathrm{sg}(g)|g|^{p\alpha} * \Delta^{1-\alpha}\eta)(x)dx, \quad \alpha \in A, \tag{4.1}$$

i.e.

$$H_\eta(\alpha) = \int\int \xi(x)(\mathrm{sg}(g)|g|^{p\alpha})(y)\Delta^{1-\alpha}(y^{-1}x)\eta(y^{-1}x)dydx \tag{4.2}$$

(later we shall recognize $H_\eta(\alpha)$ as simply $(\xi_\alpha|\overline{\eta})$).

Note that

$$\begin{aligned}\forall \alpha \in A :&\| \, |\mathrm{sg}(g)|g|^{p\alpha}| * |\Delta^{1-\alpha}\eta| \, \|_2 \\ &\leq \| \, |g|^{p\mathrm{Re}(\alpha)}\|_1 \|\Delta^{1-\mathrm{Re}(\alpha)}|\eta| \|_2 \\ &\leq K < \infty,\end{aligned} \tag{4.3}$$

where K is a constant independent of $\alpha \in A$. In particular, this allows us to apply Fubini's theorem to the double integral (4.2). We find

$$\begin{aligned}H_\eta(\alpha) &= \int\int \xi(x)(\mathrm{sg}(g)|g|^{p\alpha})(y^{-1})\Delta^{1-\alpha}(yx)\eta(yx)\Delta^{-1}(y)dydx \\ &= \int\int \xi(y^{-1}x)(\mathrm{sg}(g)|g|^{p\alpha})(y^{-1})\Delta^{1-\alpha}(x)\eta(x)\Delta^{-1}(y)dxdy \\ &= \int\int (\mathrm{sg}(f)|f|^{p\alpha})(y)\Delta^{1-\alpha}(y^{-1}x)\xi(y^{-1}x)\eta(x)dydx;\end{aligned}$$

it also follows that the convolution integral

$$\xi_\alpha(x) = \int (\mathrm{sg}(f)|f|^{p\alpha})(y)\Delta^{1-\alpha}(y^{-1}x)\xi(y^{-1}x)dy$$

exists, so that we can write

$$H_\eta(\alpha) = \int \xi_\alpha(x)\eta(x)dx.$$

Now we shall prove that there exists a constant $C \geq 0$ independent of α such that

$$\forall \eta \in \mathcal{K}(G) : \left| \int \xi_\alpha(x)\eta(x)dx \right| \leq C\|\eta\|_2. \tag{4.4}$$

This will imply that each $\xi_\alpha,\ \alpha \in A$, is in $L^2(G)$ with $\|\xi_\alpha\|_2 \leq C$, i.e. (*i*) and (*ii*) will be proved.

Let us prove (4.4). Without loss of generality, we may assume that $\|f\|_p = 1$. We want to show then that

$$\forall \eta \in \mathcal{K}(G) : |H_\eta(\alpha)| \le (\|\lambda(\xi)\| + \|\xi\|_2)\|\eta\|_2. \tag{4.5}$$

To do this, we shall apply the Phragmen–Lindelöf principle [24, p.93].

Fix $\eta \in \mathcal{K}(G)$. By (4.2), H_η is continuous on A and analytic in the interior of A (the integrand in (4.2) can be majorized by an integrable function that is independent of α). Furthermore, H_η is bounded (use (4.3) and (4.1)). Finally, we shall estimate H_η on the boundaries of A.

Let $t \in \mathbb{R}$. Then $\Delta^{-it}\xi \in \mathfrak{A}_l$ and $\|\lambda(\Delta^{-it}\xi)\| \le \|\lambda(\xi)\|$.

Now

$$\begin{aligned}&\mathcal{P}(\mathrm{sg}(f)|f|^{p(\frac{1}{2}+it)})(\Delta^{-it}\xi)\\ &\qquad = \mathrm{sg}(f)|f|^{p(\frac{1}{2}+it)} * \Delta^{1-(\frac{1}{2}+it)}\xi = \xi_{\frac{1}{2}+it},\end{aligned}$$

so that $\xi_{\frac{1}{2}+it} \in L^2(G)$ with

$$\begin{aligned}\|\xi_{\frac{1}{2}+it}\|_2 &\le \|\mathcal{P}(\mathrm{sg}(f)|f|^{p(\frac{1}{2}+it)})\|_2\|\lambda(\Delta^{-it}\xi)\|\\ &\le \|\mathrm{sg}(f)|f|^{p(\frac{1}{2}+it)}\|_2\|\lambda(\xi)\|\\ &= \|\,|f|^{\frac{p}{2}}\|_2\|\lambda(\xi)\|\\ &= \|\lambda(\xi)\|\end{aligned}$$

(where we have used Proposition 2.11, the fact that $\mathcal{P}$ is unitary, and the hypothesis $\|f\|_p = 1$). Similarly,

$$\mathcal{F}(\mathrm{sg}(f)|f|^{p(1+it)})(\Delta^{-it}\xi) = \mathrm{sg}(f)|f|^{p(1+it)} * \Delta^{1-(1+it)}\xi = \xi_{1+it},$$

so that $\xi_{1+it} \in L^2(G)$ with

$$\begin{aligned}\|\xi_{1+it}\|_2 &\le \|\mathcal{F}(\mathrm{sg}(f)|f|^{p(1+it)})\|_\infty\|\Delta^{-it}\xi\|_2\\ &\le \|\mathrm{sg}(f)|f|^{p(1+it)}\|_1\|\xi\|_2\\ &= \|\,|f|^p\|_1\|\xi\|_2\\ &= \|\xi\|_2\end{aligned}$$

(where we have used that $\mathcal{F} : L^1(G) \to L^\infty(\psi_0)$ is norm-decreasing).

It follows that

$$\begin{aligned}\forall t \in \mathbb{R} : |H_\eta(\frac{1}{2}+it)| &= |\int \xi_{\frac{1}{2}+it}(x)\eta(x)dx|\\ &\le \|\xi_{\frac{1}{2}+it}\|_2\|\eta\|_2 \le \|\lambda(\xi)\|\,\|\eta\|_2\end{aligned}$$

and

$$\begin{aligned}\forall t \in \mathbb{R} : |H_\eta(1+it)| &= |\int \xi_{1+it}(x)\eta(x)dx|\\ &\le \|\xi_{1+it}\|_2\|\eta\|_2 \le \|\xi\|_2\|\eta\|_2.\end{aligned}$$

Then by the Phragmen–Lindelöf principle, we have established (4.5) and thus (i) and (ii).

Finally, (iii) is easy. Indeed, since $\alpha \mapsto \xi_\alpha$ as is bounded, each $\alpha \mapsto (\xi_\alpha|\eta)$, where $\eta \in L^2(G)$, can be uniformly approximated by functions $\alpha \mapsto (\xi_\alpha|\zeta)$ with $\zeta \in \mathcal{K}(G)$, so we just have to prove (iii) in the case of $\eta \in \mathcal{K}(G)$. This is already done since $(\xi_\alpha|\eta) = H_{\overline{\eta}}(\alpha)$. □

Lemma 4.4. *Let $p \in [1,2]$. Let $f \in \mathcal{K}(G)$ and $S \in L^p(\psi_0)$. Then for all $\xi \in \mathfrak{A}_l$ and $\eta \in \mathfrak{A}_l \cap D(S)$ we have*

$$|(\mathcal{F}_p(f)\xi|S\eta)| \leq \|f\|_p \|S\|_p \|\lambda(\xi)\| \, \|\lambda(\eta)\|.$$

Note that $\xi \in D(\mathcal{F}_p(f))$ by Lemma 4.3.

Proof. We may assume that $\|f\|_p = 1$ and $\|S\|_p = 1$. Furthermore, by Lemma 2.9, we need only consider $\eta \in \mathfrak{A}_l \cap D(|S|^p)$.

