Adv. Oper. Theory 3 (2018), no. 4, 763–780
https://doi.org/10.15352/aot.1801-1303
ISSN: 2538-225X (electronic)
https://projecteuclid.org/aot

ESTIMATIONS OF THE LEHMER MEAN BY THE HERON MEAN AND THEIR GENERALIZATIONS INVOLVING REFINED HEINZ OPERATOR INEQUALITIES

MASATOSHI ITO

Communicated by R. Drnovšek

Abstract. As generalizations of the arithmetic and the geometric means, for positive real numbers a and b, the power difference mean $J_q(a,b) = \frac{q}{q+1}\frac{a^{q+1}-b^{q+1}}{a^q-b^q}$, the Lehmer mean $L_q(a,b) = \frac{a^{q+1}+b^{q+1}}{a^q+b^q}$ and the Heron mean $K_q(a,b) = (1-q)\sqrt{ab} + q\frac{a+b}{2}$ are well known.

In this paper, concerning our recent results on estimations of the power difference mean, we obtain the greatest value $\alpha = \alpha(q)$ and the least value $\beta = \beta(q)$ such that the double inequality for the Lehmer mean

$$K_\alpha(a,b) < L_q(a,b) < K_\beta(a,b)$$

holds for any $q \in \mathbb{R}$. We also obtain an operator version of this estimation. Moreover, we discuss generalizations of the results on estimations of the power difference and the Lehmer means.This argument involves refined Heinz operator inequalities by Liang and Shi.

References

1. R. Bhatia, *Interpolating the arithmetic-geometric mean inequality and its operator version*, Linear Algebra Appl. **413** (2006), 355–363.
2. J. I. Fujii and Y. Seo, *On parameterized operator means dominated by power ones*, Sci. Math. **1** (1998), 301–306.
3. M. Fujii, S. Furuichi, and R. Nakamoto, *Estimations of Heron means for positive operators*, J. Math. Inequal. **10** (2016), no. 1, 19–30.

Date: Received: Jan. 25, 2018; Accepted: Apr. 24, 2018.

2010 *Mathematics Subject Classification.* Primary 47A63; Secondary 26E60, 47A64.

Key words and phrases. Lehmer mean, Heron mean, power difference mean, operator mean, Heinz operator inequalities.

4. S. Furuichi and K. Yanagi, *Bounds of the logarithmic mean*, J. Inequal. Appl. **2013**, 2013:535, 11 pp.
5. T. Furuta, *Concrete examples of operator monotone functions obtained by an elementary method without appearing to Löwner integral representation*, Linear Algebra Appl.**429** (2008), 972–980.
6. F. Hiai and H. Kosaki, *Means for matrices and comparison of their norms*, Indiana Univ. Math. J. **48** (1999), 899–936.
7. M. Ito, *Estimations of power difference mean by Heron mean*, J. Math. Inequal. **11** (2017), no. 3, 831–843.
8. M. Kawasaki and M. Nagisa, *Transforms on operator monotone functions*, arXiv:0706.1234.
9. F. Kittaneh and M. Krnić, *Refined Heinz operator inequalities*, Linear Multilinear Algebra **61** (2013), 1148–1157.
10. F. Kubo and T. Ando, *Means of positive linear operators*, Math. Ann. **246** (1980), 205–224.
11. J. Liang and G. Shi, *Refinements of the Heinz operator inequalities*, Linear Multilinear Algebra **63** (2015), 1337–1344.
12. M. Nagisa and S. Wada, *Operator monotonicity of some functions*, Linear Algebra Appl. **486** (2015), 389–408.
13. Y. Nakamura, *Classes of operator monotone functions and Stieltjes functions*, The Gohberg anniversary collection, Vol. II (Calgary, AB, 1988), Oper. Theory Adv. Appl., 41, Birkhäuser, Basel, 1989, pp. 395–404.
14. C. P. Niculescu and L.-E. Persson, *Convex functions and their applications: A Contemporary Approach*, CMS Books in Mathematics, vol. 23. Springer, New York (2006).
15. V. E. S. Szabó, *A class of matrix monotone functions*, Linear Algebra Appl. **420** (2007), 79–85.
16. Y. Udagawa, S. Wada, T. Yamazaki, and M. Yanagida, *On a family of operator means involving the power difference means*, Linear Algebra Appl. **485** (2015), 124–131.
17. W.-F. Xia, S.-W. Hou, G.-D. Wang, and Y.-M. Chu, *Optimal one-parameter mean bounds for the convex combination of arithmetic and geometric means*, J. Appl. Anal **18** (2012), 197–207.
18. L. Zou, *Matrix versions of the classical Pólya inequality*, ScienceAsia **39** (2013), no. 2, 204–207.

Maebashi Institute of Technology, 460-1 Kamisadorimachi, Maebashi, Gunma 371-0816, Japan.

E-mail address: m-ito@maebashi-it.ac.jp