Advances in Operator Theory

Adv. Oper. Theory 4 (2019), no. 2, 462–480
https://doi.org/10.15352/aot.1809-1423
ISSN: 2538-225X (electronic)
https://projecteuclid.org/aot

A VARIATIONAL INEQUALITY THEORY FOR CONSTRAINED PROBLEMS IN REFLEXIVE BANACH SPACES

T. M. ASFAW

Communicated by D. M. Pellegrino

Abstract. Let X be a real locally uniformly convex reflexive Banach space with the locally uniformly convex dual space X^*, and let K be a nonempty, closed, and convex subset of X. Let $T : X \supseteq D(T) \to 2^{X^*}$ be maximal monotone, let $S : K \to 2^{X^*}$ be bounded and of type (S_+), and let $C : X \supseteq D(C) \to X^*$ with $D(T) \cap D(\partial\phi) \cap K \subseteq D(C)$. Let $\phi : X \to (-\infty, \infty]$ be a proper, convex, and lower semicontinuous function. New existence theorems are proved for solvability of variational inequality problems of the type $\mathrm{VIP}(\mathrm{T}+\mathrm{S}+\mathrm{C}, \mathrm{K}, \phi, \mathrm{f}^*)$ if C is compact and $\mathrm{VIP}(\mathrm{T}+\mathrm{C}, \mathrm{K}, \phi, \mathrm{f}^*)$ if T is of compact resolvent and C is bounded and continuous. Various improvements and generalizations of the existing results for $T+S$ and ϕ are obtained. The theory is applied to prove existence of solution for nonlinear constrained variational inequality problems.

References

1. T. M. Asfaw, *New variational inequality and surjectivity theories for perturbed noncoercive operators and application to nonlinear problems*, Adv. Math. Sci. Appl. **24** (2014), 611–668.
2. T. M. Asfaw, *New surjectivity results for perturbed weakly coercive operators of monotone type in reflexive Banach space*, Nonlinear Anal. **113** (2015), 209–229.
3. T. M. Asfaw, *Maximality theorems on the sum of two maximal monotone operators and application to variational inequality problems*, Abstr. Appl. Anal. 2016, Art. ID 7826475, 10pp.

Date: Received: Sep. 26, 2018; Accepted: Oct. 14, 2018.
2010 *Mathematics Subject Classification.* Primary 47H20; Secondary 47H14.
Key words and phrases. Variational inequality, compact resolvent, constrained problems, elliptic and parabolic problems.

4. T. M. Asfaw, *A degree theory for compact perturbations of monotone type operators and application to nonlinear parabolic problem*, Abstr. Appl. Anal. 2017, Art. ID 7236103, 13pp.
5. T. M. Asfaw and A. G. Kartsatos, *A Browder topological degree theory for multivalued pseudomonotone perturbations of maximal monotone operators in reflexive Banach spaces*, Adv. Math. Sci. Appl. **22** (2012), 91–148.
6. T. M. Asfaw and A. G. Kartsatos, *Variational inequalities for perturbations of maximal monotone operators in reflexive Banach spaces*, Tohoku Math. J. **66** (2014), 171–203.
7. V. Barbu, *Nonlinear differential equations of monotone types in Banach spaces*, Springer Monographs in Mathematics, New York, 2010.
8. H. Brézis, M. G. Crandall, and A. Pazy, *Perturbations of nonlinear maximal monotone sets in Banach spaces*, Comm. Pure Appl. Math. **23** (1970), 123–144.
9. H. Brèzis and L. Nirenberg, *Characterizations of the ranges of some nonlinear operators and applications to boundary value problems*, Ann. Scuola Norm. Sup. Pisa **5** (1978), 225–326.
10. F. E. Browder, *Nonlinear operators and nonlinear equations of evolution in Banach spaces*, Proc. Sympos. Pure Math. **18** (1976), 1–308.
11. F. E. Browder and P. Hess, *Nonlinear mappings of monotone type in Banach spaces*, J. Funct. Anal. **11** (1972), 251–294.
12. D. Goeleven and D. Motreanu, *Variational and hemivariational inequalities: Theory, methods and applications Vol. II. Unilateral problems. Nonconvex optimization and its applications*, 70., Kluwer Academic Publishers, Boston, 2003.
13. A. G. Kartsatos, *New results in the perturbation theory of maximal monotone operators in Banach spaces*, Trans. Amer. Math. Soc. **348** (1996), 1663–1707.
14. N. Kenmochi, *Nonlinear operators of monotone type in reflexive Banach spaces and nonlinear perturbations*, Hiroshima Math. J. **4** (1974), 229–263.
15. N. Kenmochi, *Pseudomonotone operators and nonlinear ellpitic boundary value problems*, J. Math. Soc. Japan **27** (1975), 121–149.
16. N. Kenmochi, *Monotonicity and compactness methods for nonlinear variational inequalities*, Handbook of differential equations, IV, Elsevier/North-Holland, Amsterdam, 2007, 203–298.
17. A. A. Khan and D. Motreanu, *Existence theorems for elliptic and evolutionary variational and quasi-variational inequalities*, J. Optim. Theory Appl. **167** (2015), 1136–1161.
18. D. Kinderlehrer and G. Stampacchia, *An introduction to variational inequalities and their applications*, *Pure Appl. Math.* **88**, Academic Press, New York, 1980.
19. J. Kobayashi and M. Otani, *Topological degree for* $(\mathrm{S})_+$*-mappings with maximal monotone perturbations and its applications to variational inequalities*, Nonlinear Anal. **59** (2004), no. 1-2, 147–172.
20. V. K. Le, *A range and existence theorem for pseudomonotone perturbations of maximal monotone operators*, Proc. Amer. Math. Soc. **139** (2011), 1645–1658.
21. Z. Naniewicz and P. D. Panagiotopoulos, *Mathematical theory of hemivariational inequalities and applications*, Monographs and Textbooks in Pure and Applied Mathematics, **188**, Marcel Dekker, Inc., New York, 1995.
22. D. Pascali and S. Sburlan, *Nonlinear mappings of monotone type*, Sijthoff and Noordhoff, Bucharest, 1978.
23. R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc. **149** (1970), 75–88.
24. R. E. Showalter, *Monotone operators in Banach space and nonlinear partial differential equations.* Mathematical Surveys and Monographs, 49. American Mathematical Society, Providence, RI, 1997.
25. S. Troyanski, *On locally uniformly convex and differentiable norms in certain non-separable Banach spaces*, Studia Math. **37** (1971), 173–180.
26. E. Zeidler, *Nonlinear functional analysis and its applications*, **II/B**, Springer-Verlag, New York, 1990.

Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, VA24061, USA.
E-mail address: teffera6@vt.edu