Let $\xi \in \mathfrak{A}_l$ and $\eta \in \mathfrak{A}_l \cap D(|S|^p)$. For each α in the closed strip $A = \{\alpha \in \mathbb{C} | \frac{1}{2} \leq \mathrm{Re}(\alpha) \leq 1\}$, put $\xi_\alpha = \mathrm{sg}(f)|f|^{p\alpha} * \Delta^{1-\alpha}\xi$ as in Lemma 4.3. Note that for all $\alpha \in A$ we have (by the spectral theory) $\eta \in D(U|S|^{p\alpha})$ and

$$\|U|S|^{p\alpha}\eta\|_2^2 \leq \| \, |S|^{\frac{p}{2}}\eta\|_2^2 + \| \, |S|^p\eta\|_2^2,$$

where $S = U|S|$ is the polar decomposition of S. For each $\alpha \in A$, put

$$\eta_\alpha = U|S|^{p\alpha}\eta.$$

Then the function $\alpha \mapsto \eta_\alpha$ with values in $L^2(G)$ is bounded on A. Furthermore, by [22, 9, 15], it is continuous on A and analytic in the interior of A.

Now for each $\alpha \in A$, let

$$H(\alpha) = (\xi_\alpha|\eta_{\overline{\alpha}}).$$

Then obviously H is bounded on A (by Lemma 4.3 (ii), $\alpha \mapsto \xi_\alpha$ is bounded). Furthermore, H is continuous on A. To see this, note that

$$\forall \alpha, \alpha_0 \in A : (\xi_\alpha|\eta_{\overline{\alpha}}) - (\xi_{\alpha_0}|\eta_{\overline{\alpha}_0}) = (\xi_\alpha|\eta_{\overline{\alpha}} - \eta_{\overline{\alpha}_0}) + (\xi_\alpha - \xi_{\alpha_0}|\eta_{\overline{\alpha}_0}),$$

the continuity follows since $\alpha \mapsto \xi_\alpha$ is bounded and weakly continuous (Lemma 4.3 (iii)). Finally, we claim that H is analytic in the interior of A. First note that for each $\zeta \in L^2(G)$ the function $\alpha \mapsto (\zeta|\eta_{\overline{\alpha}})$, being equal to $\alpha \mapsto \overline{(\eta_{\overline{\alpha}}|\zeta)}$, is analytic. Next, recall that $\alpha \mapsto \xi_\alpha$ is actually analytic as a function with values is $L^2(G)$ (by Lemma 4.3 (iii) and [19, Theorem 3.31]). Then, writing

$$\frac{(\xi_\alpha|\eta_{\overline{\alpha}}) - (\xi_{\alpha_0}|\eta_{\overline{\alpha}_0})}{\alpha - \alpha_0} = (\frac{1}{\alpha - \alpha_0}(\xi_\alpha - \xi_{\alpha_0})|\eta_{\overline{\alpha}}) + \frac{(\xi_{\alpha_0}|\eta_{\overline{\alpha}}) - (\xi_{\alpha_0}|\eta_{\overline{\alpha}_0})}{\alpha - \alpha_0},$$

we find that H has a derivative at each point α_0 in the interior of A.

Now suppose that

$$\forall t \in \mathbb{R} : |H(\frac{1}{2} + it)| \leq \|\lambda(\xi)\| \, \|\lambda(\eta)\| \tag{4.6}$$

and

$$\forall t \in \mathbb{R} : |H(1 + it)| \leq \|\lambda(\xi)\| \, \|\lambda(\eta)\|. \tag{4.7}$$

Then by the Phragmen–Lindelöf principle [24, p. 93] we infer that

$$\forall \alpha \in A : |H(\alpha)| \leq \|\lambda(\xi)\| \, \|\lambda(\eta)\|,$$

in particular,

$$|(\mathcal{F}_p(f)\xi|S\eta)| \leq \|\lambda(\xi)\| \, \|\lambda(\eta)\|$$

as desired, since

$$H(\frac{1}{p}) = (f * \Delta^{1-\frac{1}{p}}\xi|U|S|\eta) = (\mathcal{F}_p(f)|S\eta).$$

So we just have to prove (4.6) and (4.7).

Since $S \in L^p(\psi_0)$ with $\|S\|_p = 1$ we have

$$U|S|^{\frac{p}{2}} \in L^2(\psi_0) \quad \text{with} \quad \|U|S|^{\frac{p}{2}}\|_2 = 1 \tag{4.8}$$

and

$$U|S|^p \in L^1(\psi_0) \quad \text{with} \quad \|U|S|^p\|_1 = 1. \tag{4.9}$$

Now let $t \in \mathbb{R}$. Then by Lemma 2.5, we have

$$|S|^{-pit}\eta \in \mathfrak{A}_l \quad \text{with} \quad \|\lambda(|S|^{-pit}\eta)\| \leq \|\lambda(\eta)\|. \tag{4.10}$$

Using this, Proposition 2.11, the estimate $\|\xi_{\frac{1}{2}+it}\|_2 \leq \|\lambda(\xi)\|$ given in the proof of Lemma 4.3, and (4.8), we get

$$\begin{aligned} |H(\tfrac{1}{2}+it)| &= |(\xi_{\frac{1}{2}+it}|U|S|^{\frac{p}{2}}|S|^{-pit}\eta)| \\ &\leq \|\xi_{\frac{1}{2}+it}\|_2 \|U|S|^{\frac{p}{2}}|S|^{-pit}\eta\|_2 \\ &\leq \|\lambda(\xi)\| \, \|U|S|^{\frac{p}{2}}\|_2 \|\lambda(|S|^{-pit}\eta)\|_2 \\ &\leq \|\lambda(\xi)\| \, \|\lambda(\eta)\|, \end{aligned}$$

i.e. (4.6) is proved. To prove (4.7), note that

$$\begin{aligned} \xi_{1+it} &= \mathrm{sg}(f)|f|^{p(1+it)} * \Delta^{1-(1+it)}\xi \\ &= \lambda(\mathrm{sg}(f)|f|^{p(1+it)})\Delta^{-it}\xi \in \mathfrak{A}_l \end{aligned}$$

and

$$\begin{aligned} \|\lambda(\xi_{1+it})\| &\leq \|\lambda(\mathrm{sg}(f)|f|^{p(1+it)})\| \, \|\lambda(\Delta^{-it}\xi)\| \\ &\leq \|\mathrm{sg}(f)|f|^{p(1+it)}\|_1 \|\lambda(\xi)\| \\ &\leq \|\lambda(\xi)\|, \end{aligned}$$

since $\|\mathrm{sg}(f)|f|^{p(1+it)}\|_1 = \| \, |f|^p\|_1 = 1$. Using this together with (4.10), Proposition 2.10, and (4.9), we find

$$\begin{aligned} |H(1+it)| &= |(\xi_{1+it}|U|S|^p|S|^{-pit}\eta)| \\ &\leq \|\lambda(\xi_{1+it})\| \, \|U|S|^p\|_1 \|\lambda(|S|^{-pit}\eta)\| \\ &\leq \|\lambda(\xi)\| \, \|\lambda(\eta)\|, \end{aligned}$$

so that (4.7) is proved. □

In the formulation of the following theorem we include the case $p = 2$. Note however that the proof is based on the results for this special case (they were used for the preceding lemmas).

Theorem 4.5 (Hausdorff–Young). *Let $p \in]1,2]$ and $\frac{1}{p}+\frac{1}{q}=1$.*

1) *Let $f \in L^p(G)$. Then $\mathcal{F}_p(f) \in L^q(\psi_0)$ and*
$$\|\mathcal{F}_p(f)\|_q \leq \|f\|_p.$$
2) *The mapping*
$$\mathcal{F}_p : L^p(G) \to L^q(\psi_0)$$
is linear, norm-decreasing, injective, and has dense range.
3) *For all $h \in L^1(G)$ and $f \in L^p(G)$, we have*
$$\mathcal{F}_p(h * f) = [\lambda(h)\mathcal{F}_p(f)].$$
4) *For all $f \in L^p(G)$, we have*
$$\mathcal{F}_p(J_p f) = \mathcal{F}_p(f)^*.$$

Proof. 1) First suppose that $f \in \mathcal{K}(G)$. Then, using Proposition 2.12, we conclude from Lemma 4.4 that $\mathcal{F}_p(f) \in L^q(\psi_0)$ with $\|\mathcal{F}_p(f)\|_q \leq \|f\|_p$. Thus we have defined a norm-decreasing mapping
$$\mathcal{F}_p|_{\mathcal{K}(G)} : L^p(G) \to L^q(\psi_0).$$
Furthermore $\mathcal{F}_p|_{\mathcal{K}(G)}$ is linear: for all $f_1, f_2 \in \mathcal{K}(G)$ and all $\xi \in \mathcal{K}(G)$ we have
$$(f_1+f_2) * \Delta^{\frac{1}{q}}\xi = f_1 * \Delta^{\frac{1}{q}}\xi + f_2 * \Delta^{\frac{1}{q}}\xi$$
so that $\mathcal{F}_p(f_1+f_2) = [\mathcal{F}_p(f_1)+\mathcal{F}_p(f_2)]$ by Proposition 2.15. Now $\mathcal{F}_p|_{\mathcal{K}(G)}$ extends by continuity to a norm-decreasing linear mapping
$$\mathcal{F}'_p : L^p(G) \to L^q(\psi_0).$$
We claim that for all $f \in L^p(G)$, we have
$$\mathcal{F}'_p(f) = \mathcal{F}_p(f).$$
this will prove 1).

Let $f \in L^p(G)$. Then $\mathcal{F}'_p(f) \in L^q(\psi_0)$ and $\mathcal{K}(G) \subseteq D(\mathcal{F}'_p(f))$ by Lemma 2.16. On the other hand, by the remarks at the beginning of this section, $\mathcal{F}_p(f)$ is closed, densely defined, and $(-\frac{1}{q})$-homogeneous, an $\mathcal{K}(G) \subseteq D(\mathcal{F}_p(f))$. Thus ny Lemma 2.7, to conclude that $\mathcal{F}'_p(f) = \mathcal{F}_p(f)$ we just have to show that
$$\forall \xi \in \mathcal{K}(G) : \ \mathcal{F}'_p(f)\xi = \mathcal{F}_p(f)\xi.$$
Now, take $f_n \in \mathcal{K}(G)$ such that $f_n \to f$ in $L^p(G)$. Then for all $\xi \in \mathcal{K}(G)$, we have
$$\begin{aligned}\mathcal{F}_p(f_n)\xi &= f_n * \Delta^{\frac{1}{q}}\xi \\ &\to f * \Delta^{\frac{1}{q}}\xi = \mathcal{F}_p(f)\xi \text{ in } L^p(G).\end{aligned}$$
On the other hand, since $\mathcal{F}'_p$ is continuous,
$$\mathcal{F}_p(f_n)\xi = \mathcal{F}'_p(f_n)\xi \to \mathcal{F}'_p(f)\xi \text{ in } L^2(G)$$
by Lemma 2.16. We conclude that $\mathcal{F}_p(f)\xi = \mathcal{F}'_p(f)\xi$ as desired. Thus 1) is proved.

2) By the proof of 1), we just have to show that $\mathcal{F}_p$ is injective and has dense range. The injectivity is evident: if $\mathcal{F}_p(f) = 0$ for some $f \in L^p(G)$, then $f * \Delta^{\frac{1}{q}}\xi = 0$ for all $\xi \in \mathcal{K}(G)$, and thus $f = 0$. That $\mathcal{F}_p(L^p(G))$ is dense will be proved later.

3) For all $h \in L^1(G)$, $f \in L^p(G)$, and $\xi \in \mathcal{K}(G)$ we have

$$h * (f * \Delta^{\frac{1}{q}}\xi) = (h * f) * \Delta^{\frac{1}{q}}\xi$$

(in $L^p(G)$). Thus by Proposition 2.15,

$$\lambda(h)\mathcal{F}_p(f)] = \mathcal{F}_p(h * f).$$

4)Let $f \in \mathcal{K}(G)$. Then for $\xi, \eta \in \mathcal{K}(G)$ we have

$$\begin{aligned}(\mathcal{F}_p(J_pf)\xi|\eta) &= (J_pf * \Delta^{\frac{1}{q}}\xi|\eta)\\ &= (\Delta^{\frac{1}{q}}\xi|\Delta^{-1}(J_pf)\breve{} * \eta)\\ &= (\xi|\Delta^{\frac{1}{q}}(\Delta^{-1}\Delta^{\frac{1}{p}}f * \eta))\\ &= (\xi|f * \Delta^{\frac{1}{q}}\eta)\\ &= (\xi|\mathcal{F}_p(f)\eta),\end{aligned}$$

so that $\mathcal{F}_p(J_pf)|_{\mathcal{K}(G)} \subseteq (\mathcal{F}_p(f)|_{\mathcal{K}(G)})^*$. By Proposition 2.15, we conclude that

$$\mathcal{F}_p(J_pf) = \mathcal{F}_p(f)^*.$$

By the continuity of J_p, $\mathcal{F}_p$, and $*$, this holds for all $f \in L^p(G)$.

Finally, let us show that $\mathcal{F}_p(L^p(G))$ is dense in $L^q(\psi_0)$. By the duality between $L^q(\psi_0)$ and $L^p(\psi_0)$, this is equivalent to proving that if $T \in L^p(\psi_0)$ satisfies $\int[\mathcal{F}_p(f)T]d\psi_0 = 0$ for all $f \in L^p(G)$ for all $f \in L^p(G)$, then $T = 0$.

Suppose that $T \in L^p(\psi_0)$ is such that

$$\forall f \in L^p(G) : \int[\mathcal{F}_p(f)T]d\psi_0 = 0.$$

Let $f \in L^p(G)$. Then for all $h \in L^1(G)$ we have

$$\int[\mathcal{F}_p(h * f)T]d\psi_0 = 0.$$

Alternatively stated, since $[\mathcal{F}_p(h * f)T] = [[\lambda(h)\mathcal{F}_p(f)]T] = [\lambda(h)[\mathcal{F}_p(f)T]]$, we have

$$\forall h \in L^1(G) : \int[\lambda(h)[\mathcal{F}_p(f)T]]d\psi_0 = 0.$$

We conclude that the normal functional on M defined by $[\mathcal{F}_p(f)T] \in L^1(\psi_0)$ is 0, so that

$$[\mathcal{F}_p(f)T] = 0.$$

Changing f into J_pf and using 4) this gives

$$\forall f \in L^p(G) : \ [\mathcal{F}_p(f) * T] = 0.$$

Now let $\xi \in D(T)$. Then using [12, II, Proposition 5],[1] we find that

$$\begin{aligned}\forall f, \eta \in \mathcal{K}(G) : (T\xi | f * \Delta^{\frac{1}{q}}\eta) &= (T\xi | \mathcal{F}_p(f)\eta) \\ &= \langle [\mathcal{F}_p(f) * T], \lambda(\xi)\lambda(\eta)^* \rangle = 0.\end{aligned}$$

Thus $T\xi = 0$. This proves that $T = 0$ as wanted. ☐

Proposition 4.6. *Let $p \in [1, 2]$. Let $f \in L^p(G)$ and $\mathcal{F}_p(f) \geq 0$ if and only if*

$$\forall \xi \in \mathcal{K}(G) : \int f(x)(\xi * J_p\xi)(x)dx \geq 0.$$

Proof. We have

$$(\mathcal{F}_p(f)\xi|\xi) = \int (f * \Delta^{\frac{1}{p}}\xi)(x)\overline{\xi(x)}dx = \int f(x)(\overline{\xi} * \Delta^{-\frac{1}{p}}\check{\xi})(x)dx$$

for all $\xi \in \mathcal{K}(G)$. The result follows by changing ξ into $\overline{\xi}$ and recalling that $\mathcal{F}_p(f) = [\mathcal{F}_p(f)|_{\mathcal{K}(G)}]$. ☐

The L^p Fourier transformations are well-behaved with respect to convolution as the following proposition shows. The result generalizes 3) of theorem.

Proposition 4.7. *Let $p_1, p_2, p \in [1, 2]$ such that $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p} = 1$. Define $q_1 \in [2, \infty]$ by $\frac{1}{p_1} + \frac{1}{q_1} = 1$. Let $f_1 \in L^{p_1}(G)$ and $f_2 \in L^{p_2}(G)$. Then*

$$\mathcal{F}_p(f_1 * \Delta^{\frac{1}{q_1}} f_2) = [\mathcal{F}_{p_1}(f_1)\mathcal{F}_{p_2}(f_2)].$$

Proof. By Lemma 1.1, we have $f_1 * \Delta^{\frac{1}{q_1}} f_2 \in L^p(G)$, and $(f_1, f_2) \mapsto \mathcal{F}_p(f_1 * \Delta^{\frac{1}{q_1}} f_2)$ maps $L^{p_1}(G) \times L^{p_2}(G)$ continuously into $L^q(\psi_0)$ (where $\frac{1}{p} + \frac{1}{q} = 1$). Also $[\mathcal{F}_{p_1}(f_1)\mathcal{F}_{p_2}(f_2)]$ is continuous as a function of $(f_1, f_2) \in L^{p_1}(G) \times L^{p_2}(G)$ with values in $L^q(\psi_0)$. Thus we need only prove the statement for $f_1, f_2 \in \mathcal{K}(G)$. Since

$$(f_1 * \Delta^{\frac{1}{q_1}} f_2) * \Delta^{\frac{1}{q_1}}\xi = f_1 * \Delta^{\frac{1}{q_1}}(f_2 * \Delta^{\frac{1}{q_2}}\xi)$$

(where $\frac{1}{p_2} + \frac{1}{q_2} = 1$) for all $f_1, f_2, \xi \in \mathcal{K}(G)$, the result follows by Proposition 2.15 as usual. ☐

We conclude this section by the following characterization of the image of $L^p(G)$ under $\mathcal{F}_p$:

Proposition 4.8. *Let $p \in]1, 2]$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $T \in L^q(\psi_0)$.*

1) *If $T = \mathcal{F}_p(f)$ for some $f \in L^p(G)$, then for any approximate identity $(\xi_i)_{i \in I}$ in $\mathcal{K}(G)_+$ we have*

$$T\xi_i \to f \text{ in } L^p(G).$$

In particular, $\lim_{i \in I} \|T\xi_i\|_p = \|f\|_p < \infty$.

2) *Conversely, suppose that for some approximate identity $(\xi_i)_{i \in I}$ in $\mathcal{K}(G)_+$ we have $T\xi_i \in L^p(G)$ for all $i \in I$ and*

$$\liminf_{i \in I} \|T\xi_i\|_p < \infty.$$

Then $T \in \mathcal{F}_p(L^p(G))$.

Proof. The first part is obvious since $T\xi_i = f * \Delta^{\frac{1}{q}}\xi_i \to f$ in $L^p(G)$ and therefore $\|T\xi_i\|_p \to \|f\|_p$. Now suppose that the hypothesis of 2) holds for some $(\xi_i)_{i\in I}$. We the proceed as in the proof of the surjectivity of $\mathcal{P}$ (Theorem 3.2). for all $\eta, \zeta \in \mathcal{K}(G)$ we have

$$\begin{aligned}(\eta * \Delta^{-\frac{1}{q}}\tilde{\zeta}|T\xi_i) =&(\eta|(T\xi_i) * \Delta^{\frac{1}{q}}\zeta)\\ =&(\eta|T(\xi_i * \zeta))\\ =&(T * \eta|\xi_i * \zeta)\\ &\to (T * \eta|\zeta) = (\eta|T\zeta).\end{aligned}$$

Thus we can define a linear functional F on $\mathcal{K}(G) * \mathcal{K}(G)$ by

$$F(\xi) = \lim_{i\in I} \int \xi(x)\overline{(T\xi_i)(x)}dx.$$

Since

$$\left|\int \xi(x)\overline{(T\xi_i)(x)}dx\right| \le \|\xi\|_q\|T\xi_i\|_p$$

we have

$$|F(\xi)| \le (\liminf_{i\in I} \|T\xi_i\|_p) \cdot \|\xi\|_q.$$

Now since $\mathcal{K}(G) * \mathcal{K}(G)$ is dense in $L^q(G)$, F extends to a bounded functional on $L^q(G)$ and therefore is given by some $\overline{f} \in L^p(G)$:

$$F(\xi) = \int \xi(x)\overline{f(x)}dx.$$

In particular,

$$(\eta|T\zeta) = F(\eta * \Delta^{-\frac{1}{q}}\tilde{\zeta}) = \int (\eta * \Delta^{-\frac{1}{q}}\tilde{\zeta})(x)\overline{f(x)}dx$$

for all $\eta, \zeta \in \mathcal{K}(G)$. Since

$$\int (\eta * \Delta^{-\frac{1}{q}}\tilde{\zeta})(x)\overline{f(x)}dx = \int \eta(x)\overline{(f * \Delta^{\frac{1}{q}}\zeta)(x)}dx = (\eta|\mathcal{F}_p(f)\zeta),$$

this implies that

$$\forall \zeta \in \mathcal{K}(G) : T\zeta = \mathcal{F}_p(f)\zeta,$$

and we conclude by Proposition 2.15 that $T = \mathcal{F}_p(f)$. □

Remark 4.9. For $p = 1$, part 2) of the above proposition fails. (for counter-example, take $T = \lambda(x)$, $x \in G$.)

5. The L^p Fourier contransformation

Definition 5.1. Let $p \in [1, 2]$ and $\frac{1}{p} + \frac{1}{q} = 1$. For each $T \in L^p(\psi_0)$, denote by $\overline{\mathcal{F}}_p(T)$ the unique function in $L^q(G)$ such that

$$\int h(x)\overline{\mathcal{F}}_p(T)(x)dx = \int [\mathcal{F}_p(h)T]d\psi_0$$

for all $h \in L^p(G)$ (or just $h \in \mathcal{K}(G)$, or $h \in \mathcal{K}(G)$, or $h \in \mathcal{K}(G) * \mathcal{K}(G)$). The mapping

$$\overline{\mathcal{F}}_p : L^p(\psi_0) \to L^q(G)$$

thus define will be called the L^p fourier transformation. For $p = 1$, we write $\overline{\mathcal{F}} = \overline{\mathcal{F}}_1$.

Note that if $1 < p \leq 2$, then $\overline{\mathcal{F}}_p$ is simply the transpose of $\mathcal{F}_p : L^p(G) \to L^q(\psi_0)$ when we identify the dual spaces of $L^p(G)$ and $L^q(\psi_0)$ with $L^q(G)$ and $L^p(\psi_0)$, respectively.

The mapping $\overline{\mathcal{F}}$ takes an element $T \in L^1(\psi_0)$ into the unique function $\varphi \in A(G)$ that defines the same element of M_* as T does; in particular,

$$\overline{\mathcal{F}}\left(\frac{d\varphi}{d\psi_0}\right) = \varphi$$

for all $\varphi \in (M_*)^+ \simeq A(G)_+$.

In view of these remarks, we obviously have

Theorem 5.2. 1) *Let* $p \in]1, 2]$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\overline{\mathcal{F}}_p : L^p(\psi_0) \to L^q(G)$$

is linear, norm-decreasing, injective, and has dense range.

2) *The mapping*

$$\overline{\mathcal{F}} : L^1(\psi_0) \to A(G)$$

is an isometry of $L^1(\psi_0)$ *onto* $A(G)$.

Remark 5.3. With our definition of the contransformations, $\overline{\mathcal{F}}_2$ is not exactly the inverse of $\mathcal{P}$; they are related by the formula

$$\forall T \in L^2(\psi_0) : \overline{\mathcal{F}}_2(T) = \overline{\mathcal{P}^{-1}(T^*)}$$

(since for all $h \in L^2(G)$ we have

$$\int h(x)\overline{\mathcal{F}}_2(T)(x)dx = \int [\mathcal{F}_2(h)T]d\psi_0 = (\mathcal{F}_2(h)|T^*)_{L^2(\psi_0)}$$
$$= \left(h|\mathcal{P}^{-1}(T^*)\right)_{L^2(G)} = \int h(x)\overline{\mathcal{P}^{-1}(T^*)(x)}dx.$$

It follows that $\overline{\mathcal{F}}_2 : L^2(\psi_0) \to L^2(G)$ is unitary.

The classical Hausdorff–Young theorem [24, p.101] has a second part, stating that with each $c \in l_p(\mathbb{Z})$, $1 \leq p \leq 2$, we can associate a function $f \in L^q(\mathbb{T})$ with $\|f\|_q \leq \|c\|_p$, such that c is the sequence of Fourier coefficients of f. Theorem 5.2 is a generalization of this result. Indeed, let $T \in L^p(\psi_0)$ and put $g = \Delta^{-\frac{1}{q}}\overline{\mathcal{F}}_p(T)\check{}$. Then $g \in L^q(G)$ and $\|g\|_q = \|\overline{\mathcal{F}}_p(T)\|_q \leq \|T\|_p$, and we shall see that T is close to being the "L^q Fourier transform" of g in the sense that $T\xi = g * \Delta^{\frac{1}{p}}\xi$ for certain ξ (note that we do not in general define L^q Fourier transforms for $q \geq 2$).

Proposition 5.4. *Let* $p \in [1, 2]$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *Then for all* $T \in L^p(\psi_0)$, *we have*

$$\overline{\mathcal{F}}_p(T^*) = J_q(\overline{\mathcal{F}}_p(T)).$$

Proof. For all $h \in L^p(G)$ we have

$$\begin{aligned}\int h(x)\overline{\mathcal{F}}_p(T^*)(x)dx &= \int [\mathcal{F}_p(h)T^*]d\psi_0 \\ &= \overline{\int [T\mathcal{F}_p(h)^*]d\psi_0} = \overline{\int [T\mathcal{F}_p(J_p h)]d\psi_0} \\ &= \overline{\int \mathcal{F}_p(T)(x)\Delta^{-\frac{1}{q}}(x)\overline{h(x^{-1})}dx} \\ &= \int \Delta^{-\frac{1}{q}}(x)\overline{\mathcal{F}_p(T)(x^{-1})}h(x)dx.\end{aligned}$$

□

Lemma 5.5. *Let $h, k \in \mathcal{K}(G)$ and put $\varphi = h * \tilde{k}$. Then $[\lambda(\varphi)\Delta] \in L^1(\psi_0)$ and*

$$\int [\lambda(\varphi)\Delta]\psi_0 = \varphi(e).$$

Proof. Since

$$\begin{aligned}\lambda(\varphi)\Delta &= \lambda(h)\lambda(\tilde{k})\Delta^{\frac{1}{2}}\Delta^{\frac{1}{2}} \\ &\subseteq \lambda(h)\Delta^{\frac{1}{2}}\lambda(\Delta^{-\frac{1}{2}}\tilde{k})\Delta^{\frac{1}{2}} \subseteq \mathcal{P}(h)\mathcal{P}(k)^*,\end{aligned}$$

the closure $[\lambda(\varphi)\Delta]$ exists and $[\lambda(\varphi)\Delta] \subseteq [\mathcal{P}(h)\mathcal{P}(k)^*]$. One easily checks that for all $x \in G$ we have $\rho(x)\lambda(\varphi)\Delta \subseteq \Delta(x)\lambda(\varphi)\Delta\rho(x)$, i.e. that $\lambda(\varphi\Delta)$ is (-1)-homogeneous. Then also $[\lambda(\varphi)\Delta]$ is (-1)-homogeneous, and we conclude by Proposition 2.15 that $[\lambda(\varphi)\Delta] = [\mathcal{P}(h)\mathcal{P}(k)^*]$, so that $[\lambda(\varphi)\Delta] \in L^1(\psi_0)$ and

$$\begin{aligned}\int [\lambda(\varphi)\Delta]d\psi_0 &= (\mathcal{P}(h)|\mathcal{P}(k))_{L^2(\psi_0)} \\ &= \int h(x)\overline{k(x)}dx = (h * \tilde{k})(e) = \varphi(e).\end{aligned}$$

□

Suppose that $f_1 \in L^{p_1}(G)$ and $f_2 \in L^{p_2}(G)$, where $p_1, p_2 \in [1, 2]$. In Proposition 4.7, a formula relating $f_1 * \Delta^{\frac{1}{q_1}} f_2$ and $[\mathcal{F}_{p_1}(f_1)\mathcal{F}_{p_2}(f_2)]$ was given in the case where $\frac{1}{p_1} + \frac{1}{p_2} \geq \frac{3}{2}$ (under this assumption, $p \in [1, 2]$ satisfying $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p} = 1$ exists). The following proposition takes care of the case where $\frac{1}{p_1} + \frac{1}{p_2} \leq \frac{3}{2}$.

Proposition 5.6. *Let $p_1, p_2 \in [1, 2]$ and $q \in [2, \infty]$ such that $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{q} = 1$. Let $f_1 \in L^{p_1}(G)$ and $f_2 \in L^{p_2}(G)$. Then*

$$\overline{\mathcal{F}}_p([\mathcal{F}_{p_1}(f_1)\mathcal{F}_{p_2}(f_2)]) = \Delta^{-\frac{1}{q}}(f_1 * \Delta^{\frac{1}{q_1}} f_2)\check{},$$

where $\frac{1}{p} + \frac{1}{q}$ and $\frac{1}{p_1} + \frac{1}{q_1} = 1$.

Proof. Both expressions exist, belong to $L^q(G)$, and are continuous as functions of $(f_1, f_2) \in L^{p_1}(G) \times L^{p_2}(G)$. Thus we need only prove the formula for $f_1, f_2 \in \mathcal{K}(G)$. In this case, for all $h \in \mathcal{K}(G)$ and $\xi \in \mathcal{K}(G)$ we have

$$h * \Delta^{\frac{1}{q}}(f_1 * \Delta^{\frac{1}{q_1}}(f_2 * \Delta^{\frac{1}{q_2}}\xi)) = h * \Delta^{\frac{1}{q}}(f_1 * \Delta^{\frac{1}{q_1}} f_2) * \Delta\xi,$$

where $\frac{1}{p_2} + \frac{1}{q_2} = 1$. We conclude by Proposition 2.15 that

$$\forall h \in \mathcal{K}(G) : [\mathcal{F}_p(h)[\mathcal{F}_{p_1}(f_1)\mathcal{F}_{p_2}(f_2)]] = [\lambda(h * \Delta^{\frac{1}{q}} f)\Delta],$$

where we have written $f = f_1 * \Delta^{\frac{1}{q_1}} f_2$. Using this and Lemma 5.5, we find

$$\begin{aligned}\forall h \in \mathcal{K}(G) : &\int [\mathcal{F}_p(h)[\mathcal{F}_{p_1}(f_1)\mathcal{F}_{p_2}(f_2)]] d\psi_0 \\ &= \int [\lambda(h * \Delta^{\frac{1}{q}} f)\Delta] d\psi_0 \\ &= (h * \Delta^{\frac{1}{q}} f)(e) \\ &= \int h(x)\Delta^{\frac{1}{q}}(x^{-1}) f(x^{-1}) dx.\end{aligned}$$

We conclude that

$$\overline{\mathcal{F}}_p([\mathcal{F}_{p_1}(f_1)\mathcal{F}_{p_2}(f_2)]) = \Delta^{-\frac{1}{q}} \check{f}$$

as desired. □

Corollary 5.7. *Let $f, g \in L^2(G)$. Then*

$$f * \tilde{g} = \overline{\mathcal{F}}([\mathcal{P}(\overline{g}\mathcal{P}(\overline{f})^*]).$$

Proof. Letting $p_1 = p_2 = 2$ and $q = \infty$ in Proposition 5.6, we obtain

$$\mathcal{F}([\mathcal{P}(\overline{g})\mathcal{P}(\overline{f})^*]) = \overline{\mathcal{F}}([\mathcal{F}_2(\overline{g})\mathcal{F}_2(J\overline{f})]) = (\overline{g} * \Delta^{\frac{1}{2}} J\overline{f})\check{} = f * \tilde{g}.$$

□

Remark 5.8. Since $A(G) = \overline{\mathcal{F}}(L^1(\psi_0))$ and since every $T \in L^1(\psi_0)$ can be written $T = [RS^*]$ where $R, S \in L^2(\psi_0) = \mathcal{P}(L^2(G))$ (just put $R = U|T|^{\frac{1}{2}}$ and $S^* = |T|^{\frac{1}{2}}$, where $T = U|T|$ is the polar decomposition of T), we have reproved the fact [6] that $A(G) = \{f * \tilde{g} | f, g \in L^2(G)\}$. It also follows that $\|\varphi\|_{A(G)} \leq \|f\|_2\|g\|_2$ whenever $\varphi = f * \tilde{g}$, $f, g \in L^2(G)$ (since $\|[\mathcal{P}(\overline{g})\mathcal{P}(\overline{f})^*]\|_1 \leq \|\mathcal{P}(\overline{g})\|_2\|\mathcal{P}(\overline{f})\|_2$), and that, given $\varphi \in A(G)$, there exist $f, g \in L^2(G)$ with $\varphi = f * \tilde{g}$ such that $\|\varphi\|_{A(G)} = \|f\|_2\|g\|_2$ (use that $\|T\|_1 = \|U|T|^{\frac{1}{2}}\|_2\||T|^{\frac{1}{2}}\|_2$ for $T \in L^1(\psi_0)$).

Proposition 5.9. *Let $p \in [1, 2]$ and $q_1, q_2 \in [2, \infty]$ such that $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{p}$. Let $T \in L^{q_1}(\psi_0)$ and $S \in L^{q_2}(\psi_0)$. Then*

$$\langle T\xi | S\eta \rangle = \int \overline{\mathcal{F}}_p([S * T])(x)(\xi * J_p\eta)(x) dx$$

for all $\xi, \eta \in \mathcal{K}(G)$.

Proof. By Lemma 2.16, the left hand side of the equation to be proved is a continuous function of T and S. The same is true of the right hand side. Therefore it is enough to prove the statement for T and S belonging to the (dense) sets $\mathcal{F}_{p_1}(\mathcal{K}(G))$ and $\mathcal{F}_{p_2}(\mathcal{K}(G))$ (where, as usual, $\frac{1}{p_1} + \frac{1}{q_1} = 1$, $\frac{1}{p_2} + \frac{1}{q_2} = 1$).

Now suppose that $T = \mathcal{F}_{p_1}(h)$ and $S = \mathcal{F}_{p_2}(k)$ where $h, k \in \mathcal{K}(G)$. Then

$$\begin{aligned}(T\xi|S\eta) &= (h * \Delta^{\frac{1}{q_1}}\xi|k * \Delta^{\frac{1}{q_2}}\eta)\\ &= (\Delta^{\frac{1}{q_1}}\xi * \Delta^{-\frac{1}{q_2}}\tilde{\eta}|\Delta^{-1}\tilde{h} * k)\\ &= (\xi * \Delta^{-\frac{1}{q_1}-\frac{1}{q_2}}\tilde{\eta}|\Delta^{-\frac{1}{p_1}}\tilde{h} * \Delta^{-\frac{1}{q_1}}k)\\ &= \int (\xi * J_p\eta)(x)(\Delta^{-\frac{1}{p_1}}\check{h} * \Delta^{-\frac{1}{q_1}}\overline{k})(x)dx.\end{aligned}$$

Since

$$\begin{aligned}\overline{\mathcal{F}}_p([S * T]) &= \overline{\mathcal{F}_p}([\mathcal{F}_{p_2}(J_{p_2}k)\mathcal{F}_{p_1}(h)])\\ &= \Delta^{-\frac{1}{q}}(J_{p_2}k * \Delta^{\frac{1}{q_2}}h)\check{}\\ &= \Delta^{-1+\frac{1}{p}}\Delta^{-\frac{1}{q_2}}\check{h} * \Delta^{-1+\frac{1}{p}}\Delta^{\frac{1}{p_2}}\overline{k}\\ &= \Delta^{-\frac{1}{p_1}}\check{h} * \Delta^{-\frac{1}{q_1}}\overline{k}\end{aligned}$$

we have proved the formula. □

Proposition 5.10. *Let $p \in [1, 2]$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $T \in L^p(\psi_0)$ with polar decomposition $T = U|T|$. Put $g = \Delta^{-\frac{1}{q}}\overline{\mathcal{F}}_p(T)\check{}$.*
Then

$$(|T|^{\frac{1}{2}}\xi\,|\,|T|^{\frac{1}{2}}U^*\eta) = \int (g * \Delta^{\frac{1}{p}}\xi)(x)\overline{\eta(x)}dx$$

for all $\xi, \eta \in \mathcal{K}(G)$.

Proof. Put $q_1 = q_2 = 2p$. then $|T|^{\frac{1}{2}} \in L^{q_1}(\psi_0)$ and $|T|^{\frac{1}{2}}U^* \in L^{q_2}(\psi_0)$, and by Proposition 5.9 we get

$$\begin{aligned}(|T|^{\frac{1}{2}}\xi|\,|T|^{\frac{1}{2}}U^*\eta) &= \int \overline{\mathcal{F}}_p(T)(x)(\xi * J_p\eta)(x)dx\\ &= \int \overline{\mathcal{F}}(T)(x^{-1}(\Delta^{\frac{1}{p}}\overline{\eta} * \check{\xi})(x^{-1})\Delta^{-1}(x)dx\\ &= \int g(x)(\overline{\eta} * \Delta^{-\frac{1}{p}}\check{\xi})(x)dx\\ &= \int (g * \Delta^{\frac{1}{p}}\xi)(x)\overline{\eta(x)}dx.\end{aligned}$$

□

Proposition 5.11. *Let $p \in [1, 2]$ and $T \in L^p(\psi_0)$. Put $g = \Delta^{-\frac{1}{q}}\overline{\mathcal{F}}_p(T)\check{}$. Let $\xi \in \mathcal{K}(G)$. Then $\xi \in D(T)$ if and only if $g * \Delta^{\frac{1}{p}}\xi \in L^2(G)$, and if this is the case, we have*

$$T\xi = g * \Delta^{\frac{1}{p}}\xi.$$

Proof. First suppose that $\xi \in D(T)$. Then for all $\eta \in \mathcal{K}(G)$ we have

$$\int (T\xi)(x)\overline{\eta(x)}dx = (T\xi|\eta) = (|T|^{\frac{1}{2}}\xi\,|\,|T|^{\frac{1}{2}}U^*\eta) = \int (g * \Delta^{\frac{1}{p}}\xi)(x)\overline{\eta(x)}dx.$$

Hence $g * \Delta^{\frac{1}{p}}\xi = T\xi$ and thus $g * \Delta^{\frac{1}{p}}\xi \in L^2(G)$.

Conversely, if $g * \Delta^{\frac{1}{p}}\xi \in L^2(G)$, then

$$\begin{aligned}(|T|^{\frac{1}{2}}\xi \,|\, |T|^{\frac{1}{2}}U^*\eta|) &= \left|\int (g * \Delta^{\frac{1}{p}}\xi)(x)\overline{\eta(x)}dx\right| \\ &\leq \|g * \Delta^{\frac{1}{p}}\xi\|_2\|\eta\|_2\end{aligned}$$

for all $\eta \in \mathcal{K}(G)$.
We conclude that $|T|^{\frac{1}{2}}\xi \in D([|T|^{\frac{1}{2}}U^*|_{\mathcal{K}(G)}]^*)$. Now $[|T|^{\frac{1}{2}}U^*|_{\mathcal{K}(G)}]^* = [|T|^{\frac{1}{2}}U^*]^* = U|T|^{\frac{1}{2}}$, so that $|T|^{\frac{1}{2}}\xi \in D(U|T|^{\frac{1}{2}})$, whence $\xi \in D(T)$. □

Theorem 5.12. *Let $p \in [1, 2]$ and $T \in L^p(\psi_0)$. Put $g = \Delta^{-\frac{1}{q}}\overline{\mathcal{F}}_p(T)\check{}$. Suppose that $g \in L^2(G)$. Then T is the closure of the operator*

$$\xi \mapsto g * \Delta^{\frac{1}{p}}\xi, \qquad \xi \in \mathcal{K}(G).$$

Proof. When $g \in L^2(G)$, we have $g * \Delta^{\frac{1}{p}} \in L^2(G)$ for all $\xi \in \mathcal{K}(G)$. Thus, by Proposition 5.11, $\mathcal{K}(G) \subseteq D(T)$, and $T\xi = g * \Delta^{\frac{1}{p}}\xi$ for all $\xi \in \mathcal{K}(G)$. Since $T = [T|_{\mathcal{K}(G)}]$ by Proposition 2.15, the theorem is proved. □

As a corollary, we have

Theorem 5.13 (Fourier inversion)**.** *Let $p \in [1, 2]$ and $\frac{1}{p} + \frac{1}{q} = 1$.*

1) *Let $T \in L^p(\psi_0)$. Put $g = \Delta^{-\frac{1}{q}}\overline{\mathcal{F}}_p(T)\check{}$. If $g \in L^r(G)$ for some $r \in [1, 2]$, then $\mathcal{F}_r(q)\Delta^{\frac{1}{r}-\frac{1}{q}}$ is closable, and*
$$T = \left[\mathcal{F}_r(q)\Delta^{\frac{1}{r}-\frac{1}{q}}\right].$$
2) *Let $f \in L^p(G)$. If for some $r \in [1, 2]$, the closure $S = \left[\mathcal{F}_p(f)\Delta^{\frac{1}{r}-\frac{1}{q}}\right]$ exists and belongs to $L^r(\psi_0)$, then*
$$f = \Delta^{-\frac{1}{s}}\overline{\mathcal{F}}_r(S)\check{},$$
where $\frac{1}{r} + \frac{1}{s} = 1$.

Proof. 1) Since $g \in L^r(G) \cap L^q(G)$, we also have $g \in L^2(G)$. Then by Theorem 5.12 we have

$$T\xi = g * \Delta^{\frac{1}{p}}\xi = g * \Delta^{\frac{1}{s}}\Delta^{-1} = \frac{1}{r} + \frac{1}{p}\xi = \mathcal{F}_r(g)\Delta^{\frac{1}{r}-\frac{1}{q}}\xi$$

for all $\xi \in \mathcal{K}(G)$. Thus $T|_{\mathcal{K}(G)} \subseteq \mathcal{F}_r(g)\Delta^{\frac{1}{r}-\frac{1}{q}}$. As is easily seen $\mathcal{F}_r(g)\Delta^{\frac{1}{r}-\frac{1}{q}}$ is $(-\frac{1}{p})$-homogeneous. It is also closable, since its adjoint is densely defined (indeed, $(\mathcal{F}_r(g)\Delta^{\frac{1}{r}-\frac{1}{q}})^* \subseteq (T|_{\mathcal{K}(G)})^* = T^*$ so that $(\mathcal{F}_r(g)\Delta^{\frac{1}{r}-\frac{1}{q}})^* = T^*$). We conclude that $T = [\mathcal{F}_r(g)\Delta^{\frac{1}{r}-\frac{1}{q}}]$ (since $T \subseteq [\mathcal{F}_r(g)\Delta^{\frac{1}{r}-\frac{1}{q}}]$).

2) For all $\xi \in \mathcal{K}(G)$, we have $\xi \in D(S)$ and by Proposition 5.11,

$$f * \Delta^{\frac{1}{r}}\xi = \mathcal{F}_p(f)\Delta^{\frac{1}{r}-\frac{1}{q}}\xi = S\xi = \Delta^{-\frac{1}{s}}\overline{\mathcal{F}}_r(S)\check{} * \xi.$$

The result follows. □

Putting $p = r = 1$ in the first part of Theorem 5.12 and recalling that $\overline{\mathcal{F}}\left(\frac{d\varphi}{d\psi_0}\right) = \varphi$ for $\varphi \in A(G)_+$ we obtain

Corollary 5.14. *Let $\varphi \in A(G)_+$. If $\check{\varphi} \in L^1(G)$, then*

$$\frac{d\varphi}{d\psi_0} = [\lambda(\check{\varphi})\Delta].$$

Finally we shall give some results on positive operators $T \in L^p(\psi_0)$ valid without any restriction on $\mathcal{F}_p(T)$.

Note that for all $f \in L^q(G)$ and $\xi, \eta \in \mathcal{K}(G)$ we have

$$\begin{aligned}\int f(x)(\xi * J_p\eta)(x)dx &= \int\int f(x)\xi(y)\Delta^{-\frac{1}{p}}(y^{-1}x)\tilde{\eta}(y^{-1}x)dydx \\ &= \int\int f(yx)\xi(y)\Delta^{-\frac{1}{p}}(x)\tilde{\eta}(x)dxdy \\ &= \int\int f(yx^{-1})\xi(y)\Delta^{\frac{1}{q}}(x)\overline{\eta(x)}dxdy.\end{aligned}$$

Proposition 5.15. *Let $p \in [1, 2]$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $T \in L^p(\psi_0)_+$. Put $f = \overline{\mathcal{F}}_p(T)$. Let*

$$q(\xi) = \int f(x)(\xi * J_p\xi)(x)dx = \int\int f(yx^{-1})\Delta^{\frac{1}{q}}(x)\xi(y)\overline{\xi(x)}dydx$$

for all $\xi \in \mathcal{K}(G)$. Then q is a closable positive quadratic form, and the positive self-adjoint operator associated with its closure is T.

Proof. By (the proof of) Proposition 5.10, we have

$$(T^{\frac{1}{2}}\xi|T^{\frac{1}{2}}\xi) = \int f(x)(\xi * J_p\xi)(x)dx = q(\xi)$$

for all $\xi \in \mathcal{K}(G)$, and $T^{\frac{1}{2}} = [T^{\frac{1}{2}}|_{\mathcal{K}(G)}]$. Thus q is a closable positive quadratic form with closure corresponding to T. □

Corollary 5.16. *Let $\varphi \in A(G)_+$. Then $\frac{d\varphi}{d\psi_0}$ is the positive self-adjoint operator associated with the closure of the positive quadratic form q given by*

$$q(\xi) = \int \varphi(x)(\xi * \xi^*)(x)dx = \int\int \varphi(yx^{-1})\xi(y)\overline{\xi(x)}dydx$$

for all $\xi \in \mathcal{K}(G)$.

Remark 5.17. This result also follows directly from the definition of $\frac{d\varphi}{d\psi_0}$. Indeed,

$$\left\|\left(\frac{d\varphi}{d\psi_0}\right)^{\frac{1}{2}}\xi\right\|^2 = \varphi(\lambda(\xi)\lambda(\xi)^*) = \int \varphi(x)(\xi * \xi^*)(x)dx$$

for all $\xi \in \mathcal{K}(G)$, and we have $\left(\frac{d\varphi}{d\psi_0}\right)^{\frac{1}{2}} = \left[\left(\frac{d\varphi}{d\psi_0}\right)^{\frac{1}{2}}\Big|_{\mathcal{K}(G)}\right]$ by Proposition 2.15 (or, alternatively, by an application of [9, Theorem] together with the fact that $\left(\frac{d\varphi}{d\psi_0}\right)^{\frac{1}{2}} = \left[\left(\frac{d\varphi}{d\psi_0}\right)^{\frac{1}{2}}\Big|_{\mathfrak{A}_l}\right]$).

Actually, the property of defining closable quadratic forms on $\mathcal{K}(G)$ characterizes $A(G)_+$-functions among all positive definite continuous functions. The precise statement is as follows:

Theorem 5.18. *Let φ be a positive definite continuous function. define q on $\mathcal{K}(G)$ by*

$$q(\xi) = \int \varphi(x)(\xi * \xi^*)(x)dx = \int\int \varphi(yx^{-1})\xi(y)\overline{\xi(x)}dydx, \qquad \xi \in \mathcal{K}(G).$$

Then q is a positive quadratic form on $\mathcal{K}(G)$, and q is closable if and only if $\varphi \in A(G)$.

Proof. That q is a quadratic form is obvious, and since φ is positive definite, q is positive.

Now suppose that q is closable. Denote by T the positive self-adjoint operator associated with its closure; Then T is characterized by the properties $\mathcal{K}(G) \subseteq D(T^{\frac{1}{2}})$, $T^{\frac{1}{2}} = [T^{\frac{1}{2}}|_{\mathcal{K}(G)}]$, and

$$\forall \xi \in \mathcal{K}(G): \ \|T^{\frac{1}{2}}\xi\|^2 = q(\xi).$$

Let us show that T is (-1)-homogeneous. Suppose that $x \in G$. Then $T_x = \Delta^{-1}(x)\rho(x)T\rho(x^{-1})$ is positive self-adjoint and $T_x^{\frac{1}{2}} = \Delta^{-\frac{1}{2}}(x)\rho(x)T^{\frac{1}{2}}\rho(x^{-1})$. Therefore $\mathcal{K}(G) \subseteq D(T_x^{\frac{1}{2}})$ and $T_x^{\frac{1}{2}} = [T_x^{\frac{1}{2}}|_{\mathcal{K}(G)}]$. Furthermore, for all $\xi \in \mathcal{K}(G)$ we have

$$\begin{aligned}
\|T_x^{\frac{1}{2}}\xi\|^2 &= \|\Delta^{-\frac{1}{2}}(x)\rho(x)T^{\frac{1}{2}}\rho(x^{-1})\xi\| \\
&= \Delta^{-1}(x)\|T^{\frac{1}{2}}\rho(x^{-1})\xi\|^2 \\
&= \Delta^{-1}(x)q(\rho(x^{-1})\xi) \\
&= \Delta^{-1}(x)\int\int \varphi(yz^{-1})(\rho(x^{-1})\xi)(y)\overline{(\rho(x^{-1})\xi)(z)}dydz \\
&= \int\int \Delta^{-1}(x)\varphi(yz^{-1})\Delta^{\frac{1}{2}}(x^{-1})\xi(yx^{-1})\Delta^{\frac{1}{2}}(x^{-1})\overline{\xi(zx^{-1})}dydz \\
&= \Delta^{-1}(x)\int\int \varphi(yxz^{-1})\xi(y)\overline{\xi(zx^{-1})}dydz \\
&= \int\int \varphi(yz^{-1})\xi(y)\overline{\xi(z)}dzdy \\
&= q(\xi).
\end{aligned}$$

We conclude from the characterization of T that $T_x = T$, so that

$$\forall x \in G: \ \Delta^{-1}(x)\rho(x)T\rho(x^{-1}) = T,$$

i.e. T is (-1)-homogeneous.

Now let $(\xi_i)_{i\in I}$ be an approximate identity in $\mathcal{K}(G)_+$. Then

$$\begin{aligned}\|T^{\frac{1}{2}}\xi_i\|^2 &= q(\xi_i)\\ &= \int \varphi(x)(\xi_i * \xi_i^*)(x)dx\\ &\le \sup\{|\varphi(x)|\ |x \in \mathrm{supp}(\xi_i * \xi_i^*)\} \cdot \|\xi_i * \xi_i^*\|_1\\ &\le \sup\{|\varphi(x)|\ |x \in \mathrm{supp}(\xi_i * \xi_i^*)\}.\end{aligned}$$

Since φ is continuous and the support of the $\xi_i * \xi_i^*$ tend to $\{e\}$, we get

$$\liminf_{i\in I} \|T^{\frac{1}{2}}\xi_i\|^2 \le \varphi(e).$$

By Proposition 2.10, this shows that $T \in L^1(\psi_0)$.

Put $\varphi_1 = \overline{\mathcal{F}}(T) \in A(G)$. Then

$$\int \varphi_1(x)(\xi * \xi^*)(x)dx = \|T^{\frac{1}{2}}\xi\|^2 = q(\xi) = \int \varphi(x)(\xi * \xi^*)(x)dx$$

for all $\xi \in \mathcal{K}(G)$. We conclude that $\varphi = \varphi_1$ and thus $\varphi \in A(G)$. □

Acknowledgment. The author is indebted to Uffe Haagerup for many helpful discussions on the subject of this paper, in the period 1979-1980 when it was first drafted. The paper has remained unpublished until now, as it only came out as a preprint (University of Copenhagen, Matematisk Institut Preprint Series 1980, no.11). The author thanks "Advances in Operator Theory (AOT)" for assistance in the publication process, including the transformation of the original typewritten format into TEX.

References

1. F. Combes, *Poids associé a une algèbra hilbertienne a gauche*, Compositio Math. **23** (1971), 49–77.
2. A. Connes, *On the spatial theory of von Neumann algebras*, J. Funct. Anal. **35** (1980), 153–164.
3. A. van Daele, *A new approach to the Tomita-Takesaki theory of generalized Hilbert algebras*, J. Funct. Anal. **15** (1974), 378–393.
4. J. Dixmier, *Les C^*-algebres et leurs representations*, (French) Cahiers Scientifiques, Fasc. XXIX Gauthier-Villars & Cie, Éditeur-Imprimeur, Paris 1964.
5. J. Dixmier, *Formes linéaires sur un anneau d'opérateurs*, Bull. Soc. Math. France **81** (1953), 9–39.
6. P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France **92** (1964), 181–236.
7. P. Eymard and M. Terp, *La transformation de Fourier et son inverse sur le groupe des $ax + b$ d'un corps local*, Lecture Notes in Mathematics 739 (Analyse Harmonique sur les Groupes de Lie II, Séminaire Nancy-Strasbourg 1976-78, fid.: P. Eymard et al.), 207-248. Springer, Berlin-Heidelberg New York, 1979.
8. U. Haagerup, *The standard form of von Neumann algebras*, Math. Scand. **37** (1975), 271–283.
9. U. Haagerup, *A density theorem for left Hilbert algebras*, Lecture Notes in Mathematics 725 (Algèbres d'Opérateurs, Séminaire, Les Plans-sur-Bex, Suisse 1978, fid.: P. de la Harps), 170-179. Springer, Berlin-Heidelberg-New York, 1979.

10. U. Haagerup, *L^p-spaces associated with an arbitrary von Neumann algebra*, Algèbres d'opérateurs et leurs applications en physique mathématique (Colloques internationaux du CNRS, No- 274, Marseille 20-24 juin 1977), 175-184. Editions du CNRS, Paris 1979.
11. E. Hewitt and K. A. Ross, *Abstract harmonic analysis I*, Springer, Berlin-Göttingen Heidelberg 1963.
12. M. Hilsum, *Les espaces L^p d'une algèbre de von Neumann définies par la derivée spatiale*, (French) [L^p-spaces of a von Neumann algebra defined by the spatial derivative], J. Funct. Anal. 40 (1981), no. 2, 151–169.
13. A. Klein and B. Russo, *Sharp inequalities for Weyl operators and Heisenberg groups*, Math. Ann. **235** (1978), 175–194.
14. R. A. Kunze, *L^p Fourier transforms on locally compact unimodular groups*, Trans. Amer. Math. Soc. **89** (1958), 519-540.
15. E. Nelson, *Notes on non-commutative integration*, J. Funct. Anal. **15** (1974), 103–116.
16. G. K. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, London-New York-San Francisco 1979.
17. G. K. Pedersen and M. Takesaki, T*he Radon-Nikodym theorem for von Neumann algebras*, Acta Math. **130** (1973), 53–87.
18. J. Phillips, *Positive integrable elements relative to a left Hilbert algebra*, J. Funct. Anal. **13** (1973), 390–409.
19. W. Rudin, *Functional analysis*, McGraw-Hill Series in Higher Mathematics. McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973.
20. B. Russo, *On the Hausdorff–Young theorem for integral operators*. Pacific J. Math. **68** (1977), 241–253.
21. I. E. Segal, *A non-commutative extension of abstract integration*, Ann. of Math. **57** (1953), 401–457.
22. S. Strătilă and L. Zsidó, *Lectures on von Neumann algebras*, Abacus Press, Kent 1979.
23. A. Weil, *L'intégration dans les groupes topologiques et ses applications*. Hermann, Paris 1940.
24. A. Zygmund, *Trigonometric series II*, Cambridge University Press, New York, 1959.

Department of Mathematical Sciences, Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark.
Present address: Livjægergade 43, st.tv, DK-2100 Copenhagen Ø, Denmark.
E-mail address: mtgmohr@gmail.com